FUNCTORIALITY AND UNIFORMITY IN HRUSHOVSKI'S GROUPOID-COVER CORRESPONDENCE

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ABSTRACT. The correspondence between definable connected groupoids in a theory T and internal generalised imaginary sorts of T, established by Hrushovski in ["Groupoids, imaginaries and internal covers," *Turkish Journal of Mathematics*, 2012], is here extended in two ways: First, it is shown that the correspondence is in fact an equivalence of categories, with respect to appropriate notions of morphism. Secondly, the equivalence of categories is shown to vary uniformly in definable families, with respect to an appropriate relativisation of these categories. Some elaborations on Hrushovki's original constructions are also included.

Contents

1. INTRODUCTION

The notion of a definable set X being *internal* to another definable set Y has been of central significance in geometric stability theory, as part of the study of the fine structure of definable sets. Roughly speaking, it means that X is definably isomorphic to a definable set in Y^{eq}, though it is crucial that the isomorphism may involve more parameters than those required to define X and Y. For example, in differentially closed fields (DCF_0), we can take Y to be the constant field and X to be the set of solutions to the algebraic differential equation $x' = 1$. Then X is definably isomorphic to Y after we name any one solution to the equation. This dependence on additional parameters is controlled by the *liaison* or *binding group*, an interpretable group acting definably on X and agreeing with the action of the group of automorphisms of the universe that fix Y pointwise. The existence and importance of the liaison group was recognised already by Zilber [10] in the late nineteen-seventies. Poizat [8] realised that this theory of definable automorphism groups, when applied to DCF_0, could be used to recover Kolchin's differential Galois theory. Hrushovski gave the full modern development of the subject, first in the context of stable theories and later in complete generality; in [4] the liaison group

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is constructed from internality assuming only that Y is *stably embedded*, i.e., that the full induced structure agrees with the 0-definable induced structure on Y.

In [5], Hrushovski addresses the question of how to describe X from the point of view of the induced structure on Y. The liaison group lives naturally and without additional parameters in $(X, Y)^{\text{eq}}$, but does not in general have a canonical manifestation in Y^{eq}. Hrushovski introduces *liaison groupoids*, establishing a bijective correspondence (up to a natural notion of equivalence) between definable connected groupoids in a theory T and *internal covers* of T; i.e., expansions of T that are internal to T and in which T is stably embedded. In the above notation, T is the theory of Y^{eq} and the internal cover is the theory of $(X, Y)^{\text{eq}}$. So, what is effected by Hrushovski's correspondence is an encoding of internal covers of T as definable connected groupoids in T, in very much the same way that the sorts of T^{eq} are encoded in T as definable equivalence relations.

In fact there is a small mistake in [5] so that the correspondence does not encompass all definable connected groupoids but only those satisfying the additional technical condition of *finite faithfulness*.[1] This is explained and corrected in Section 3 below, where we also present a brief description of Hrushovski's constructions, elaborating on a few aspects.

This paper extends Hrushovski's work in two ways. First, we show that his correspondence comes from an equivalence of categories. There is a more or less natural way to make internal covers into a category: a morphism between two internal covers should be a function that is definable in a common cover. This is made precise in §4.1, but for example, a definable function between definable sets in DCF_0 that are internal to the constants will give rise to a morphism between the corresponding internal covers of ACF_0. With this notion of morphism, isomorphic internal covers are precisely those that are equivalent in Hrushovski's sense. On the other hand, it is not so clear what we should be regarding as morphisms between definable groupoids. The answer, given in §4.2, takes a somewhat unexpected form. In any case, we are able to show, in the rest of Section 4, that Hrushovski's correspondence is an equivalence of these categories.

Secondly, we pursue a suggestion in [5] and address the question of uniformity in definable families. Allowing connected groupoids to vary in a definable family amounts to dropping the assumption of connectedness, and allowing internal covers to vary definably amounts to replacing internality with 1-*analysability*. We prove that Hrushovski's constructions – of the liaison groupoid associated to an internal cover and of the internal cover associated to a finitely faithful definable connected groupoid – do vary uniformly with the parameters, but only under the (necessary) additional assumption that there is no significant interaction amongst the varying internal covers (called *independence of fibres* below). This is done in Section 5.

2. Preliminaries on covers

We review in this brief section the notion of a cover of a theory. Details on the various model-theoretic notions that come up can be found in [9].

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[1]This mistake in [5, Theorem 3.2] does not invalidate any of the other theorems of that paper, as they relate to contexts where finite faithfulness is automatic; namely when the theory is stable or when the automorphism groups in the groupoid are finite.

Suppose M is a (multi-sorted) structure. Throughout this paper, *definable* means definable with parameters, unless stated otherwise. A 0-definable set X is said to be *stably embedded in M* if for every $n > 0$, every definable subset of X^n is definable with parameters from X. A *stably embedded reduct* of M is a reduct N with the property that every subset of a finite cartesian product of sorts in N that is definable in M is definable in N. So X is stably embedded in M if and only if the structure induced on X by the 0-definable sets of M is a stably embedded reduct of M. Here are some other useful characterisations; see the Appendix of [1] for details.

Fact 2.1. *Suppose M is uncountable and saturated and X is a 0-definable set in M. The following are equivalent:*
  (i) *X is stably embedded.*
  (ii) *For any tuple a of small length from M there is a small subset $B \subset X$ such that $\mathrm{tp}(a/B) \vdash \mathrm{tp}(a/X)$.*
  (iii) *If a and b are tuples from M of small length with $\mathrm{tp}(a/X) = \mathrm{tp}(b/X)$ then there exists $\sigma \in \mathrm{Aut}(M)$ that fixes X pointwise and such that $\sigma(a) = b$.*

Fact 2.2. *Suppose M is uncountable and saturated and N is a reduct of M. The following are equivalent:*
  (i) *N is stably embedded.*
  (ii) *The restriction map $\mathrm{Aut}(M) \to \mathrm{Aut}(N)$ is surjective. We denote the kernel of this map by $\mathrm{Aut}(M/N)$.*

The fact that we have worked with 0-definable sets is not so important. If $A \subset M$ is a small set of parameters, and X is A-definable, then X is said to be *stably embedded over A* if it is stably embedded in M_A, the structure obtained by naming the elements of A as new constants.

Definition 2.3 (Hrushovski [5])**.** Let T be a complete multi-sorted first order theory with elimination of imaginaries in a language L. Consider an expansion L' of L. Let T' be a complete L'-theory that extends T. We say that T' is a *cover* of T if in every model of T' the reduct to L is stably embedded.

Note that if L' does not involve new sorts then any cover of T in L' is a definitional expansion of T. So we are really interested in expansions involving new sorts.

As is the tradition in model theory, we work in a monster model of T. We fix a cardinal λ greater than the size of all objects that we are interested in and assume that $2^\kappa \leq \lambda$ for all $\kappa < \lambda$. In this way all theories (in small languages) that we encounter have a saturated model of cardinality λ, which will serve as a monster model. Such a cardinal can be a strongly inaccessible one. The reader who is uncomfortable assuming the existence of strongly inaccessible cardinals, can, as suggested for example in [3], use forcing to add a cardinal with $2^\kappa = \kappa^+$ while keeping the universe of small sets intact. Other approaches to the monster model include special models of [2] and class-size saturated models as in [9]. Each of these approaches has its own technicalities that need to be overcome.

Thus we fix a monster model $\mathbb{U}$ of T and work inside $\mathbb{U}$. Accordingly we call an L'-expansion $\mathbb{U}'$ a *cover* of $\mathbb{U}$ if it is saturated and $\mathbb{U}$ is stably embedded in $\mathbb{U}'$. This is an abuse of notation as two such expansions $\mathbb{U}_1$ and $\mathbb{U}_2$ of $\mathbb{U}$ may be elementarily equivalent and therefore determine the same cover of T. But by saturation this would mean there is an isomorphism $\sigma : \mathbb{U}_1 \to \mathbb{U}_2$. Stable embeddability implies, by Fact 2.2(ii), that $\sigma|_{\mathbb{U}}$ extends to an automorphism α of $\mathbb{U}_2$, and so $\alpha^{-1}\sigma$ yields

an isomorphism that is the identity on $\mathbb{U}$. That is, $\mathbb{U}_1$ and $\mathbb{U}_2$ give the same cover of T if and only if they are isomorphic over $\mathbb{U}$.

In later sections we will want to consider unions of covers. Suppose $\mathbb{U}_1 = (\mathbb{U}, (S_{1j})_{j \in J_1}, \dots)$ and $\mathbb{U}_2 = (\mathbb{U}, (S_{2j})_{j \in J_2}, \dots)$ are covers of $\mathbb{U}$, where $(S_{ij})_{j \in J_i}$ are the new sorts of $\mathbb{U}_i$ for $i = 1, 2$. Then $\mathbb{U}_1 \cup \mathbb{U}_2$ is by definition the expansion of $\mathbb{U}$ that has new sorts $(S_{ij} : i = 1, 2, j \in J_i)$ and whose structure is as dictated by $\mathbb{U}_1$ and $\mathbb{U}_2$ on the appropriate sorts. The language in which this structure is considered is L together with the disjoint union of $L_1 \setminus L$ and $L_2 \setminus L$.

Lemma 2.4. *If $\mathbb{U}_1$ and $\mathbb{U}_2$ are covers of $\mathbb{U}$ then $\mathbb{U}_1 \cup \mathbb{U}_2$ is a cover of $\mathbb{U}_1$ and $\mathbb{U}_2$, and $\mathrm{Th}(\mathbb{U}_1 \cup \mathbb{U}_2) = \mathrm{Th}(\mathbb{U}_1) \cup \mathrm{Th}(\mathbb{U}_2)$.*

Proof. A saturated (in the same cardinality as $\mathbb{U}_1, \mathbb{U}_2$) model of $\mathrm{Th}(\mathbb{U}_1) \cup \mathrm{Th}(\mathbb{U}_2)$ will be of the form $\mathbb{V}_1 \cup \mathbb{V}_2$ with $\mathbb{V}_i$ a saturated model of $\mathrm{Th}(\mathbb{U}_i)$. We therefore have isomorphisms $\sigma_i : \mathbb{U}_i \to \mathbb{V}_i$. Now $(\sigma_2|_{\mathbb{U}})^{-1}\sigma_1|_{\mathbb{U}} \in \mathrm{Aut}(\mathbb{U})$, and as $\mathbb{U}$ is stably embedded in $\mathbb{U}_2$ we can extend it to an automorphism α of $\mathbb{U}_2$ by Fact 2.2(ii). Replacing σ_2 by $\sigma_2\alpha$, we may therefore assume that σ_1 and σ_2 agree on $\mathbb{U}$. So $\sigma_1 \cup \sigma_2$ is an isomorphism from $\mathbb{U}_1 \cup \mathbb{U}_2$ to $\mathbb{V}_1 \cup \mathbb{V}_2$. This shows that $\mathbb{U}_1 \cup \mathbb{U}_2$ is saturated and $\mathrm{Th}(\mathbb{U}_1 \cup \mathbb{U}_2) = \mathrm{Th}(\mathbb{U}_1) \cup \mathrm{Th}(\mathbb{U}_2)$.

It remains to show that each $\mathbb{U}_i$ is stably embedded in $\mathbb{U}_1 \cup \mathbb{U}_2$. Suppose $\sigma \in \mathrm{Aut}(\mathbb{U}_1)$. Then $\sigma|_{\mathbb{U}} \in \mathrm{Aut}(\mathbb{U})$ and hence extends to an automorphism σ' of $\mathbb{U}_2$. Then $\sigma \cup \sigma' \in \mathrm{Aut}(\mathbb{U}_1 \cup \mathbb{U}_2)$. A similar argument shows that every automorphism of $\mathbb{U}_2$ extends to $\mathbb{U}_1 \cup \mathbb{U}_2$. $\square$

3. Internal covers and definable groupoids

This section is primarily a survey of the correspondence that Hrushovski established in the first few sections of [5] between definable connected groupoids and internal covers. We take this opportunity, however, to correct an error in the published statement and proof of that correspondence. We have also streamlined some of the arguments and elaborated on some of the ideas. Our presentation is largely self-contained, and we have made an effort to be explicit about where we have diverged from Hrushovski's notations and conventions.

We will focus on covers which involve only a finite increase in the language.

Definition 3.1. A cover T' of T is said to be in a language that is *finitely generated over* L if there exists a sublanguage $L'_\circ \subseteq L'$ such that $L'_\circ \setminus L$ is finite and every L'-formula is equivalent modulo T' to an $L'_\circ$-formula.

We will be interested in *internal* covers $\mathbb{U}'$ of $\mathbb{U}$. Since T eliminates imaginaries this can be characterised by saying that each new sort S of $\mathbb{U}'$ is definably isomorphic (over possibly additional parameters) with a definable set in $\mathbb{U}$.

Our primary focus is on internal covers that involve only one new sort.

Definition 3.2 (Hrushovski [5]). An *internal generalised imaginary sort* of $\mathbb{U}$ is an internal cover $\mathbb{U}'$ with only one new sort S, such that the language of $\mathbb{U}'$ is finitely generated over the language of $\mathbb{U}$.

There is a canonical example of such internal covers.

Example 3.3. Let $\mathbb{U} = (G, \cdot, e)$ be a group and S a principal homogeneous space for G with the action denoted by $\rho : G \times S \to S$. Then $\mathbb{U}' = (G, S, \cdot, e, \rho)$ is an internal cover with new sort S witnessed by fixing any $a \in S$ and considering the definable bijection $\rho_a : G \to S$ given by $g \mapsto \rho(g, a)$.

If we fix $a \in S$ then we have another, a-definable, action of G on S: each $g \in G$ gives rise to the definable bijection $\alpha_g : S \to S$ given by $\alpha_g(s) = \rho_a((\rho_a)^{-1}(s)g^{-1})$. Note that $\alpha_g \cup \mathrm{id}_G$ preserves ρ and so is an automorphism of $\mathbb{U}'$. The correspondence $g \mapsto \alpha_g \cup \mathrm{id}_G$ is an isomorphism between G and $\mathrm{Aut}(\mathbb{U}'/\mathbb{U})$, as groups acting on S.

As is well known, this phenomenon occurs under very broad conditions: if $\mathbb{U}'$ is an internal generalised imaginary sort then $\mathrm{Aut}(\mathbb{U}'/\mathbb{U})$ together with its action on S is interpretable in $\mathbb{U}'$. It is called the liaison group. However, in general we need parameters to realise this group in $\mathbb{U}$. (In the above example parameters were not needed to define the group, just to define the action). The parameter-free construction of Hrushovski [5] gives instead a definable liaison *groupoid* in $\mathbb{U}$. Moreover, this groupoid captures precisely the internal cover $\mathbb{U}'$.

