THREE RECENT APPLICATIONS OF MODEL THEORY

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This spring's MSRI program *Model Theory, Arithmetic Geometry, and Number Theory* is centered on recent interactions between model theory (a branch of mathematical logic) and other parts of mathematics. To give some idea of what these interactions are, I will discuss here three particular examples of applications of model theory: to Berkovich spaces, to approximate subgroups, and to the André-Oort Conjecture for $\mathbb{C}^n$. Except for some concluding remarks on model theory, I will say almost nothing about the techniques and ideas that are behind the proofs of these theorems, and only hope that the interested reader will pursue his or her own further investigations.

Each of the applications I will discuss was the subject of tutorials (available on the MSRI web page) in the Introductory Workshop of our program as well as Séminaire Bourbaki articles. The tutorials were by Martin Hils, Lou van den Dries, and Kobi Peterzil, respectively, and the corresponding articles are by Antoine Ducros, Lou van den Dries, and Thomas Scanlon. I have relied heavily on these sources, and it is to them that I direct the reader for further expository details.

1. Berkovich Spaces

In a recent manuscript entitled "Non-archimedean tame topology and stably dominated types" Hrushovski and Loeser use model theory to develop a framework for studying the analytic geometry associated to an algebraic variety over a non-archimedean valued field. As a consequence they deduce several new results on Berkovich spaces.

Fix a complete non-archimidean absolute valued field $(K, |\cdot|)$. *Non-archimedean* refers to the fact that $|\cdot| : K \to \mathbb{R}_{\geq 0}$ satisfies the *ultrametric inequality*

$$|a + b| \leq \max\{|a|, |b|\}$$

and *complete* means with respect to the induced metric. The prototypical examples are: the field of p-adic numbers $\mathbb{Q}_p$, the completion of the algebraic closure of $\mathbb{Q}_p$, and the Laurent series fields $k((t))$. Now consider an algebraic variety V over K. In analogy with real or complex algebraic varieties, one would like to use the metric structure on K to consider $V(K)$ from the point of view of analytic geometry. The problem is that the topology that $|\cdot|$ induces on $V(K)$ is totally disconnected. In the early nineties Berkovich proposed to resolve this deficiency by considering an enriched space V_K^{an} whose points are pairs (x, ν) where x is a scheme-theoretic point of V and $\nu : K(x) \to \mathbb{R}_{\geq 0}$ is an absolute value extending that of K. More concretely, in the case when V is affine, V_K^{an} can be canonically identified with the set of *multiplicative seminorms* on the co-ordinate ring $K[V]$; that is, multiplicative maps $\nu : K[V] \to \mathbb{R}_{\geq 0}$ that extend the absolute value on

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K and satisfy the ultrametric inequality. The topology induced on V_K^{an} from the product topology on $\mathbb{R}^{K[V]}$ is then locally path connected and locally compact.

Berkovich spaces have proved to have many and diverse applications. They have led to the development of p-adic analogues of classical notions from complex analysis including spectral theory, harmonic analysis, equidistribution, and dynamics. There have been applications to the Langlands program in arithmetic geometry via the development of étale cohomology of analytic spaces. Finally, by endowing a given ground field with the trivial absolute value (which, note, is complete and non-archimedean), Berkovich spaces have also been useful in general algebraic geometry.

Hrushovski and Loeser use model theory to show that Berkovich spaces exhibit very tame topological behaviour, generalisating and strengthening what was known before. Here are some of their results.

Theorem 1 (Hrushovski, Loeser)**.** *Suppose V is a quasi-projective variety over K. Then*

(1) *V_K^{an} admits a strong deformation retraction to a closed subspace that is homeomorphic to a finite simplicial complex,*
(2) *V_K^{an} is locally contractible, and*
(3) *given a morphism $f : V \to W$ to an algebraic variety W over K, among the fibres of $f^{\mathrm{an}} : V_K^{\mathrm{an}} \to W_K^{\mathrm{an}}$ there are only finitely many homotopy types.*

2. Approximate Groups

Given a positive integer K, a *K-approximate group* is a finite subset X of a group G such that $1 \in X$, $X^{-1} = X$, and $X^2 := \{xy : x, y \in X\}$ is covered by K left translates of X. This is supposed to say that X is almost closed under multiplication; so one should think of K as being fixed and of $|X|$ as being large compared to K. A 1-approximate group is a subgroup, and an easy example of a 2-approximate group that is not a subgroup is the set $\{-N, \dots, N\}$ in $\mathbb{Z}$, for any $N > 0$. But the interest here is really when G is not commutative; approximate subgroups were introduced by Tao while studying the extension of additive combinatorics to the non-commutative setting.