A groupoid $\mathcal{G}$ is a category where all morphisms are invertible. It is called *connected* if it is nonempty and there is a morphism between any two objects. If a, b are objects in $\mathcal{G}$, then we denote by $\mathrm{Hom}_{\mathcal{G}}(a, b)$ the set of morphisms from a to b and also denote $\mathrm{Aut}_{\mathcal{G}}(a) = \mathrm{Hom}_{\mathcal{G}}(a, a)$. If $\mathcal{G}$ is clear, we may skip it in the subscript. Note that if $\mathcal{G}$ is connected then all these automorphism groups are isomorphic.

A *definable groupoid* consists of a definable family $(O_i)_{i \in I}$ of definable sets and a definable family $(f_m)_{m \in M}$ of definable bijections between them, closed under composition and taking inverses. It is worth being more explicit here. There is a definable set I, which encodes the set of objects, and a definable $M \to I \times I$, which encodes the set of morphisms. There is a definable set O and a surjective definable map $\pi : O \to I$ such that $O_i = \pi^{-1}(i)$, and a definable subset $f \subseteq M \times_{I \times I} (O \times O)$ such that $f_m = \{(x, y) \in O^2 : (m, x, y) \in f\}$ is the graph of a bijection between O_i and O_j where m lives over (i, j). Further we require that distinct $m \in M$ define distinct bijections. The groupoid structure on (I, M) (i.e. the composition of two morphism, or the domain or the range of a morphism) is then definable from this data. We denote by $M(i, j)$ the fibre of $M \to I \times I$ above (i, j), so that $(f_m)_{m \in M(i,j)}$ are the morphisms from O_i to O_j.

Remark 3.4. Our terminology is slightly different from that of Hrushovski [5]. There the structure (I, M) (together with functions of partial composition, domain and range) is called a definable groupoid. Then he considers a faithful definable functor from (I, M) to the category of definable sets and definable functions. This gives rise to $(O_i, f_m)_{i \in I, m \in M}$ which he called a *concrete definable groupoid*. So all our definable groupoids are concrete. This has the slightly awkward consequence that whereas for Hrushovski a definable groupoid with one object is nothing more than a definable group, for us it is a definable group action.

We present below Hrushovski's theorem [5, Proposition 2.5] that associates a liaison groupoid to an internal cover. We include a proof here mostly for the sake of completeness; ours differs superficially from Hrushovski's in that we do not go via $*$-definable groupoids but rather use finite-generatedness of the language to construct directly a definable groupoid.

Theorem 3.5 (Hrushovski)**.** *Let* $\mathbb{U}' = (\mathbb{U}, S, \dots)$ *be an internal generalised imaginary sort of* $\mathbb{U}$. *There is a 0-definable connected groupoid* $\mathcal{G}'$ *in* $(\mathbb{U}')^{\mathrm{eq}}$ *such that:*

- *The objects of* $\mathcal{G}'$ *are definable sets in* $\mathbb{U}$ *together with* S.
  In particular, the restriction $\mathcal{G}$ *of* $\mathcal{G}'$ *to* $\mathbb{U}$ – *i.e., the full subcategory generated by all objects excepts* S – *is a 0-definable connected groupoid in* $\mathbb{U}$. *It is called the* liaison groupoid *of* $\mathbb{U}'$.
- $\mathrm{Aut}_{\mathcal{G}'}(S)$, *together with its action on* S, *is isomorphic to* $\mathrm{Aut}(\mathbb{U}'/\mathbb{U})$.

Proof. Since $\mathbb{U}$ eliminates imaginaries, there is a definable subset O_b in $\mathbb{U}$ and a definable bijection $f_c : S \to O_b$, where $b \in \mathbb{U}$ and $c \in \mathbb{U}'$ are the parameters used in the respective definitions. Given (b', c') we can consider the corresponding formulas with parameters (b', c') in place of (b, c); and when this again gives rise to a definable function we will write it as $f_{c'} : S \to O_{b'}$.

Since $\mathbb{U}$ is stably embedded, there is a small set $B \subseteq \mathbb{U}$ – which we may assume contains b – such that $\mathrm{tp}(c/B) \vdash \mathrm{tp}(c/\mathbb{U})$. Let $p(x) = \mathrm{tp}(c/B)$. If $d, e \models p$, then there is an automorphism $\sigma \in \mathrm{Aut}(\mathbb{U}'/\mathbb{U})$ such that $\sigma(e) = d$. Hence for any $a \in S$ we have

$$f_e(a) = \sigma(f_e(a)) = f_{\sigma(e)}(\sigma(a)) = f_d(\sigma(a)).$$

Thus $f_d^{-1} \circ f_e = \sigma|_S$ and so $f_d^{-1} \circ f_e \cup \mathrm{id}_{\mathbb{U}}$ is elementary. But since the language of $\mathbb{U}'$ is finitely generated over that of $\mathbb{U}$, there is a formula $\theta(d, e)$ asserting this. Moreover, we have shown that $p(x) \cup p(y) \vdash \theta(x, y)$. Hence by compactness there is a formula $\phi(x, g) \in p(x)$ such that

$$\phi(x, g) \wedge \phi(y, g) \vdash \theta(x, y).$$

By extending b and g we may assume that $b = g$. Again by compactness there is a formula $\psi(z) \in \mathrm{tp}(b)$ such that

$$\psi(z) \vdash \forall x, y (\phi(x, z) \wedge \phi(y, z) \to \theta(x, y))$$

Now as objects of $\mathcal{G}'$ we take S together with $O_{b'}$ where b' realises $\psi(z)$. The morphisms from S to $O_{b'}$ are $f_{c'}$ where c' realises $\phi(x, b')$. Note that if $f_{c'}, f_{c''} \in \mathrm{Hom}(S, O_{b'})$, then $\theta(c', c'')$ holds and therefore $f_{c'}^{-1} \circ f_{c''} \cup \mathrm{id}_{\mathbb{U}} \in \mathrm{Aut}(\mathbb{U}'/\mathbb{U})$. Conversely if $\sigma \in \mathrm{Aut}(\mathbb{U}'/\mathbb{U})$ then $f_{c'} \circ \sigma|_S = f_{\sigma^{-1}(c')} \in \mathrm{Hom}(S, O_b')$. Therefore, for any b' realising $\psi(z)$, and for any c' realising $\phi(x, b')$,

$$(3.1) \qquad\qquad \mathrm{Hom}(S, O_{b'}) = f_{c'} \circ \mathrm{Aut}(\mathbb{U}'/\mathbb{U}).$$

The rest of the groupoid can be defined from this. We set

$$
\begin{aligned}
\mathrm{Hom}(O_{b'}, S) &= \{f^{-1} : f \in \mathrm{Hom}(S, O_{b'})\} \\
\mathrm{Hom}(O_{b'}, O_{b''}) &= \{f \circ g^{-1} : f \in \mathrm{Hom}(S, O_{b''}), g \in \mathrm{Hom}(S, O_{b'})\} \\
\mathrm{Hom}(S, S) &= \{f^{-1} \circ g : f, g \in \mathrm{Hom}(S, O_{b'}) \text{ for some } b'\}
\end{aligned}
$$

It is not hard to verify that $\mathcal{G}'$ is indeed a 0-definable connected groupoid in $\mathbb{U}'$. Note also that from (3.1) it follows that $\mathrm{Hom}(S, S) = \mathrm{Aut}(\mathbb{U}'/\mathbb{U})$. $\qquad\square$

The following proposition shows that finite-generatedness of the language is a necessary hypothesis of the above theorem. It will also be of use later.

Proposition 3.6. *Let* $\mathbb{U}'$ *be an internal cover of* $\mathbb{U}$ *with one new sort* S, *such that* $\mathrm{Aut}(\mathbb{U}'/\mathbb{U})$ *together with its action on* S *is interpretable in* $\mathbb{U}'$. *Then the language of* $\mathbb{U}'$ *is finitely generated over* $\mathbb{U}$.

Proof. An immediate consequence of the definability of the action of $\mathrm{Aut}(\mathbb{U}'/\mathbb{U})$ on S is that, working in $(\mathbb{U}')^{\mathrm{eq}}$, for any formula $\phi(x, y, z)$, where x and y are from the sort S, there is a formula $\theta(z)$ such that $\models \theta(e)$ if and only if $\phi(x, y, e)$ defines the graph of a bijection $S \to S$ which extends by identity to an automorphism of $\mathbb{U}'$. Indeed, $\theta(z)$ asserts that $\phi(x, y, z)$ agrees with some element of $\mathrm{Aut}(\mathbb{U}'/\mathbb{U})$ on S.

Now, by internality, for some $c \in \mathbb{U}'$, there is a c-definable bijection $f_c : S \to O_c$, where O_c is a definable set in $\mathbb{U}$. Let $\psi(x, u, v)$ be an L'-formula such that $\psi(x, u, c)$ defines the graph of f_c, and $\psi(x, u, c')$ defines the graph of a bijection $f_{c'} : S \to O_{c'}$ whenever $\phi(x, u, c')$ is consistent. Let $z = (v, w)$ where w is of the same arity as v, and set

$$\phi(x, y, z) := \exists u \big(\psi(x, u, v) \wedge \psi(y, u, w) \big).$$

Applying the property described in the paragraph above to $\phi(x, y, z)$, we have an L'-formula $\theta(v, w)$ such that $\models \theta(e, d)$ if and only if the bijection $f_d^{-1} \circ f_e : S \to S$ extends by identity to an automorphism of $\mathbb{U}'$. (At first θ may be an $(L')^{\mathrm{eq}}$-formula, but as v and w belong to sorts in $\mathbb{U}'$ it can be replaced by an L'-formula.) It follows by compactness that there must be a finite sublanguage $\Sigma \subseteq L'$ such that $f_d^{-1} \circ f_e \cup \mathrm{id}_{\mathbb{U}}$ is L'-elementary if and only if it is Σ-elementary. Extending Σ we may also assume that $\psi(x, u, v)$ is a Σ-formula. Now set $L'_\circ := L \cup \Sigma$.

We claim that this $L'_\circ$ works. That is, every 0-definable set in $\mathbb{U}'$ is 0-definable in the reduct to $L'_\circ$. Letting $\mathbb{U}'_\circ$ be the reduct of $\mathbb{U}'$ to the language $L'_\circ$, it suffices to show that $\mathrm{Aut}(\mathbb{U}'_\circ) = \mathrm{Aut}(\mathbb{U}')$, see for example [2, §10.5]. Since $L'_\circ$ extends L, this in turn reduces to showing that $\mathrm{Aut}(\mathbb{U}'_\circ/\mathbb{U}) = \mathrm{Aut}(\mathbb{U}'/\mathbb{U})$, see for example [5, Lemma 1.3]. Suppose $\sigma \in \mathrm{Aut}(\mathbb{U}'_\circ/\mathbb{U})$. Then for any $a \in S$,

$$
\begin{aligned}
f_c(a) &= \sigma(f_c(a)) && \text{as } f_c(a) \in \mathbb{U} \\
&= f_{\sigma(c)}(\sigma(a)) && \text{as } \psi(x, u, v) \text{ is an } L'_\circ\text{-formula}
\end{aligned}
$$

so that $\sigma = f_{\sigma(c)}^{-1} \circ f_c \cup \mathrm{id}_{\mathbb{U}}$. Hence, as σ is Σ-elementary, it must be L'-elementary, as desired. $\square$

Corollary 3.7. *If $\mathbb{U}'$ is an internal cover of $\mathbb{U}$ with one new sort S, and $\mathbb{U}'$ is totally transcendental, then the language of $\mathbb{U}'$ is finitely generated over $\mathbb{U}$.*

Proof. The usual liaison group theorem, see for example [7, Theorem 7.4.8], applied to totally transcendental theories, tells us that $\mathrm{Aut}(\mathbb{U}'/\mathbb{U})$ together with its action on S is interpretable in $\mathbb{U}'$. Now apply Proposition 3.6. $\square$

It follows that instances of internality in a totally transcendental theory can be understood in terms of internal generalised imaginary sorts.

Example 3.8. Suppose $\mathbb{M}$ is a saturated differentially closed field in characteristic zero and $\mathbb{U}$ is its field of constants with the induced structure (which we know is that of a pure algebraically closed field). Suppose S is a definable set in $\mathbb{M}$ that is internal to the constants, for example defined by $\delta x = 1$ or $\delta x = x$. Let $\mathbb{U}'$ be the 0-definable structure induced by $\mathbb{M}$ on $(\mathbb{U}, S)$. Then, by Corollary 3.7, $\mathbb{U}'$ is an internal generalised imaginary sort of $\mathbb{U}$. One gets similar examples by working with the theory of compact complex manifolds rather than differentially closed fields, in which case $\mathbb{U}$ is the induced structure on the projective line (again essentially a pure algebraically closed field).

Having proved the existence of liaison groupoids, Hrushovski goes on to show in [5, Theorem 3.2] that every 0-definable connected groupoid in $\mathbb{U}$ comes from an

internal generalised imaginary sort. This is not precisely correct. Note that the groupoid $\mathcal{G}'$ of Theorem 3.5 has an additional property: if $c \in \mathbb{U}'$ is as in the proof, then any automorphism $\sigma \in \operatorname{Aut}(\mathbb{U}'/\mathbb{U})$ that fixes c must be the identity. Indeed, for any $a \in S$,

$$
\begin{aligned}
\sigma(a) &= \sigma(f_c^{-1} f_c(a)) \\
&= f_{\sigma(c)}^{-1}(\sigma(f_c(a))) \\
&= f_c^{-1}(f_c(a)) \\
&= a.
\end{aligned}
$$

By connectedness, a similar property then holds for $\mathcal{G}$, the restriction of $\mathcal{G}'$ to $\mathbb{U}$. So the definable groupoids in $\mathbb{U}$ that arise this way are special.

Definition 3.9. A definable groupoid $\mathcal{G}$ is called *finitely faithful*[2] if for any object O of $\mathcal{G}$, there is a finite $c \subset O$ such that the pointwise stabiliser of c in $\operatorname{Aut}_{\mathcal{G}}(O)$ is trivial.

Note that if $\mathcal{G}$ is connected then it suffices to check the condition for some (rather than every) object O.

Of course, not all definable groupoids (even group actions) are finitely faithful.

From a 0-definable connected groupoid $\mathcal{G}$ in $\mathbb{U}$, not necessarily finitely faithful, Hrushovski does construct an associated internal cover of $\mathbb{U}$, but it is not an internal generalised imaginary sort as it involves two new sorts (rather than one), neither of which can be eliminated in general. We review his construction and then explain how finite faithfulness enters the picture.

Let $\mathcal{G} = (O_i, f_m)_{i \in I, m \in M}$ be a 0-definable connected groupoid in $\mathbb{U}$. We want to associate to it an internal cover $\mathbb{U}'$. The idea is to construct $\mathbb{U}'$ by adding a single new object to $\mathcal{G}$. We just duplicate one of the objects and add the necessary morphisms to make this into a connected groupoid. Fix some $i \in I$. There are two new sorts O_* and M_*. The sort O_* as a set is a copy of O_i, for concreteness let $O_* = O_i \times \{0\}$. The sort M_* is a copy of the set of morphisms in M whose domain or range is O_i, plus two additional copies of $M(i, i)$, for concreteness say

$$
M_* = (M(i, i) \times \{0, 1, 2\}) \cup \bigcup_{j \neq i \in J} \big(M(i, j) \cup M(j, i)\big) \times \{0\}.
$$

The idea is that

- $M(i, i) \times \{0\}$ will encode morphisms $O_* \to O_*$,
- $M(i, i) \times \{1\}$ will encode morphisms $O_i \to O_*$,
- $M(i, i) \times \{2\}$ will encode morphisms $O_* \to O_i$,
- $M(i, j) \times \{0\}$ will encode morphisms $O_* \to O_j$, and
- $M(j, i) \times \{0\}$ will encode morphisms $O_j \to O_*$.

Next we add a ternary relation R that interprets these morphisms as bijection. Formally R is interpreted as the set $R^1 \cup R^2 \cup R^3$ where

- $R^1 \subset M_* \times O \times O_*$ is given by

$$
\begin{aligned}
R^1 \quad := \quad &\big\{((m, 1), a, (b, 0)) : m \in M(i, i), a, b \in O_i, f_m(a) = b\big\} \ \cup \\
&\big\{((m, 0), a, (b, 0)) : m \in M(j, i), j \neq i, a \in O_j, b \in O_i, f_m(a) = b\big\}
\end{aligned}
$$

---

[2]This terminology was suggested to us by Hrushovski in private communication.

- $R^2 \subset M_* \times O_* \times O$ is given by

$$
\begin{aligned}
R^2 \quad := \quad & \big\{ \big((m,2), (a,0), b \big) : m \in M(i,i), a, b \in O_i, f_m(a) = b \big\} \ \cup \\
& \big\{ \big((m,0), (a,0), b \big) : m \in M(i,j), j \neq i, a \in O_i, b \in O_j, f_m(a) = b \big\}
\end{aligned}
$$

- $R^3 \subset M_* \times O_* \times O_*$ is given by

$$
R^3 \quad := \quad \big\{ \big((m,0), (a,0), (b,0) \big) : m \in M(i,i), a, b \in O_i, f_m(a) = b \big\}
$$

Remark 3.10. There is a notational detail here: the arity of R is not well defined as the sequence of sorts are different in R^1, R^2 and R^3. So technically we either view this as three seperate relations, or what we prefer, is to work in $(\mathbb{U}')^{\mathrm{eq}}$ where $O \sqcup O_*$ can be identified with a sort, and then view $R \subseteq M_* \times (O \sqcup O_*) \times (O \sqcup O_*)$.

We define $\mathcal{G}'$ to be the groupoid with objects $(O_i)_{i \in I} \cup \{O_*\}$ and morphisms $(f_m)_{m \in M} \cup (R_{m'})_{m' \in M_*}$. It is a connected one-object extension of $\mathcal{G}$. Working in $(\mathbb{U}')^{\mathrm{eq}}$, we set $I' := I \cup \{*\}$, $O' := O \sqcup O_*$, $M' := M \sqcup M_*$, $\pi' : O' \to I'$ the natural extension of $\pi : O \to I$ by O_*, and $f' \subseteq O' \times O' \times M'$ the natural extension of $f \subseteq O \times O \times M$ by R. It is easily seen that this presents $\mathcal{G}'$ as a 0-definable connected groupoid in $(\mathbb{U}')^{\mathrm{eq}}$.

Theorem 3.11 (Hrushovski). *Given $\mathbb{U}$ and 0-definable connected groupoid $\mathcal{G}$, the structure $\mathbb{U}'$ constructed above determines an internal cover of $\mathbb{U}$ that depends only on $\mathcal{G}$ (so not on the choice of i). Furthermore, $\mathrm{Aut}_{\mathcal{G}'}(O_*)$ together with its action on O_* is isomorphic to $\mathrm{Aut}(O_*/\mathbb{U}) := \{\sigma|_{O_*} : \sigma \in \mathrm{Aut}(\mathbb{U}') \text{ and } \sigma|_{\mathbb{U}} = \mathrm{id}_{\mathbb{U}}\}$.*

We leave the reader to consult [5, Lemma 3.1] for a proof. It is, however, worth clarifying that although O_* is a copy of O_i, this fact is not part of the theory of $\mathbb{U}'$. In particular the canonical parameter of O_i (that is i) does not become 0-definable in $\mathbb{U}'$ despite the fact that the canonical paramter of O_* is 0-definable.

We wish to eliminate the sort M_* so as to get an internal generalised imaginary sort. Hrushovski erroneously claims that this is always possible (see the parenthetical final sentence of the first paragraph of the proof of Theorem 3.2 of [5]). The next result explains when in fact this can be done.

Proposition 3.12. *Let $\mathbb{U}'$ be as constructed above. The following are equivalent.*

1. *The groupoid $\mathcal{G}$ is finitely faithful.*
2. *$M_* \subseteq \mathrm{dcl}(\mathbb{U}, O_*)$ where dcl is calculated in $\mathbb{U}'$.*
3. *$(\mathbb{U}, O_*)$ is stably embedded in $\mathbb{U}'$.*

Proof. If $\mathcal{G}$ is finitely faithful then, as $\mathcal{G}'$ is a connected one-object extension of $\mathcal{G}$, we get that $\mathcal{G}'$ is finitely faithful too. For each $i \in I'$, let $c_i \subset O_i'$ be a finite tuple whose pointwise stabiliser in $\mathrm{Aut}_{\mathcal{G}'}(O_i')$ is trivial. Then for any $m \in M'$, say $m \in M'_{i,j}$ for some $i, j \in I'$, we have that $f'_m : O_i' \to O_j'$ is determined by its action on c_i, and so $m \in \mathrm{dcl}(f'_m(c_i), c_i)$. Hence (1) $\implies$ (2).

That (2) $\implies$ (3) is clear.

Let us show that (3) $\implies$ (1). By connectedness, to show that $\mathcal{G}$ is finitely faithful it suffices to show that $\mathcal{G}'$ is, and for this we need only to check that O_* has a finite subtuple whose pointwise stabiliser is trivial.

Fix $i \in I$. Note that for any $m, n \in M'_{*,i}$, $(f'_n)^{-1} f'_m \in \mathrm{Aut}(O_*/\mathbb{U})$ by Theorem 3.11, and so extends to some $\sigma \in \mathrm{Aut}(\mathbb{U}'/\mathbb{U})$. Moreover, for any $b \in O_*$

$$
\begin{aligned}
f'_{\sigma(m)}(b) &= \sigma(f'_m(\sigma^{-1}(b))) \\
&= f'_m(\sigma^{-1}(b)) \quad \text{as } f'_m(\sigma^{-1}(b)) \in O_i \subseteq \mathbb{U} \\
&= f'_m((f'_m)^{-1} f'_n(b))) \\
&= f'_n(b).
\end{aligned}
$$

So $\sigma(m) = n$.

Now, fix $m \in M'_{*,i}$. By stable embeddedness there is a small $A \subset O_*$ such that $\mathrm{tp}(m/\mathbb{U}A) \vdash \mathrm{tp}(m/\mathbb{U}O_*)$ (see [1]). If f'_m and f'_n agree on A then the σ constructed above fixes A pointwise, and so m and n have the same type over $\mathbb{U} \cup O_*$, which means that f'_m and f'_n agree on all of O_*, and hence $m = n$. It follows that m is fixed by all automorphism in $\mathrm{Aut}(\mathbb{U}'/Af'_m(A))$. By saturation, m is fixed by all automorphism in $\mathrm{Aut}(\mathbb{U}'/af'_m(a))$ for some finite tuple a from A. If f'_m and f'_n agree on a then $\sigma \in \mathrm{Aut}(\mathbb{U}'/af'_m(a))$, and so we must have $n = m$. But this means that the pointwise stabiliser of a in $\mathrm{Aut}_{\mathcal{G}'}(O')$ is the identity, else one could precompose by a nontrivial element of this stabiliser. So $\mathcal{G}'$ is finitely faithful. $\square$

Corollary 3.13. *If $\mathbb{U}$ is stable then all definable connected groupoids are finitely faithful.*

Proof. Note that an internal cover of a stable theory is stable as any formula with the order property in the cover can be pushed into the reduct using any definable bijection witnessing the internality. Now if $\mathbb{U}'$ is stable then condition (3) of Proposition 3.12 holds automatically. $\square$

In any case, we now can refine Theorem 3.11 to

Theorem 3.14. *Suppose $\mathcal{G}$ is a 0-definable finitely faithful connected groupoid in $\mathbb{U}$, and let $\mathbb{U}'$ be the internal cover constructed in Theorem 3.11. Let $\mathbb{U}''$ be the structure induced on $(\mathbb{U}, O_*)$ by $\mathbb{U}'$. Then $\mathbb{U}''$ is an internal generalised imaginary sort of $\mathbb{U}$, $\mathcal{G}'$ is 0-definable in $(\mathbb{U}'')^{\mathrm{eq}}$, and $\mathrm{Aut}_{\mathcal{G}'}(O_*)$ with its action on O_* is isomorphic to $\mathrm{Aut}(\mathbb{U}''/\mathbb{U})$.*

Proof. Clearly $\mathbb{U}''$ is an internal cover involving a single new sort O_*. Note also that by condition (2) of Proposition 3.12, and our assumption that $\mathcal{G}$ is finitely faithful, $\mathbb{U}'$ is interpretable in $\mathbb{U}''$. Hence $\mathcal{G}'$ is interpretable in $\mathbb{U}''$, and $\mathrm{Aut}(\mathbb{U}''/\mathbb{U}) = \mathrm{Aut}(O_*/\mathbb{U})$ where the latter is in the sense of $\mathbb{U}'$. So the liaison group statement follows from Theorem 3.11. The only thing left to check is that the language of $\mathbb{U}''$ is finitely generated over that of $\mathbb{U}$. But this follows from Proposition 3.6 since the liaison group and its action is interpretable. $\square$

This yields a bijective correspondence which appears as Theorem 3.2 in [5], though with the assumption of finite faithfulness omitted.

Theorem 3.15 (Hrushovski). *The constructions of Theorems 3.5 and 3.14 form a bijective correspondence between internal generalised imaginary sorts of $\mathbb{U}$ and 0-definable finitely faithful connected groupoids in $\mathbb{U}$, up to equivalence on both sides.*

We leave the reader to consult [5] for a proof, but we should at least spell out what these equivalences are. Two generalised imaginary sorts $\mathbb{U}_1 = (\mathbb{U}, S_1, \dots)$ and $\mathbb{U}_2 = (\mathbb{U}, S_2, \dots)$ are called *equivalent* if there is a bijection $g : S_1 \to S_2$ such that

$g \cup \mathrm{id}_{\mathbb{U}}$ is a bi-interpretation; that is, X is 0-definable in $\mathbb{U}_1$ if and only if its image is 0-definable in $\mathbb{U}_2$.

For equivalence of definable groupoids, recall from [5] that a definable embedding of $\mathcal{G}_1$ into $\mathcal{G}_2$ consists of the following:

- a definable injection $\iota : I_1 \to I_2$;
- a definable family of definable bijections $(h_i : O_{1i} \to O_{2\iota(i)})_{i \in I_1}$ such that for every $i, j \in I_1$ we have $h_j \circ \mathrm{Hom}_{\mathcal{G}_1}(O_{1i}, O_{1j}) = \mathrm{Hom}_{\mathcal{G}_2}(O_{2\iota(i)}, O_{2\iota(j)}) \circ h_i$.

Now, two 0-definable connected groupoids $\mathcal{G}_1$ and $\mathcal{G}_2$ are called *equivalent* if there is a 0-definable connected groupoid $\mathcal{G}$ and 0-definable embeddings $\mathcal{G}_i \to \mathcal{G}$ for $i = 1, 2$.

4. FUNCTORIALITY

In this section we categorify the correspondence of 3.15. Our goal here is twofold. Firstly, one rarely studies internal covers in isolation. For example in the monster model of DCF_0 one looks at definable sets internal to the constants, but also at definable maps between them. So a question arises as to what kind of object do these maps induce between the corresponding groupoids in the constants. Secondly, the correspondence of 3.15 is defined up to equivalence; we would like to realise these equivalences as isomorphisms.

For a general reference on terminology from category theory we suggest [6].

4.1. **Morphisms of covers.** The idea is that morphisms should be definable maps in a common cover. So a *morphism* between covers $\mathbb{U}_1 = (\mathbb{U}, S_1, \dots)$ and $\mathbb{U}_2 = (\mathbb{U}, S_2, \dots)$, each with a single new sort, is given by a common cover of $\mathbb{U}_1$ and $\mathbb{U}_2$ of the form $(\mathbb{U}_1 \cup \mathbb{U}_2, g)$ where $g : S_1 \to S_2$ is a new basic function. (See §2 for a discussion of unions of covers.)

It may be worth being careful about the language here. Suppose L is the language of $\mathbb{U}$ and L_i is the langauge of $\mathbb{U}_i$ for $i = 1, 2$. We form a copy of L_1, which we denote by L_1^{dom}, by replacing each symbol $R \in L_1 \setminus L$ by the symbol R^{dom}. Similarly we make a duplicate of L_2 which we denote by L_2^{ran}. We then have that $L_1^{\mathrm{dom}} \cap L_2^{\mathrm{ran}} = L$. The morphism $(\mathbb{U}_1 \cup \mathbb{U}_2, g)$ is a structure in the language $L_1^{\mathrm{dom}} \cup L_2^{\mathrm{ran}} \cup \{g\}$, where g is a new function symbol from S_1^{dom} to S_2^{ran}. So when we say that $(\mathbb{U}_1 \cup \mathbb{U}_2, g)$ is a cover of $\mathbb{U}_1$ and $\mathbb{U}_2$, we really mean modulo the natural identification of L_1 with L_1^{dom} and L_2 with L_2^{ran}.