In his 2012 paper entitled "Stable group theory and approximate subgroups", Hrushovski studies the structure of K-approximate groups as the cardinality $|X|$ goes to infinity by applying model-theoretic techniques to the logical limits (i.e. *ultraproducts*) of sequences of K-approximate groups. His main achievement is to model such a limit of approximate groups by a compact neighbourhood of the identity in a Lie group. This is reminiscent of the proof of Gromov's theorem on groups of polynomial growth, and indeed, one of the striking applications of Hrushovski's work is a strengthening (and new proof) of Gromov's theorem. Another application is an extension of the Freiman-Ruzsa theorem to the non-commutative setting: in a group of finite exponent every K-approximate group is *commensurable* to an actual subgroup, commensurable here in the sense that each is contained in finitely many left translates of the other where the number of translates is bounded in terms of K. But the most celebrated application is the theorem of Breuillard, Green, and Tao saying roughly that approximate groups are in general controlled by nilpotent groups. This appears in their 2012 paper "The structure of approximate groups", where they also give alternative proofs of some of Hrushovski's results. Here is a weak version of their theorem that is simple to state.

Theorem 2 (Breuillard, Green, Tao)**.** *Given $K \geq 1$ there exists $L \geq 1$ such that for any K-approximate group $X \subseteq G$ there is a finite set $Y \subseteq \langle X \rangle$ such that X is covered by L left translates of Y, Y is covered by L left translates of X, and $\langle Y \rangle$ has a nilpotent subgroup of finite index.*

Among the applications of this theorem is a finitary version of Gromov's theorem and a generalised Margulis Lemma that was conjectured by Gromov.

3. André-Oort for $\mathbb{C}^n$

Model theory's first spectacular application to diophantine geometry was Hrushovski's solution in the early nineties to the function-field Mordell-Lang Conjecture in all characteristics. This was one of the central themes of the 1998 MSRI program on the model theory of fields. In recent years there has been another round of diophantine applications, this time to the André-Oort Conjecture, in which model theory plays a very different role. The model theory behind these latest interactions stems from the 2006 paper of Pila and Wilkie that used model theory to count rational points on a certain class of subsets of $\mathbb{R}^n$ with tame topological properties. Following a general strategy proposed by Zannier, there are now a number of applications of this result in various directions. I will focus on what is possibly the most striking one thus far: Pila's solution to the André-Oort Conjecture for $\mathbb{C}^n$.

Recall that to each point τ in the upper half plane $\mathbb{H} := \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$ we can associate the elliptic curve $E_\tau := \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\tau)$. The elliptic curve E_τ is said to have *complex multiplication* if its endomorphism ring is strictly bigger than $\mathbb{Z}$, which is equivalent to τ belonging to an imaginary quadratic extension of $\mathbb{Q}$. Now, there is a holomorphic surjection $j : \mathbb{H} \to \mathbb{C}$ with the property that $j(\tau_1) = j(\tau_2)$ if and only if E_{τ_1} and E_{τ_2} are isomorphic. We are interested in the affine varieties $X \subseteq \mathbb{A}^n_{\mathbb{C}}$ which have a Zariski dense set of points of the form $\big(j(\tau_1), \ldots, j(\tau_n)\big)$ where each E_{τ_i} has complex multiplication. One thinks of the set of these points, called *special points*, as being in some way arithmetical, roughly analogous to the set of torsion points on a semiabelian variety. It is a fact that the special points are Zariski dense in $\mathbb{A}^n_{\mathbb{C}}$, so affine space itself gives us examples of such varieties X. More interesting examples are obtained by considering the Hecke correspondences

$$T_N := \{\big(j(\tau), j(N\tau)\big) : \tau \in \mathbb{H}\}$$

for each positive integer N. It turns out that T_N is an algebraic curve in $\mathbb{A}^2$. It has a Zariski dense set of special points since if τ is in a quadratic imaginary extension of $\mathbb{Q}$, then so is $N\tau$. The André-Oort conjecture for $\mathbb{C}^n$, proved by Pila in 2011, says that all examples come from the above two types. More precisely:

Theorem 3 (Pila)**.** *Suppose $X \subseteq \mathbb{A}^n_{\mathbb{C}}$ is an irreducible subvariety containing a Zariski dense set of special points. Then X is an irreducible component of an intersection of varieties of the form:*

- $S_{i,\tau} := \{(z_1, \ldots, z_n) : z_i = j(\tau)\}$ *where E_τ has complex multiplication, and*
- $T_{i,j,N} := \{(z_1, \ldots, z_n) : (z_i, z_j) \in T_N\}$ *where $N > 0$.*

4. And behind them all: Model Theory

To the reader unfamiliar with model theory it may be surprising that the above theorems are all applications of a single subject, and at that a branch of mathematical logic. In fact, model theory often plays the role of recognising, formalising, and facilitating analogies between different mathematical settings. In this final section I would like to say a few words about what model theory is.

The fundamental notion in model theory is that of a structure. A *structure* consists of an underlying set M together with a set of distinguished subsets of various cartesian powers of M called the *basic relations*. It is assumed that equality is a basic (binary) relation in every structure. One could also allow *basic functions* from various cartesian powers of M to M, but by replacing them with their graphs we can restrict to relational structures. For example, a ring can be viewed as a structure where the underlying set is the set of elements of the ring and there are, besides equality, two basic relations: the ternary relations given by the graphs of addition and multiplication. If the ring also admits an ordering that we are interested in, then we can consider the new structure where we add the ordering as another basic binary relation. The *definable sets* of a structure are those subsets of cartesian powers of M that are obtained from the basic relations in finitely many steps using the following operations: intersection, union, complement, cartesian product, image under a co-ordinate projection, and fibre of a co-ordinate projection. When $(R, +, \times)$ is a commutative unitary ring, for example, one sees immediately that if $f_1, \dots, f_\ell$ are polynomials in $R[x_1, \dots, x_n]$ then their set of common zeros in R^n, is definable. Hence the Zariski constructible subsets of R^n are all definable. It is an important fact that if R is an algebraically closed field then these are the only definable sets. This is *quantifier elimination* for algebraically closed fields, or equivalently Chevalley's theorem that over an algebraically closed field the projection of a constructible set is again constructible.

In any case, given a structure, model theory is concerned with this associated class of definable sets. Of course, starting with an arbitrary structure one cannot expect to say much. A key aspect is the isolation of *tameness* conditions under which the definable sets are in some way tractable. For example, algebraically closed fields are *strongly minimal* because the definable subsets of the field itself are all uniformly finite or cofinite. Strongly minimal structures admit a very well-behaved notion of *dimension* for definable sets. Real closed fields, on the other hand, display a different kind of tameness: they are *o-minimal* in that every definable subset of the line is a finite union of intervals and points – and this too leads to a (differently) well-behaved notion of dimension on the cartesian powers. Strong minimality and o-minimality are only at the beginning of extensive hierarchies of tameness notions. Algebraically closed valued fields, for example, with their strongly minimal residue field and o-minimal value group, involve a certain comingling of the two.

Behind Pila's proof of the André-Oort Conjecture for $\mathbb{C}^n$ is the definability of the j-function (restricted to a suitable fundamental set) in some o-minimal structure on the reals and the Pila-Wilkie theorem on counting rational points on definable sets in such structures. The theorems of Hrushovki and Loeser on Berkovich spaces use the tameness of definable sets in algebraically closed valued fields. The structure that lies behind the work of Hrushovski and that of Breuillard, Green and Tao on approximate groups is an ultraproduct of K-approximate groups. In each of the applications that I have discussed, the model theoretic techniques and ideas that are brought to bear on the problem are quite specialised, and it would be misleading to suggest some underlying or overarching principle. Nevertheless, they all stem from the perspective that model theory offers, and it is this perspective that brings together the themes, and participants, of our program.