Remark 4.1. Of course the morphism is really the theory of $(\mathbb{U}_1 \cup \mathbb{U}_2, g)$, so that different functions may produce the same morphism. But note that $(\mathbb{U}_1 \cup \mathbb{U}_2, g) \equiv (\mathbb{U}_1 \cup \mathbb{U}_2, h)$ if and only if $h = \alpha g$ for some $\alpha \in \mathrm{Aut}(\mathbb{U}_2/\mathbb{U})$. Indeed, by saturation there is an ismorphism $\tau : (\mathbb{U}_1 \cup \mathbb{U}_2, g) \to (\mathbb{U}_1 \cup \mathbb{U}_2, h)$, and by stable embeddedness of $\mathbb{U}_1$ in $(\mathbb{U}_1 \cup \mathbb{U}_2, g)$ we can take τ to be the identity on $\mathbb{U}_1$. Then $\alpha := \tau|_{\mathbb{U}_2}$ works.

Composition is defined naturally: given morphisms $(\mathbb{U}_1 \cup \mathbb{U}_2, g)$ and $(\mathbb{U}_2 \cup \mathbb{U}_3, h)$, the composition is $(\mathbb{U}_1 \cup \mathbb{U}_3, h \circ g)$. To see that this is a well-defined morphism we need to check that it is a cover of $\mathbb{U}_1$ and $\mathbb{U}_3$ and that its theory does not depend on the particular choice of g and h (just on the morphisms that g and h determine). We do this by applying Lemma 2.4 to $(\mathbb{U}_1 \cup \mathbb{U}_2, g)$ and $(\mathbb{U}_2 \cup \mathbb{U}_3, h)$, which are both covers of $\mathbb{U}_2$, at least after re-identifying L_2, L_2^{ran}, and L_2^{dom}, as say L_2^{mid}. We get that $(\mathbb{U}_1 \cup \mathbb{U}_2 \cup \mathbb{U}_3, g, h)$ has a theory that depends only on the theories of $(\mathbb{U}_1 \cup \mathbb{U}_2, g)$ and $(\mathbb{U}_2 \cup \mathbb{U}_3, h)$, and is a cover of both $\mathbb{U}_1$ and $\mathbb{U}_3$. Now, $(\mathbb{U}_1 \cup \mathbb{U}_3, h \circ g)$

is a reduct of $(\mathbb{U}_1 \cup \mathbb{U}_2 \cup \mathbb{U}_3, g, h)$ expanding $\mathbb{U}_1$ and $\mathbb{U}_3$, and so it too has these desired properties.

The identity morphism on a cover $\mathbb{U}' = (\mathbb{U}, S, \dots)$ is $(\mathbb{U}' \cup \mathbb{U}', \mathrm{id}_S)$, where id_S is here viewed as a function from S^{dom} to S^{ran}, which are the two new sort symbols in the language of $\mathbb{U}' \cup \mathbb{U}'$, both of which are interpreted in $\mathbb{U}' \cup \mathbb{U}'$ as the set S. To see that $(\mathbb{U}' \cup \mathbb{U}', \mathrm{id}_S)$ is saturated, let $(\mathbb{U}_1 \cup \mathbb{U}_2, g)$ be a saturated model of its theory. Note that while it is no longer the case that on underlying sets g is the identity – indeed the underlying sets of the new sorts of $\mathbb{U}_1$ and $\mathbb{U}_2$ need not be identical – it is the case that modulo the natural identification of L'^{dom} with L'^{ran}, $g \cup \mathrm{id}_{\mathbb{U}}$ is an isomorphism from $\mathbb{U}_1$ to $\mathbb{U}_2$. Also, $\mathbb{U}_i$ is a cover of $\mathbb{U}$ of the same theory as $\mathbb{U}'$, so that by Lemma 2.4, we have an isomorphism $\sigma_1 \cup \sigma_2 : \mathbb{U}' \cup \mathbb{U}' \to \mathbb{U}_1 \cup \mathbb{U}_2$. In order to be an isomorphism between $(\mathbb{U}' \cup \mathbb{U}', \mathrm{id}_S)$ and $(\mathbb{U}_1 \cup \mathbb{U}_2, g)$ we would need the following diagram to commute:

$$
\begin{array}{ccc}
S & \xrightarrow{\ \mathrm{id}_S\ } & S \\
\downarrow{\scriptstyle \sigma_1|_S} & & \downarrow{\scriptstyle \sigma_2|_S} \\
S_1 & \xrightarrow{\ g\ } & S_2
\end{array}
$$

While this may not be the case, $\alpha := (\sigma_2|_S)^{-1} \circ g \circ \sigma_1|_S$ is such that $\alpha \cup \mathrm{id}_{\mathbb{U}}$ is an L'^{ran}-automorphism of $\mathbb{U}'$. So replacing σ_2 by $\sigma_2 \circ (\alpha \cup \mathrm{id}_{\mathbb{U}})$, we get that the diagram does commute. Hence $(\mathbb{U}' \cup \mathbb{U}', \mathrm{id}_S)$ is isomorphic to $(\mathbb{U}_1 \cup \mathbb{U}_2, g)$, and is therefore saturated. Similar arguments show that $\mathbb{U}'$, under both the identification of L' with L'^{dom} and with L'^{ran}, is stably embedded in $(\mathbb{U}' \cup \mathbb{U}', \mathrm{id}_S)$.

This defines the category of covers of $\mathbb{U}$ with one extra sort. The following identifies the isomorphisms of this category.

Proposition 4.2. *A morphism $(\mathbb{U}_1 \cup \mathbb{U}_2, g)$ in this category of covers of $\mathbb{U}$ by a single sort is an isomorphism if and only if $g \cup \mathrm{id}_{\mathbb{U}} : \mathbb{U}_1 \to \mathbb{U}_2$ is a bijective bi-interpretation.*

Proof. It is not hard to verify that in this category a morphism $(\mathbb{U}_1 \cup \mathbb{U}_2, g)$ is an isomorphism if and only if $g : S_1 \to S_2$ is bijective.

Suppose $(\mathbb{U}_1 \cup \mathbb{U}_2, g)$ is an isomorphism and $X \subseteq S_1^n \times \mathbb{U}^m$ is 0-definable. Then its image $B = (g \cup \mathrm{id}_{\mathbb{U}})(A) \subseteq S_2^n \times \mathbb{U}^m$ is 0-definable in $(\mathbb{U}_1 \cup \mathbb{U}_2, g)$. Hence by stable embeddedness and saturation it is 0-definable in $\mathbb{U}_2$. Similarly, preimages of 0-definable sets in $\mathbb{U}_2$ are 0-definable in $\mathbb{U}_1$. So $g \cup \mathrm{id}_{\mathbb{U}}$ is a bijective bi-interpretation.

Conversely, suppose $g \cup \mathrm{id}_{\mathbb{U}}$ be a bijective bi-interpretation. We need to show that g is a morphism, i.e. $(\mathbb{U}_1 \cup \mathbb{U}_2, g)$ is a cover of both $\mathbb{U}_1$ and $\mathbb{U}_2$.

For saturation, we let $(\overline{\mathbb{U}}_1 \cup \overline{\mathbb{U}}_2, \overline{g})$ be a saturated model of its theory. Then Lemma 2.4 gives us an $(L_1^{\mathrm{dom}} \cup L_2^{\mathrm{ran}})$-isomorphism $\sigma_1 \cup \sigma_2 : \mathbb{U}_1 \cup \mathbb{U}_2 \to \overline{\mathbb{U}}_1 \cup \overline{\mathbb{U}}_2$. Again, to be an isomorphism from $(\mathbb{U}_1 \cup \mathbb{U}_2, g)$ to $(\overline{\mathbb{U}}_1 \cup \overline{\mathbb{U}}_2, \overline{g})$ we would need

$$
\begin{array}{ccc}
S_1 & \xrightarrow{\ g\ } & S_2 \\
\downarrow{\scriptstyle \sigma_1|_{S_1}} & & \downarrow{\scriptstyle \sigma_2|_{S_2}} \\
\overline{S}_1 & \xrightarrow{\ \overline{g}\ } & \overline{S}_2
\end{array}
$$

to commute. We can make the diagram commute by replacing σ_2 by $\sigma_2 \circ (\alpha \cup \mathrm{id}_{\mathbb{U}})$ where $\alpha := (\sigma_2|_{S_2})^{-1} \circ \overline{g} \circ \sigma_1|_{S_1} \circ g^{-1}$. But note that after such an alteration $\sigma_1 \cup \sigma_2$ is no longer elementary with respect to L_2^{ran}. However, $\sigma_1 \cup \sigma_2 : (\mathbb{U}_1 \cup \mathbb{U}_2, g) \to$

$(\overline{\mathbb{U}}_1 \cup \overline{\mathbb{U}}_2, \overline{g})$ still has the property that a set is 0-definable if and only if its image is. This is enough to conclude that $(\mathbb{U}_1 \cup \mathbb{U}_2, g)$ is saturated since $(\overline{\mathbb{U}}_1 \cup \overline{\mathbb{U}}_2, \overline{g})$ is.

Next we show that $\mathbb{U}_1$ is stably embedded in $(\mathbb{U}_1 \cup \mathbb{U}_2, g)$, modulo the identification of L_1 with L_1^{dom}. Given $\sigma_1 \in \mathrm{Aut}(\mathbb{U}_1)$ we can extend it to $\sigma_1 \cup \sigma_2$ on $\mathbb{U}_1 \cup \mathbb{U}_2$ by Lemma 2.4. To make it commute with g we would precompose σ_2 with $(\sigma_2|_{S_2})^{-1} \circ g \circ \sigma_1|_{S_1} \circ g^{-1}$. This time the new $\sigma_1 \cup \sigma_2$ is elementary, as we are conjugating by the bi-interpretation g. Similarly $(\mathbb{U}, S_2)$ is stably embedded in $(\mathbb{U}_1 \cup \mathbb{U}_2, g)$, modulo the identification of L_2 with L_2^{ran}. $\qquad\square$

We are interested not in arbitrary covers, but internal generalised imaginaries.

Definition 4.3. We denote by $\mathbf{IGI}_T$ the full subcategory generated by the internal generalised imaginary sorts.

Proposition 4.2 says that isomorphisms in $\mathbf{IGI}_T$ are precisely equivalences of internal generalised imaginary sorts as defined by Hrushovski [5] (and discussed in the previous section).

4.2. **Morphisms of definable groupoids.** If we think of a definable groupoid as a category we might be tempted to have the morphisms be functors, maybe satisfying additional properties, but in any case involving a definable function between the sets of objects. If we focus on the definable groupoid as generalising definable group actions we may expect morphisms to induce definable group homomorphisms between the automorphism groups. As it turns out, both of these approaches are misleading. If, as is the case, we view definable groupoids as the intrinsic avatars of internal generalised imaginaries, then we should proceed as follows.

Fix connected groupoids $\mathcal{G}_1 = (O_{1i}, f_{1m})_{i \in I_1, m \in M_1}$ and $\mathcal{G}_2 = (O_{2i}, f_{2m})_{i \in I_2, m \in M_2}$ definable in $\mathbb{U}$. A *definable morphism* $h : \mathcal{G}_1 \to \mathcal{G}_2$ is a nonempty set of uniformly definable functions $\left(h_n : O_{1i_1} \to O_{2i_2} \right)_{n \in N}$ satisfying conditions (A) and (B) below. More precisely, we have a nonempty definable set N equipped with a definable map $N \to I_1 \times I_2$, and a definable set $h \subseteq N \times O_1 \times O_2$, such that for every $n \in N$ living above $(i_1, i_2) \in I_1 \times I_2$, the fibre $h_n \subseteq O_{1i_1} \times O_{2i_2}$ is the graph of a function. We denote by $N(i_1, i_2)$ the fibre of $N \to I_1 \times I_2$ above (i_1, i_2), so that $h|_{N(i_1, i_2)}$ is a definable family of definable functions from O_{1i_1} to O_{2i_2}. The conditions we impose on the family h to be a morphism are:

(A) For all $p \in N(i_1, i_2), q \in N(j_1, j_2)$,

$$h_q \circ \mathrm{Hom}_{\mathcal{G}_1}(O_{1i_1}, O_{1j_1}) = \mathrm{Hom}_{\mathcal{G}_2}(O_{2i_2}, O_{2j_2}) \circ h_p.$$

(B) For all $p \in N(i, j)$ and $m \in M_2(j, k)$, there is $q \in N(i, k)$ such that

$$f_{2m} h_p = h_q.$$

Note that condition (B) says h is closed under compositions with bijections from $\mathrm{Hom}_{\mathcal{G}_2}$. Given (A) this implies h is also closed under precompositions with $\mathrm{Hom}_{\mathcal{G}_1}$.

Remark 4.4.   (a) To be precise, for us the definable morphism is the set of definable functions and not the definable family. That is we identify h and h' if $\{h_n : n \in N\} = \{h'_n : n \in N'\}$.

(b) We do not require each $n \in N$ to define a distinct morphism, though this could be easily achieved by elimination of imaginaries.

We now compose morphisms. Suppose $g : \mathcal{G}_1 \to \mathcal{G}_2$ and $h : \mathcal{G}_2 \to \mathcal{G}_3$ are definable morphisms between definable connected groupoids, where $g = (g_p)_{p \in P}$ and $h = (h_q)_{q \in Q}$. Then $h \circ g : \mathcal{G}_1 \to \mathcal{G}_3$ is given as follows. Let $N := P \times_{I_2} Q$ be the fibred product of $P \to I_1 \times I_2 \to I_2$ and $Q \to I_2 \times I_3 \to I_2$, and equip it with the natural map $N \to I_1 \times I_3$. Given $n \in N(i_1, i_3)$ we have that for some $i_2 \in I_2$, $n = (p, q)$ where $p \in P(i_1, i_2)$ and $q \in Q(i_2, i_3)$. Then

$$(h \circ g)_n := h_q \circ g_p : O_{1i_1} \to O_{3i_3}.$$

Lemma 4.5. *The composition $h \circ g$ is a definable morphism from $\mathcal{G}_1$ to $\mathcal{G}_3$*
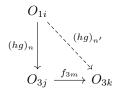
Proof. It is more or less clear that $h \circ g$ is a set of uniformly definable functions parameterised by $N \to I_1 \times I_3$ with the correct domains and codomains. We need to check conditions (A) and (B).

For (A) suppose we are given $n \in N(i_1, i_3)$ and $n' \in N(j_1, j_3)$. Then $n = (p, q)$ and $n' = (p', q')$ where $p \in P(i_1, i_2)$, $q \in Q(i_2, i_3)$, $p' \in P(j_1, j_2)$ and $q' \in Q(j_2, j_3)$, for some $i_2, j_2 \in I_2$. Now,

$$
\begin{aligned}
(hg)_{n'} \operatorname{Hom}_{\mathcal{G}_1}(O_{1i_1}, O_{1j_1}) &= h_{q'} g_{p'} \operatorname{Hom}_{\mathcal{G}_1}(O_{1i_1}, O_{1j_1}) \\
&= h_{q'} \operatorname{Hom}_{\mathcal{G}_2}(O_{2i_2}, O_{2j_2}) g_p \quad \text{by (A) for } g \\
&= \operatorname{Hom}_{\mathcal{G}_3}(O_{3i_3}, O_{3j_3}) h_q g_p \quad \text{by (A) for } h \\
&= \operatorname{Hom}_{\mathcal{G}_3}(O_{3i_3}, O_{3j_3})(hg)_n
\end{aligned}
$$

as desired.

For (B), given $(hg)_n$ and f_{3m} we seek $(hg)_{n'}$ such that the following commutes:

Write $n = (p, q)$ where $p \in P(i, \ell)$ and $q \in Q(\ell, j)$ for some $\ell \in I_2$. Then the above becomes

where the existence of $h_{q'}$ is given by (B) applied to h. So $n' := (p, q')$ will work. $\square$

The identity morphism for a definable connected groupoid $\mathcal{G} = (O_i, f_m)_{i \in I, m \in M}$ is the family $f = (f_m)_{m \in M}$ itself. That f is a definable endomorphism, namely that (A) and (B) hold, follows easily from the fact that the f_m are all bijections, and that f is closed under compositions and inverses. That it really serves as the identity is also straightforward: given a definable morphism $h : \mathcal{H} \to \mathcal{G}$, that $f \circ h \subseteq h$ is exactly (B) for h, and that $f \circ h \supseteq h$ is by the fact that we have all identities in f. A similar argument shows that $h \circ f = h$ for any definable morphism $h : \mathcal{G} \to \mathcal{H}$, using the fact observed before that (B) holds with precompositions too.

We thus have a category; let $\mathbf{CGpd}_T$ denote the 0-definable connected groupoids in T equipped with the 0-definable morphisms.

Lemma 4.6. *A 0-definable morphism $h : \mathcal{G}_1 \to \mathcal{G}_2$ is an isomorphism in $\mathbf{CGpd}_T$ if and only if each (equivalently some) h_n is a bijection.*

Proof. If some h_n is bijective then so are all the others, since by (A) they can be obtained from h_n by pre and post-composing with bijections in $\mathrm{Hom}_{\mathcal{G}_1}$ or $\mathrm{Hom}_{\mathcal{G}_2}$.

If every h_n is bijective then it is easy to see that $h^{-1} := (h_n^{-1})_{n \in N}$ is a definable morphism parameterised by $N \to I_2 \times I_1$, where this is the co-ordinate flip of the map $N \to I_1 \times I_2$ that parameterised h. That it is the inverse of h also follows readily from (A). For example, if $p \in N(i,j)$ and $q \in N(k,j)$ then from

$$h_q \circ \mathrm{Hom}_{\mathcal{G}_1}(O_{1i}, O_{1k}) = \mathrm{Hom}_{\mathcal{G}_2}(O_{2j}, O_{2j}) \circ h_p$$

we get that $h_p h_q^{-1} \in \mathrm{Hom}_{\mathcal{G}_1}(O_{1i}, O_{1k})$ and hence is one of the definable functions in the identity endomorphism on $\mathcal{G}_1$.

Conversely, suppose $g = (g_p)_{p \in P} : \mathcal{G}_2 \to \mathcal{G}_1$ is the inverse of h. Then for any $n \in N(i,j)$ and $p \in P(j,i)$ we have that $g_p h_n \in \mathrm{Aut}_{\mathcal{G}_1}(O_{1i})$ is bijective, and so h_n is injective. But also $h_n g_p \in \mathrm{Aut}_{\mathcal{G}_2}(O_{2j})$ is bijective, implying h_n is surjective. $\square$

We want to show that equivalences, as defined by Hrushovski [5] and discussed at the end of §3 above, agree with isomorphisms in $\mathbf{CGpd}_T$. Recall that two definable connected groupoids were said to be equivalent if they jointly embedded into a third definably connected groupoid. So the key is to show that embeddings in Hrushovski's sense induce isomorphisms in ours.

Recall that a definable embedding from $\mathcal{G}_1$ to $\mathcal{G}_2$ is given by a definable injection $\iota : I_1 \to I_2$ and a definable family of definable bijections $(h_i : O_{1i} \to O_{2\iota(i)})_{i \in I_1}$ such that $h_j \circ \mathrm{Hom}_{\mathcal{G}_1}(O_{1i}, O_{1j}) = \mathrm{Hom}_{\mathcal{G}_2}(O_{2\iota(i)}, O_{2\iota(j)}) \circ h_i$. We can produce from this an isomorphism $\widetilde{h} = (\widetilde{h}_n)_{n \in N} : \mathcal{G}_1 \to \mathcal{G}_2$ such that

$$\widetilde{h}|_{N(i_1, i_2)} = \mathrm{Hom}_{\mathcal{G}_2}(O_{2\iota(i_1)}, O_{2i_2}) \circ h_{i_1}$$

To do so, consider $M_2 \to I_2$ which picks out the domain of a bijection in $\mathcal{G}_2$, and let N be the fibred product of $M_2 \to I_2$ with $\iota : I_1 \to I_2$. We have the natural map $N \to I_1 \times I_2$. Given $n \in N(i_1, i_2)$ write $n = (m, i_1)$ where $m \in M_2(\iota(i_1), i_2)$, and define $\widetilde{h}_n := f_{2m} \circ h_{i_1} : O_{1i_1} \to O_{2i_2}$.

Proposition 4.7. *Suppose $\mathcal{G}_1$ and $\mathcal{G}_2$ are definable connected groupoids and $(\iota, h_i)_{i \in I_1}$ is a definable embedding of $\mathcal{G}_1$ into $\mathcal{G}_2$. The family of definable maps $\widetilde{h}$ given above is an isomorphism from $\mathcal{G}_1$ to $\mathcal{G}_2$ in $\mathbf{CGpd}_T$.*

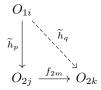*Proof.* As each $\widetilde{h}_n$ is bijective by definition, Lemma 4.6 tells us that it suffices to show that $\widetilde{h}$ is a morphism, namely that it satisfies (A) and (B).

For (A), fix $p \in N(i_1, i_2)$ and $q \in N(j_1, j_2)$. Then $p = (m, i_1)$ for some $m \in M_2(\iota(i_1), i_2)$, and $q = (m', j_1)$ for some $m' \in M_2(\iota(j_1), j_2)$, and we have

$$
\begin{aligned}
\widetilde{h}_q \, \mathrm{Hom}_{\mathcal{G}_1}(O_{1i_1}, O_{1j_1}) &= f_{2m'} h_{j_1} \mathrm{Hom}_{\mathcal{G}_1}(O_{1i_1}, O_{1j_1}) \quad \text{by definition of } \widetilde{h} \\
&= f_{2m'} \mathrm{Hom}_{\mathcal{G}_2}(O_{2\iota(i_1)}, O_{2\iota(j_1)}) h_{i_1} \quad \text{as } h \text{ is an embedding} \\
&= \mathrm{Hom}_{\mathcal{G}_2}(O_{2\iota(i_1)}, O_{2\iota(j_1)}) f_{2m} h_{i_1} \quad \text{as } f_2 \text{ satisfies (A)} \\
&= \mathrm{Hom}_{\mathcal{G}_2}(O_{2\iota(i_1)}, O_{2\iota(j_1)}) \widetilde{h}_p
\end{aligned}
$$

as desired.

To show (B), we are given $p \in N(i,j)$ and $m \in M_2(j,k)$, and we seek $q \in N(i,k)$ such that

$$
\begin{array}{ccc}
O_{1i} & & \\
\widetilde{h}_p \downarrow & \searrow^{\widetilde{h}_q} & \\
O_{2j} & \xrightarrow{f_{2m}} & O_{2k}
\end{array}
$$

commutes. Writing $p = (m', i)$ for some $m' \in M_2(\iota(i), j)$ the above becomes

$$
\begin{array}{ccc}
O_{1i} & & \\
h_i \downarrow & & \\
O_{2\iota(i)} & & \\
f_{2m'} \downarrow & \searrow^{f_\ell} & \\
O_{2j} & \xrightarrow{f_{2m}} & O_{2k}
\end{array}
$$

where $f_\ell := f_{2m} f_{2m'}$. So $q := (\ell, i)$ will work. $\qquad\square$

Corollary 4.8. *Equivalent definable connected groupoids in the sense of Hrushovski are isomorphic in $\mathbf{CGpd}_T$.*

The converse is harder to see directly. It will follow for finitely faithful definable connected groupoids from the equivalence of categories that we prove below.

We will restrict our attention to the finitely faithful case. Let $\mathbf{FCGpd}_T$ denote the full subcategory of $\mathbf{CGpd}_T$ generated by the 0-definable finitely faithful connected groupoids.

4.3. **The functor $G : \mathbf{IGI}_T \to \mathbf{FCGpd}_T$.** Given an internal generalised imaginary sort $\mathbb{U}' = (\mathbb{U}, S, \dots)$, Theorem 3.5 associated to it the liaison groupoid $\mathcal{G}$ in $\mathbb{U}$. Let us write $\mathcal{G}$ as $G(\mathbb{U}')$. Since $G(\mathbb{U}')$ is a 0-definable finitely faithful connected groupoid in $\mathbb{U}$, this makes G a function from the objects of $\mathbf{IGI}_T$ to the objects of $\mathbf{FCGpd}_T$. Our aim here is to make G into a covariant functor between these categories.

To do so, we have to also consider the one-object extension of $\mathcal{G}$ which was called $\mathcal{G}'$ in Theorem 3.5; it was a definable connected groupoid in $\mathbb{U}'$ whose additional object was S and which had the property that $\mathrm{Aut}_{\mathcal{G}'}(S) = \mathrm{Aut}(\mathbb{U}'/\mathbb{U})$ as groups actiong on S. We will denote $\mathcal{G}'$ as $G'(\mathbb{U}')$.

Remark 4.9. Actually the groupoid associated to the cover $\mathbb{U}'$ is not unique; so we are using the axiom of choice to fix a particular one. Also, the groupoid is really associated to the cover $\mathrm{Th}(\mathbb{U}')$; so denoting it by $G(\mathbb{U}')$ is an abuse of notation.

Suppose $g : S_1 \to S_2$ is a morphism of internal generalised imaginaries $\mathbb{U}_1, \mathbb{U}_2$. Let $\mathcal{G}_1 := G(\mathbb{U}_1)$ and $\mathcal{G}'_1 := G'(\mathbb{U}_1)$. We write $\mathcal{G}_1 = (O_{1i}, f_{1m})_{i \in I_1, m \in M_1}$ and $\mathcal{G}'_1 = (O_{1i}, f_{1m})_{i \in I'_1, m \in M'_1}$ where $I'_1 = I_1 \cup \{*\}$ and $O_{1*} = S_1$. We make similar notational conventions to talk about $\mathcal{G}_2 := G(\mathbb{U}_2)$ and $\mathcal{G}'_2 := G'(\mathbb{U}_2)$. We define $G(g)$ to be the family of functions $O_{1i_1} \to O_{2i_2}$, as i_1 ranges in I_1 and i_2 ranges in I_2, of the form $\alpha_2 \circ g \circ \alpha_1$ where $\alpha_1 \in \mathrm{Hom}_{\mathcal{G}'_1}(O_{1i_1}, S_1)$ and $\alpha_2 \in \mathrm{Hom}_{\mathcal{G}'_2}(S_2, O_{2i_2})$.

It is clear that $G(g)$ is a 0-definable family of definable functions in $(\mathbb{U}_1 \cup \mathbb{U}_2, g)$. But as the individual functions are between definable sets in $\mathbb{U}$, and as $\mathbb{U}$ is stably

embedded in $(\mathbb{U}_1 \cup \mathbb{U}_2, g)$, we get that the family $G(g)$ is 0-definable in $\mathbb{U}$. Let $N \to I_1 \times I_2$ be a 0-definable parameterising set for $G(g)$.

Lemma 4.10. $G(g)$ *is a morphism from* $G(\mathbb{U}_1)$ *to* $G(\mathbb{U}_2)$ *that depends only on* $\mathrm{Th}(\mathbb{U}_1 \cup \mathbb{U}_2, g)$.

Proof. Suppose $h : S_1 \to S_2$ is such that $(\mathbb{U}_1 \cup \mathbb{U}_2, g) \equiv (\mathbb{U}_1 \cup \mathbb{U}_2, h)$. Then, by 4.1, $h = \gamma g$ for some $\gamma \in \mathrm{Aut}(\mathbb{U}_2/\mathbb{U})$. By definition of the liaison groupoid we have that $\gamma \in \mathrm{Aut}_{\mathcal{G}_2'}(S_2)$. Hence, for any $i_2 \in I_2$, $\mathrm{Hom}_{\mathcal{G}_2'}(S_2, O_{2i_2})\gamma = \mathrm{Hom}_{\mathcal{G}_2'}(S_2, O_{2i_2})$, and so $G(\gamma g)$ yields the same family of functions as $G(g)$.

To show that $G(g)$ is a morphism we need to verify (A) and (B). For (A), given $p \in N(i_1, i_2)$ and $q \in N(j_1, j_2)$ we will show that

$$G(g)_q \circ \mathrm{Hom}_{\mathcal{G}_1}(O_{1i_1}, O_{1j_1}) \subseteq \mathrm{Hom}_{\mathcal{G}_2}(O_{2i_2}, O_{2j_2}) \circ G(g)_p$$

and the symmetry in the argument will take care of the other containment. Given $\gamma_1 \in \mathrm{Hom}_{\mathcal{G}_1}$ we seek $\gamma_2 \in \mathrm{Hom}_{\mathcal{G}_2}$ such that

$$
\begin{array}{ccc}
O_{1i_1} & \xrightarrow{\gamma_1} & O_{1j_1} \\
\downarrow{\scriptstyle G(g)_p} & & \downarrow{\scriptstyle G(g)_q} \\
O_{2i_2} & \dashrightarrow{\scriptstyle \gamma_2} & O_{2j_2}
\end{array}
$$

commutes. By definition of $G(g)$ this becomes

$$
\begin{array}{ccccccc}
S_1 & \xleftarrow{\alpha_1} & O_{1i_1} & \xrightarrow{\gamma_1} & O_{1j_1} & \xrightarrow{\beta_1} & S_1 \\
\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle G(g)_p} & & \downarrow{\scriptstyle G(g)_q} & & \downarrow{\scriptstyle g} \\
S_2 & \xrightarrow{\alpha_2} & O_{2i_2} & & O_{2j_2} & \xleftarrow{\beta_2} & S_2
\end{array}
$$

Now $\beta_1 \gamma_1 \alpha_1^{-1} \in \mathrm{Aut}_{\mathcal{G}_1'}(S_1) = \mathrm{Aut}(\mathbb{U}_1/\mathbb{U})$ and hence extends by stable embeddedness to some $\sigma \in \mathrm{Aut}(\mathbb{U}_1 \cup \mathbb{U}_2, g)$. So

$$
\begin{array}{ccccccc}
S_1 & \xleftarrow{\alpha_1} & O_{1i_1} & \xrightarrow{\gamma_1} & O_{1j_1} & \xrightarrow{\beta_1} & S_1 \\
\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle G(g)_p} & & \downarrow{\scriptstyle G(g)_q} & & \downarrow{\scriptstyle g} \\
S_2 & \xrightarrow{\alpha_2} & O_{2i_2} & & O_{2j_2} & \xleftarrow{\beta_2} & S_2
\end{array}
$$
$$\sigma|_{S_2}$$

commutes. Hence, $\gamma_2 := \beta_2 \sigma|_{S_2} \alpha_2^{-1}$ works. Note that $\sigma|_{S_2} \in \mathrm{Aut}(\mathbb{U}_2/\mathbb{U}) = \mathrm{Aut}_{\mathcal{G}_2'}(S_2)$ so that γ_2 is in $\mathrm{Hom}_{\mathcal{G}_2}$.

For (B), we are given $p \in N(i, j)$ and $m \in M_2(j, k)$ and seek $q \in N(i, k)$ such that

$$
\begin{array}{ccc}
 & O_{1i} & \\
{\scriptstyle G(g)_p}\downarrow & \diagdown{\scriptstyle G(g)_q} & \\
O_{2j} & \xrightarrow{f_{2m}} & O_{2k}
\end{array}
$$

commutes. Unravelling, this becomes

$$S_1 \xleftarrow{\alpha_1} O_{1i}$$

$$\begin{array}{ccc} S_1 & \xleftarrow{\alpha_1} & O_{1i} \\ {\scriptstyle g}\downarrow & {\scriptstyle G(g)_p} & \downarrow \\ S_2 & \xrightarrow{\alpha_2} O_{2j} & \xrightarrow{f_{2m}} O_{2k} \end{array}$$

Since $\beta_2 := f_{2m}\alpha_2 \in \mathrm{Hom}_{\mathcal{G}_2'}(S_2, O_{2k})$, letting q be such that $G(g)_q = \beta_2 g \alpha_1$ works. $\qquad\square$

Proposition 4.11. G *is a covariant functor from* $\mathbf{IGI}_T$ *to* $\mathbf{FCGpd}_T$.

Proof. It remians only to show that if $(\mathbb{U}_1 \cup \mathbb{U}_2, g)$ and $(\mathbb{U}_2 \cup \mathbb{U}_3, h)$ are morphisms between internal generalised imaginaries then

$$G(h \circ g) = G(h) \circ G(g).$$

A function in the left-hand-side is of the form $\alpha_3 h g \alpha_1$ while one in the right is of the form $\alpha_3 h \beta_2 \alpha_2 g \alpha_1$ – here $\alpha_1 \in \mathrm{Hom}_{G'(\mathbb{U}_1)}$ with codomain S_1, $\alpha_2 \in \mathrm{Hom}_{G'(\mathbb{U}_2)}$ with domain S_2, $\beta_2 \in \mathrm{Hom}_{G'(\mathbb{U}_2)}$ with codomain S_2, $\alpha_3 \in \mathrm{Hom}_{G'(\mathbb{U}_3)}$ with domain S_3, and all the compositions make sense. Given $\alpha_3 h g \alpha_1$ on the left-hand-side, to see that it is of the form $\alpha_3 h \beta_2 \alpha_2 g \alpha_1$ we can take any $\alpha_2 : S_2 \to O_{i_2}$ in $G'(\mathbb{U}_2)$, for any $i_2 \in I_2$, and let $\beta_2 := \alpha_2^{-1}$.

For the converse, suppose we are given $\alpha_3 h \beta_2 \alpha_2 g \alpha_1 \in G(h) \circ G(g)$. Then $\beta_2 \alpha_2 \in \mathrm{Aut}_{G'(\mathbb{U}_2)}(S_2) = \mathrm{Aut}(\mathbb{U}_2/\mathbb{U})$ and hence extends by $\mathbb{U}_2$ being stably embedded in $(\mathbb{U}_1 \cup \mathbb{U}_2, g)$ to an automorphism σ of the latter. So $\beta_2 \alpha_2 g = g \sigma|_{S_1}$. Hence $\alpha_3 h \beta_2 \alpha_2 g \alpha_1 = \alpha_3 h g \sigma|_{S_1} \alpha_1$. Since $\sigma|_{S_1} \in \mathrm{Aut}(\mathbb{U}_1/\mathbb{U}) = \mathrm{Aut}_{G'(\mathbb{U}_1)}(S_1)$, we have that $\sigma|_{S_1} \alpha_1 \in \mathrm{Hom}_{G'(\mathbb{U}_1)}$ with codomain S_1, so $\alpha_3 h g \sigma|_{S_1} \alpha_1$ has the desired form to be in $G(h \circ g)$. $\qquad\square$

4.4. **The functor** $C : \mathbf{FCGpd}_T \to \mathbf{IGI}_T$. Recall that Theorems 3.11 and 3.14 associated to any 0-definable finitely faithful connected groupoid $\mathcal{G}$ in $\mathbb{U}$ an internal generalised imaginary sort denoted there by $\mathbb{U}'' = (\mathbb{U}, O_*, \dots)$. Let us relabel this cover as $C(\mathcal{G})$, but retain O_* as the name of the new sort. So C is a function from the objects of $\mathbf{FCGpd}_T$ to the objects of $\mathbf{IGI}_T$, which we now make into a covariant functor.

Let us first recall that we have a one-object extension of $\mathcal{G}$ denoted by $\mathcal{G}'$, which is a definable connected groupoid in $C(\mathcal{G})$ whose new object is O_*, and with the property that $\mathrm{Aut}_{\mathcal{G}'}(O_*) = \mathrm{Aut}\big(C(\mathcal{G})/\mathbb{U}\big)$ as groups acting on O_*.

Now suppose we have two 0-definable finitely faithful connected groupoids in $\mathbb{U}$, $\mathcal{G}_1 = (O_{1i}, f_{1m})_{i \in I_1, m \in M_1}$ and $\mathcal{G}_2 = (O_{2i}, f_{2m})_{i \in I_2, m \in M_2}$, and $h : \mathcal{G}_1 \to \mathcal{G}_2$ is a 0-definable morphism between them. We obtain covers $C(\mathcal{G}_1) = (\mathbb{U}, O_{1*}, \dots)$ and $C(\mathcal{G}_2) = (\mathbb{U}, O_{2*}, \dots)$. Recall from the construction on page 8 that

$$O_{1*} = O_{1i_1} \times \{0\}$$

for some fixed $i_1 \in I_1$, and

$$O_{2*} = O_{2i_2} \times \{0\}$$

for some fixed $i_2 \in I_2$. Let $p \in N(i_1, i_2)$ be arbitrary so that $h_p : O_{1i_1} \to O_{2i_2}$, and define $C(h) := h_p \times \{0\} : O_{1*} \to O_{2*}$.

Lemma 4.12. $C(h)$ *determines a morphism from* $C(\mathcal{G}_1)$ *to* $C(\mathcal{G}_2)$ *that depends only on* h *and not on the choices of* i_1, i_2, p.

Proof. First let us check that $C(h)$, or rather $\mathrm{Th}\big(C(\mathcal{G}_1) \cup C(\mathcal{G}_2), C(h)\big)$, is well defined. So let $j_1 \in I_1$, $j_2 \in I_2$ and $q \in N(j_1, j_2)$ be different choices. Let $\overline{C(\mathcal{G}_1)}$ denote the internal cover constructed exactly as $C(\mathcal{G}_1)$ but using j_1 instead of i_1. We already know by Theorem 3.11 that $C(\mathcal{G}_1)$ and $\overline{C(\mathcal{G}_1)}$ are isomorphic over $\mathbb{U}$; in fact such an isomorphism is induced by fixing any $\alpha_1 \in \mathrm{Hom}_{\mathcal{G}_1}(O_{i_1}, O_{j_1})$. By (A), there is $\alpha_2 \in \mathrm{Hom}_{\mathcal{G}_2}(O_{2i_2}, O_{2j_2})$ such that $h_q \alpha_1 = \alpha_2 h_p$. Now, α_2 induces an isomorphism of $C(\mathcal{G}_2)$ and $\overline{C(\mathcal{G}_2)}$ over $\mathbb{U}$. The commutativity condition above implies that $\alpha_1 \cup \alpha_2$ induces an isomorphism from $\big(C(\mathcal{G}_1) \cup C(\mathcal{G}_2), h_p \times \{0\}\big)$ to $\big(\overline{C(\mathcal{G}_1)} \cup \overline{C(\mathcal{G}_2)}, h_q \times \{0\}\big)$.

Toward proving that $\big(C(\mathcal{G}_1) \cup C(\mathcal{G}_2), C(h)\big)$ is saturated, let us first recall from the construction that we have

$$
\begin{array}{ccc}
O_{1i_1} & \xrightarrow{\ h_p\ } & O_{2i_2} \\
{\scriptstyle \alpha_1}\downarrow & & \downarrow{\scriptstyle \alpha_2} \\
O_{1*} & \xrightarrow{\ C(h)\ } & O_{2*}
\end{array}
$$

for some $\alpha_1 \in \mathrm{Hom}_{\mathcal{G}_1'}(O_{1i_1}, O_{1*})$ and $\alpha_2 \in \mathrm{Hom}_{\mathcal{G}_2'}(O_{2i_2}, O_{2*})$. Indeed, $\alpha_1 = \mathrm{id}_{O_{1i_1}} \times \{0\}$ and $\alpha_2 = \mathrm{id}_{O_{2i_2}} \times \{0\}$, though of course this particular representation is not known to the theory of the morphism $C(h)$.

Now let $\big(\overline{C(\mathcal{G}_1)} \cup \overline{C(\mathcal{G}_2)}, \overline{C(h)}\big)$ be a saturated model of $\mathrm{Th}\big(C(\mathcal{G}_1) \cup C(\mathcal{G}_2), C(h)\big)$. By Lemma 2.4 we know that $C(\mathcal{G}_1) \cup C(\mathcal{G}_2)$ is a saturated model of the complete theory $\mathrm{Th}\big(C(\mathcal{G}_1)\big) \cup \mathrm{Th}\big(C(\mathcal{G}_2)\big)$, and so we have an isomorphism

$$
\sigma_1 \cup \sigma_2 : C(\mathcal{G}_1) \cup C(\mathcal{G}_2) \to \overline{C(\mathcal{G}_1)} \cup \overline{C(\mathcal{G}_2)}.
$$

We may as well identify the copies of $\mathbb{U}$ in both so that $\sigma_1 \cup \sigma_2$ is the identity on $\mathbb{U}$. On the other hand, the existence of the above commuting square is part of the theory of the morphism $C(h)$, and so we have

$$
\begin{array}{ccc}
O_{1i_1} & \xrightarrow{\ h_p\ } & O_{2i_2} \\
{\scriptstyle \overline{\alpha}_1}\downarrow & & \downarrow{\scriptstyle \overline{\alpha}_2} \\
\overline{O}_{1*} & \xrightarrow{\ \overline{C(h)}\ } & \overline{O}_{2*}
\end{array}
$$

for some $\overline{\alpha}_1 \in \mathrm{Hom}_{\overline{\mathcal{G}_1'}}(O_{1i_1}, \overline{O}_{1*})$ and $\overline{\alpha}_2 \in \mathrm{Hom}_{\overline{\mathcal{G}_2'}}(O_{2i_2}, \overline{O}_{2*})$. Since σ_1^{-1} is an isomorphism from $\overline{C(\mathcal{G}_1)}$ to $C(\mathcal{G}_1)$ that is the identity on $\mathbb{U}$, and $\overline{\alpha}_1 \in \mathrm{Hom}_{\overline{\mathcal{G}_1'}}$ has its domain in $\mathbb{U}$, we get that $\sigma^{-1}\overline{\alpha}_1 \in \mathrm{Hom}_{\mathcal{G}_1'}$. Hence, so is $\overline{\alpha}_1^{-1}\sigma_1$. It follows that $\widetilde{\alpha}_1 := \overline{\alpha}_1^{-1}\sigma_1 \alpha_1 \in \mathrm{Aut}_{\mathcal{G}_1}(O_{1i_1})$. By (A), there is $\widetilde{\alpha}_2 \in \mathrm{Aut}_{\mathcal{G}_2}(O_{2i_2})$ such that $h_p \widetilde{\alpha}_1 = \widetilde{\alpha}_2 h_p$. Now, since σ_2^{-1} is an isomorphism from $\overline{C(\mathcal{G}_2)}$ to $C(\mathcal{G}_2)$ that is the identity $\mathbb{U}$, we have that $\beta := \sigma_2^{-1}\overline{\alpha}_2\widetilde{\alpha}_2\alpha_2^{-1} \in \mathrm{Aut}_{\mathcal{G}_2'}(O_{2*}) = \mathrm{Aut}\big(C(\mathcal{G}_2)/\mathbb{U}\big)$. A final diagram chase, shows that if we replace σ_2 with $\sigma_2\beta$ then

$$
\begin{array}{ccc}
O_{1*} & \xrightarrow{\ C(h)\ } & O_{2*} \\
{\scriptstyle \sigma_1}\downarrow & & \downarrow{\scriptstyle \sigma_2} \\
\overline{O}_{1*} & \xrightarrow{\ \overline{C(h)}\ } & \overline{O}_{2*}
\end{array}
$$

commutes. That is,

$$\sigma_1 \cup \sigma_2 : \big(C(\mathcal{G}_1) \cup C(\mathcal{G}_2), C(h)\big) \to \big(\overline{C(\mathcal{G}_1)} \cup \overline{C(\mathcal{G}_2)}, \overline{C(h)}\big)$$

is an isomorphism, and so $\big(C(\mathcal{G}_1) \cup C(\mathcal{G}_2), C(h)\big)$ is saturated.

Note that $\mathbb{U}$ is stably embedded in $\big(C(\mathcal{G}_1) \cup C(\mathcal{G}_2), C(h)\big)$. Indeed, if $\sigma \in \mathrm{Aut}(\mathbb{U})$ then it induces an isomorphism between $\big(C(\mathcal{G}_1) \cup C(\mathcal{G}_2), C(h)\big)$ and the analogous construction with $\sigma(i_1), \sigma(i_2), \sigma(p)$ rather than i_1, i_2, p. But by the first paragraph of this proof, this is isomorphic to $\big(C(\mathcal{G}_1) \cup C(\mathcal{G}_2), C(h)\big)$ over $\mathbb{U}$. Composing these isomorphism we get an automorphism of $\big(C(\mathcal{G}_1) \cup C(\mathcal{G}_2), C(h)\big)$ that extends σ.

Finally, we show that $C(\mathcal{G}_1)$ is stably embedded in $\big(C(\mathcal{G}_1) \cup C(\mathcal{G}_2), C(h)\big)$. Suppose $\sigma \in \mathrm{Aut}\big(C(\mathcal{G}_1)\big)$. By the previous paragraph it suffices to consider $\sigma \in \mathrm{Aut}\big(C(\mathcal{G}_1)/\mathbb{U}\big) = \mathrm{Aut}_{\mathcal{G}_1'}(O_{1*})$. So σ is induced by some $\alpha_1 \in \mathrm{Aut}_{\mathcal{G}_1}(O_{1i_1})$. By (A), we can find $\alpha_2 \in \mathrm{Aut}_{\mathcal{G}_2}(O_{2i_2})$ such that $h_p \alpha_1 = \alpha_2 h_p$. Now, α_2 induces some $\sigma_2 \in \mathrm{Aut}_{\mathcal{G}_2'}(O_{2*}) = \mathrm{Aut}\big(C(\mathcal{G}_2)/\mathbb{U}\big)$. The above commuting property ensures that $\sigma \cup \sigma_2$ is an automorphism of $\big(C(\mathcal{G}_1) \cup C(\mathcal{G}_2), C(h)\big)$. A similar argument shows that $C(\mathcal{G}_2)$ is stably embedded in $\big(C(\mathcal{G}_1) \cup C(\mathcal{G}_2), C(h)\big)$. $\square$

Proposition 4.13. *C is a covariant functor from $\mathbf{FCGpd}_T$ to $\mathbf{IGI}_T$.*

Proof. It remains to show that if $g : \mathcal{G}_1 \to \mathcal{G}_2$ and $h : \mathcal{G}_2 \to \mathcal{G}_3$ are morphisms of definable connected groupoids, then $C(h \circ g) = C(h) \circ C(g)$. Using i_1, i_2, p for $C(g)$ and i_2, i_3, q for $C(h)$ we get that the right-hand-side is given by $(h_q \times \{0\}) \circ (g_p \times \{0\})$ where $g_p : O_{1i_1} \to O_{2i_2}$ and $h_q : O_{2i_2} \to O_{2i_3}$. But now we can use $i_1, i_3, n = (p, q)$ so that the left-hand-side is given by $(h \circ g)_n \times \{0\}$. As $(h \circ g)_n = h_q \circ g_p$ by definition, these agree. $\square$

4.5. The equivalence.

Theorem 4.14. *The functors G and C defined above establish an equivalence between the categories $\mathbf{IGI}_T$ and $\mathbf{FCGpd}_T$.*

Proof. We first construct a natural isomorphism of functors $\eta : \mathrm{id}_{\mathbf{IGI}_T} \to CG$. That is, for each internal generalised imaginary sort $\mathbb{V}$ of $\mathbb{U}$ (the reader will see in a minute why for the purposes of this proof we use $\mathbb{V}$ rather the usual terminology of $\mathbb{U}'$) we construct an isomorphism of covers $\eta_{\mathbb{V}} : \mathbb{V} \to CG(\mathbb{V})$ such that

$$
\begin{array}{ccc}
\mathbb{U}_1 & \xrightarrow{\eta_{\mathbb{U}_1}} & CG(\mathbb{U}_1) \\
\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle CG(g)} \\
\mathbb{U}_2 & \xrightarrow{\eta_{\mathbb{U}_2}} & CG(\mathbb{U}_2)
\end{array}
$$

commutes, for every morphism $g : \mathbb{U}_1 \to \mathbb{U}_2$ in $\mathbf{IGI}_T$.

Given $\mathbb{V} = (\mathbb{U}, S, \dots)$ let $\mathcal{G} := G(\mathbb{V}) = (O_i, f_m)_{i \in I, m \in M}$ be the liaison groupoid in $\mathbb{U}$. Then $CG(\mathbb{V}) = C(\mathcal{G})$ is another cover of $\mathbb{U}$ whose new sort is $O_* := O_i \times \{0\}$ for some fixed $i \in I$. We have two one-object groupoid extensions of $\mathcal{G}$: in $\mathbb{V}$ we have $G'(\mathbb{V})$ whose extra object is S, whereas in $CG(\mathbb{V})$ we have $G(\mathbb{V})'$ whose additional object is O_*. Now, fix $\alpha \in \mathrm{Hom}_{G'(\mathbb{V})}(S, O_i)$, and define

$$\eta_{\mathbb{V}} := \alpha \times \{0\} : S \to O_*,$$

where we mean here the function given by $x \mapsto (\alpha(x), 0)$. That $\eta_{\mathbb{V}}$ does not depend on the choice of i and α follows from by now familiar methods using that $\mathrm{Aut}_{G'(\mathbb{V})}(S) = \mathrm{Aut}(\mathbb{V}/\mathbb{U})$. Saturation of $(\mathbb{V} \cup CG(\mathbb{V}), \eta_{\mathbb{V}})$ follows very much in the same way as the saturation argument in the proof of Lemma 4.12, indeed somewhat more simply, using this time that $\mathrm{Aut}_{G(\mathbb{V})'}(O_*) = \mathrm{Aut}(CG(\mathbb{V})/\mathbb{U})$. It is similarly not hard to deduce, following the methods of the proof of Lemma 4.12, that both $\mathbb{V}$ and $CG(\mathbb{V})$ are stably embedded in $(\mathbb{V} \cup CG(\mathbb{V}), \eta_{\mathbb{V}})$. This shows that $\eta_{\mathbb{V}}$ is indeed a morphism of covers. It is an isomorphism because it is bijective.

We now check that the above square commutes for any morphism $g : S_1 \to S_2$ of internal generalised imaginaries $\mathbb{U}_1 = (\mathbb{U}, S_1, \dots)$ and $\mathbb{U}_2 = (\mathbb{U}, S_2, \dots)$. Let $G(\mathbb{U}_1) = (O_{1i}, f_{1m})_{i \in I_1, m \in M_1}$ and fix $i_1 \in I_1$ so that the new sort of $CG(\mathbb{U}_1)$ is $O_{1*} := O_{1i_1} \times \{0\}$. Let $\alpha_1 \in \mathrm{Hom}_{G'(\mathbb{U}_1)}(S_1, O_{1i_1})$ be such that $\eta_{\mathbb{U}_1} = \alpha_1 \times \{0\}$. Make similar notational conventions and choices for $G(\mathbb{U}_2)$, $CG(\mathbb{U}_2)$, and $\eta_{\mathbb{U}_2}$. Thus $\eta_{\mathbb{U}_2} g$ is the theory of $(\mathbb{U}_1 \cup CG(\mathbb{U}_2), \alpha_2 g \times \{0\})$. But now note that by definition of how the functor G acts on morphisms we have that $\alpha_2 g \alpha_1^{-1} : O_{1i_1} \to O_{2i_2}$ is one of the functions in $G(g)$. Thus, by how C was defined to act on morphisms, $CG(g)$ is the theory of $(CG(\mathbb{U}_1) \cup CG(\mathbb{U}_2), \alpha_2 g \alpha_1^{-1})$. Hence $CG(g) \eta_{\mathbb{U}_1}$ is also the theory of $(\mathbb{U}_1 \cup CG(\mathbb{U}_2), \alpha_2 g \times \{0\})$, as desired.

So we have a a natural isomorphism $\eta : \mathrm{id}_{\mathbf{IGI}_T} \to CG$.

Next we construct a natural isomorphism $\varepsilon : \mathrm{id}_{\mathbf{FCGpd}_T} \to GC$.

Let $\mathcal{G} = (O_{1i}, f_{1m})_{i \in I_1, m \in M_1}$ be a 0-definable finitely faithful connected groupoid in $\mathbb{U}$. So $C(\mathcal{G})$ is an internal generalised imaginary with new sort $O_* = O_i \times \{0\}$ for some fixed $i \in I_1$. We get a liaison groupoid $GC(\mathcal{G}) = (O_{2i}, f_{2m})_{i \in I_2, m \in M_2}$ in $\mathbb{U}$, and we wish to define a natural isomorphism $\varepsilon_{\mathcal{G}} : \mathcal{G} \to GC(\mathcal{G})$.

Recall that $\mathcal{G}'$ is a definable connected groupoid in $C(\mathcal{G})$ extending $\mathcal{G}$ by the one new object O_*. On the other hand, $G'C(\mathcal{G})$ is also a definable connected groupoid in $C(\mathcal{G})$, this time extending $GC(\mathcal{G})$ but also by the one new object O_*. Note that

$$\mathrm{Aut}_{\mathcal{G}'}(O_*) = \mathrm{Aut}\big(C(\mathcal{G})/\mathbb{U}\big) = \mathrm{Aut}_{G'C(\mathcal{G})}(O_*)$$

as groups acting on O_*. This suggests how we should define $\varepsilon_{\mathcal{G}} : \mathcal{G} \to GC(\mathcal{G})$; given $i_1 \in I_1$ and $i_2 \in I_2$, the set of function from O_{1i_1} to O_{2i_2} in $\varepsilon_{\mathcal{G}}$ is

$$\mathrm{Hom}_{G'C(\mathcal{G})}(O_*, O_{2i_2}) \circ \mathrm{Hom}_{\mathcal{G}'}(O_{1i_1}, O_*).$$

As both $\mathcal{G}'$ and $G'C(\mathcal{G})$ are 0-definable in $C(\mathcal{G})$, this is clearly uniformly 0-definable in $C(\mathcal{G})$, and hence by stable embeddability in $\mathbb{U}$. Let $N \to I_1 \times I_2$ parametrise $\varepsilon_{\mathcal{G}}$.

To see that $\varepsilon_{\mathcal{G}}$ does not depend on the choice of i, suppose $\overline{C(\mathcal{G})}$ is an isomorphic copy of $C(\mathcal{G})$ over $\mathbb{U}$, which is exactly what a different choice of i would produce. Suppose σ is the isomorphism. We need to show that

$$\mathrm{Hom}_{G'C(\mathcal{G})}(O_*, O_{2i_2}) \circ \mathrm{Hom}_{\mathcal{G}'}(O_{1i_1}, O_*) = \mathrm{Hom}_{G'\overline{C(\mathcal{G})}}(\overline{O}_*, O_{2i_2}) \circ \mathrm{Hom}_{\overline{\mathcal{G}'}}(O_{1i_1}, \overline{O}_*).$$

But σ would take $\mathrm{Hom}_{G'C(\mathcal{G})}(O_*, O_{2i_2})$ to $\mathrm{Hom}_{G'\overline{C(\mathcal{G})}}(\overline{O}_*, O_{2i_2})$, and $\mathrm{Hom}_{\mathcal{G}'}(O_{1i_1}, O_*)$ to $\mathrm{Hom}_{\overline{\mathcal{G}'}}(O_{1i_1}, \overline{O}_*)$, so that it takes the left-hand-side of the desired equality to the right-hand-side. But as σ is the identity on $\mathbb{U}$, these sets must be equal.

Next we check condition (A) of being a morphism. Fix $p \in N(i_1, i_2)$ and $q \in N(j_1, j_2)$. Write $(\varepsilon_{\mathcal{G}})_p = \beta \alpha$ where $\alpha \in \mathrm{Hom}_{\mathcal{G}'}(O_{1i_1}, O_*)$ and $\beta \in \mathrm{Hom}_{G'C(\mathcal{G})}(O_*, O_{2i_2})$. Similarly, write $(\varepsilon_{\mathcal{G}})_q = \delta \gamma$ where $\gamma \in \mathrm{Hom}_{\mathcal{G}'}(O_{1j_1}, O_*)$ and $\delta \in \mathrm{Hom}_{G'C(\mathcal{G})}(O_*, O_{2j_2})$.

Then

$$
\begin{aligned}
(\varepsilon_{\mathcal{G}})_q \operatorname{Hom}_{\mathcal{G}'}(O_{1i_1}, O_{1j_1}) &= \delta\gamma \operatorname{Hom}_{\mathcal{G}'}(O_{1i_1}, O_{1j_1}) \\
&= \delta \operatorname{Aut}_{\mathcal{G}'}(O_*)\alpha \\
&= \delta \operatorname{Aut}_{G'C(\mathcal{G})}(O_*)\alpha \\
&= \operatorname{Hom}_{G'C(\mathcal{G})}(O_{2i_2}, O_{2j_2})\beta\alpha \\
&= \operatorname{Hom}_{G'C(\mathcal{G})}(O_{2i_2}, O_{2j_2})(\varepsilon_{\mathcal{G}})_p
\end{aligned}
$$

as desired.

That condition (B) holds follows easily from the definitions and the fact that $\operatorname{Hom}_{G'C(\mathcal{G})}$ is closed under composition. So $\varepsilon_{\mathcal{G}}$ is a morphism. It is an isomorphism by Lemma 4.6 since it is made up of bijections.

Finally, it remains to show that for any morphism $h : \mathcal{G}_1 \to \mathcal{G}_2$,

$$
\begin{array}{ccc}
\mathcal{G}_1 & \xrightarrow{\varepsilon_{\mathcal{G}_1}} & GC(\mathcal{G}_1) \\
\downarrow{\scriptstyle h} & & \downarrow{\scriptstyle GC(h)} \\
\mathcal{G}_2 & \xrightarrow{\varepsilon_{\mathcal{G}_2}} & GC(\mathcal{G}_2)
\end{array}
$$

commutes. We leave this to the reader as it is notationaly rather tedious to write down and does not use any observation that we have not already made, but rather requires simply an unravelling of the definitions. □

5. Uniformity

We now investigate the extent to which these functors G and C preserve uniform definability in families. As usual we fix a monster model $\mathbb{U}$ of our complete (multi-sorted) L-theory T, which we assume admits elimination of imaginaries.

First, let us be precise about what we mean by families of connected groupoids.

Definition 5.1. Given a 0-definable set A in $\mathbb{U}$, by a 0-*definable family of connected groupoids over* A, denoted by $\mathcal{G} \to A$, we mean the following: a 0-definable set I equipped with a surjective 0-definable function $I \to A$, a 0-definable set M equipped with a surjective 0-definable function $M \to I \times_A I$, a 0-definable set O with a surjective 0-definable $O \to I$, and a 0-definable subset $f \subseteq M \times_{I \times_A I} (O \times_A O)$ such that for each $a \in A$, the fibres $I_a, M_a \to I_a \times I_a, O_a \to I_a$, and $f_a \subseteq M_a \times_{I_a \times I_a} (O_a \times O_a)$, form an a-definable connected groupoid $\mathcal{G}_a = \big((O_a)_i, (f_m)_i\big)_{i \in I_a, m \in M_a}$.

Note that this is really nothing other than a 0-definable groupoid in $\mathbb{U}$ that may not be connected. Indeed, the total space $\mathcal{G} = (O_i, f_m)_{i \in I, m \in M}$ is a 0-definable groupoid whose connected components are precisely the $\mathcal{G}_a$ for $a \in A$. Conversely, if we start with a possibly disconnected 0-definable groupoid $\mathcal{G} = (O_i, f_m)_{i \in I, m \in M}$ in $\mathbb{U}$, and we let (following [5]) $\operatorname{Iso}_{\mathcal{G}}$ denote the (0-definable) equivalence relation on I where $(i, j) \in \operatorname{Iso}_{\mathcal{G}}$ if and only if $M(i, j) \neq \emptyset$, then $\mathcal{G}$ naturally obtains the structure of a definable family of connected groupoids over $A := I/\operatorname{Iso}_{\mathcal{G}}$. So asking about definable families of connected groupoids amounts to generalising from connected groupoids to possible disconnected groupoids.

On the other side, what should we mean by an internal generalised imaginary sort *relative to A*? This is what Hrushovski [5] refers to as a 1-analysable cover over A, and we will continue to use his terminology.

Definition 5.2. Given a 0-definable set A in $\mathbb{U}$, a 1-*analysable cover over A* is a cover $\mathbb{U}' = (\mathbb{U}, S, \dots)$ of $\mathbb{U}$ by a single new sort, in a language that is finitely generated over L, equipped with a 0-definable surjective map $S \to A$ such that each fibre S_a is internal to $\mathbb{U}$.

As Hrushovski mentions, in the Remark following Lemma 3.1 of [5], the 1-analysable covers that arise from possibly disconnected definable groupoids have the additional property that there are no interesting definable relations between the distinct fibres of $S \to A$. We formalise this as follows.

Definition 5.3. We will say that a cover $\mathbb{U}' = (\mathbb{U}, S, \dots)$ by a single new sort equipped with a 0-definable surjective map $S \to A$, where A is a 0-definable set in $\mathbb{U}$, has *independent fibres* if

$$\text{Aut}(\mathbb{U}'/\mathbb{U}) = \prod_{a \in A} \text{Aut}(S_a/\mathbb{U})$$

in the sense that $\sigma \mapsto (\sigma|_{S_a} : a \in A)$ is an isomorphism. Here,

$$\text{Aut}(S_a/\mathbb{U}) := \{\sigma|_{S_a} : \sigma \in \text{Aut}\left(\mathbb{U}'/\mathbb{U}\right)\}.$$

The notation $\text{Aut}(S_a/\mathbb{U})$ could be misleading, it could be confused with the *a priori* larger group of bijections $f : S_a \to S_a$ such that $f \cup \text{id}_{\mathbb{U}}$ is a partial elementary map of $\mathbb{U}'$. The following lemma implies that in the case of independent fibres these groups do actually agree.

Lemma 5.4. *Suppose A is 0-definable in $\mathbb{U}$, and $\mathbb{U}'$ is a cover of $\mathbb{U}$ by a single new sort S equipped with a 0-definable surjective map $S \to A$ with independent fibres. Then $(\mathbb{U}, S_a)$ is stably embedded in $\mathbb{U}'$ for each $a \in A$.*

Proof. It suffices to show that every automorphism of the structure induced on $(\mathbb{U}, S_a)$ by the a-definable sets in $\mathbb{U}'$ extends to an automorphism of $\mathbb{U}'$. Let ρ be such an automorphism of the induced structure on $(\mathbb{U}, S_a)$. Since $\mathbb{U}$ is stably embedded in $\mathbb{U}'$ – this is part of being a cover – $\rho|_{\mathbb{U}}$ does extend to $\text{Aut}(\mathbb{U}')$, and hence, by precomposing with the inverse of this extension, we may assume that $\rho|_{\mathbb{U}} = \text{id}_{\mathbb{U}}$. We will show that ρ extends by identity to an automorphism of $\mathbb{U}'$. That is, we will show $\text{tp}(bc) = \text{tp}\left(\rho(b)c\right)$ for any finite tuple b from S_a and c from $\mathbb{U}' \setminus S_a$. We know that $\text{tp}(b/\mathbb{U}) = \text{tp}\left(\rho(b)/\mathbb{U}\right)$. As $\mathbb{U}$ is stably embedded in $\mathbb{U}'$ there exists $\sigma \in \text{Aut}(\mathbb{U}'/\mathbb{U})$ with $\sigma(b) = \rho(b)$. Now $\sigma|_{S_a} \in \text{Aut}(S_a/\mathbb{U})$, and so by independence of fibres there is $\tau \in \text{Aut}(\mathbb{U}'/\mathbb{U})$ that agrees with σ on S_a and is the identity everywhere else. Hence, for any finite tuple c from $\mathbb{U}' \setminus S_a$, we get

$$\text{tp}(bc) = \text{tp}\left(\tau(bc)\right) = \text{tp}\left(\sigma(b)c\right) = \text{tp}\left(\rho(b)c\right)$$

as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad$ $\square$

We are now ready to show that the liaison groupoid functor G relativises – that the construction of Theorem 3.5 is uniformly definable in families – under the assumption that the fibres are independent.

Theorem 5.5. *Suppose A is 0-definable set in $\mathbb{U}$, and $\mathbb{U}' = (\mathbb{U}, S, \dots)$ is a 1-analysable cover over A with independent fibres. Then there are 0-definable families of finitely faithful connected groupoids, $\mathcal{G} \to A$ in $\mathbb{U}$ and $\mathcal{G}' \to A$ in $(\mathbb{U}')^{\mathrm{eq}}$, such that for each $a \in A$*

- *$\mathcal{G}'_a$ is a one-object extension of $\mathcal{G}_a$ with that object being S_a, and*
- *$\mathrm{Aut}_{\mathcal{G}'_a}(S_a)$ together with its action on S_a is isomorphic to $\mathrm{Aut}(S_a/\mathbb{U})$.*

We denote $\mathcal{G} \to A$ by $G_A(\mathbb{U}')$.

Proof. The first step will be to produce for each $a \in A$ an a-definable finitely faithful connected groupoid $\mathcal{G}_a$ in $\mathbb{U}$ with a one-object extension $\mathcal{G}'_a$ in $(\mathbb{U}')^{\mathrm{eq}}$ such that $\mathrm{Aut}_{\mathcal{G}'_a}(S_a) = \mathrm{Aut}(S_a/\mathbb{U})$. We will then show that these vary uniformly with the parameter a.

Fix $a \in A$, and let M be the structure induced on $(\mathbb{U}, S_a)$ by the a-definable sets in $\mathbb{U}'$. Then M is an internal cover of $\mathbb{U}$. We claim that, because of the independnece of fibres, the language of M is finitely generated over that of $\mathbb{U}$. Indeed, the proof of Proposition 3.6 actually shows that M has a finitely generated language if for any definable family of (graphs of) definable bijections from S_a to itself in M, say $F \subseteq B \times S_a^2$, the set $\{b \in B : F_b \cup \mathrm{id}_{\mathbb{U}} \in \mathrm{Aut}(M/\mathbb{U})\}$ is definable. By stable embeddedness (Lemma 5.4), $\mathrm{Aut}(M/\mathbb{U}) = \mathrm{Aut}(S_a/\mathbb{U})$. By independence of fibres, $F_b \cup \mathrm{id}_{\mathbb{U}} \in \mathrm{Aut}(S_a/\mathbb{U})$ if and only if $F_b \cup \mathrm{id}_{\mathbb{U}' \setminus S_a} \in \mathrm{Aut}(\mathbb{U}')$. And the latter is a definable (in $\mathbb{U}'$) property of the parameter b as the language of $\mathbb{U}'$ is finitely generated over that of $\mathbb{U}$. Hence, by stable embeddedness of $(\mathbb{U}, S_a)$ in $\mathbb{U}'$ again, it is a definable property of b in M. So M is an internal generalised imaginary sort of $\mathbb{U}$, and we can apply Theorem 3.5 to M. We obtain an a-definable connected liaison groupoid $\mathcal{G}_a$ in $\mathbb{U}$ and its one-object connected extension $\mathcal{G}'_a$ in M with new object S_a. So $\mathcal{G}'_a$ is a-definable in $\mathbb{U}'$. Moreover, $\mathrm{Aut}_{\mathcal{G}'_a}(S_a) = \mathrm{Aut}(M/\mathbb{U}) = \mathrm{Aut}(S_a/\mathbb{U})$.

Note that despite the notation, we do not yet have uniformity as a varies. Our remaining goal is to make these groupoids part of uniformly 0-definable families of connected groupoids $\mathcal{G} \to A$ in $\mathbb{U}$ and $\mathcal{G}' \to A$ in $(\mathbb{U}')^{\mathrm{eq}}$.

Fix $a \in A$ and formulas (involving the parameter a) that define all the data that make up $\mathcal{G}_a$ and $\mathcal{G}'_a$. By compactness it suffices to produce a 0-definable set $a \in U(a) \subseteq A$ such that for all $b \in U(a)$, when one replaces a by b in these formulas one still gets connected groupoids $\mathcal{G}_b$ and $\mathcal{G}'_b$, where $\mathcal{G}'_b$ is a one-object extension of $\mathcal{G}_b$ with new object S_b, and such that $\mathrm{Aut}_{\mathcal{G}'_b}(S_b) = \mathrm{Aut}(S_b/\mathbb{U})$. The only part of this that is not clear is that the condition $\mathrm{Aut}_{\mathcal{G}'_b}(S_b) = \mathrm{Aut}(S_b/\mathbb{U})$ can be 0-definably imposed on the parameter b.

That $\mathrm{Aut}(S_b/\mathbb{U}) \subseteq \mathrm{Aut}_{\mathcal{G}'_b}(S_b)$ is no additional condition at all on b, it holds for all b such that $\mathcal{G}'_b$ is one-object extension of $\mathcal{G}_b$ with new object S_b. Indeed, let $\sigma \in \mathrm{Aut}(\mathbb{U}'/\mathbb{U})$. Choose $f_m \in \mathrm{Hom}_{\mathcal{G}'_b}(S_b, O_i)$ for some object O_i from $\mathcal{G}_b$. Then for all $x \in S_b$ we have

$$
\begin{aligned}
f_m(x) &= \sigma(f_m(x)) \quad \text{as } f_m(x) \in O_i \subseteq \mathbb{U} \\
&= f_{\sigma(m)}(\sigma(x))
\end{aligned}
$$

and $f_{\sigma(m)} \in \mathrm{Hom}_{\mathcal{G}'_b}(S_b, O_i)$ as $\sigma(b) = b$. Hence $\sigma|_{S_b} = f_{\sigma(m)}^{-1} f_m \in \mathrm{Aut}_{\mathcal{G}'_b}(S_b)$.

So it remains to find a formula $\theta(z)$ satisfied by a and implying that every element of $\mathrm{Aut}_{\mathcal{G}'_z}(S_z)$ extends to an element of $\mathrm{Aut}(\mathbb{U}'/\mathbb{U})$. By finite generatedness of the language over L, there is $\theta(z)$ expressing that every element of $\mathrm{Aut}_{\mathcal{G}'_z}(S_z)$ when

extended by the identity is an element of $\mathrm{Aut}(\mathbb{U}'/\mathbb{U})$. That this holds of a is by independence of the fibres plus the fact that $\mathrm{Aut}_{\mathcal{G}'_a}(S_a) = \mathrm{Aut}(S_a/\mathbb{U})$. $\square$

Next we relativise the functor C.

Theorem 5.6. *Suppose $\mathcal{G} \to A$ is a 0-definable family of finitely faithful connected groupoids over A in $\mathbb{U}$. There exists a 1-analysable cover over A, $C_A(\mathcal{G}) = (\mathbb{U}, S, \dots)$, with independent fibres, and a 0-definable family of connected groupoids over A, $\mathcal{G}' \to A$, in $C_A(\mathcal{G})$, such that for each $a \in A$,*
- *$\mathcal{G}'_a$ is a one-object extension of $\mathcal{G}_a$, the new object being S_a, and*
- *$\mathrm{Aut}_{\mathcal{G}'_a}(S_a)$ together with its action on S_a is isomorphic to $\mathrm{Aut}(S_a/\mathbb{U})$.*

Proof. As in the proof of Theorem 3.14 this goes by first following Hrushovski [5] in constructing $\mathcal{G}' \to A$ in a cover $\mathbb{U}'$ that involves *two* new sorts, and then using the finite faithfulness assumption to get rid of one of them. In addition one has to establish the finite-generatedness of the language (part of being a 1-analysable cover) and the independence of the fibres.

The first step is a straightforward relativisation of the construction given on page 8. This is actually done in [5, Lemma 3.1] where Hrushovski does not assume the definable groupoid is connected. Instead of fixing a single $i \in I$, we fix a section $\nu : A \to I$ and let $S_a := O_{\nu(a)} \times \{a\}$. We can then construct $\mathcal{G}' \to A$ so that for each $a \in A$ and $i \in I_a$, $\mathrm{Hom}_{\mathcal{G}'}(S_a, O_i)$ is a copy of $\mathrm{Hom}_{\mathcal{G}}(O_{\nu(a)}, O_i)$, $\mathrm{Hom}_{\mathcal{G}'}(O_i, S_a)$ is a copy of $\mathrm{Hom}_{\mathcal{G}}(O_i, O_{\nu(a)})$, and $\mathrm{Aut}_{\mathcal{G}'}(S_a)$ is a copy of $\mathrm{Aut}_{\mathcal{G}}(O_{\nu(a)})$. Hrushovski proves that this groupoid lives 0-definably in a cover $\mathbb{U}'$ of $\mathbb{U}$ with new sorts $S = \bigcup_{a \in A} S_a$ and M_*, the latter equipped with a map to A such that $(M_*)_a$ encodes the morphisms that involve S_a as either domain or codomain or both. That $\mathrm{Aut}_{\mathcal{G}'_a}(S_a) = \{\sigma|_{S_a} : \sigma \in \mathrm{Aut}(\mathbb{U}') \text{ and } \sigma|_{\mathbb{U}} = \mathrm{id}_{\mathbb{U}}\}$, is part of [5, Lemma 3.1].

For the second step, since $\mathcal{G}$ is finitely faithful, the proof of Proposition 3.12 shows that $M_* \subseteq \mathrm{dcl}(\mathbb{U}, S)$, and so $(\mathbb{U}, S)$ is stably embedded in $\mathbb{U}'$. We define $C_A(\mathcal{G})$ to be the structure induced on $(\mathbb{U}, S)$ by $\mathbb{U}'$. Hence $\mathcal{G}' \to A$ is a 0-definable family of connected groupoids over A in $C_A(\mathcal{G})$ and $\mathrm{Aut}_{\mathcal{G}'_a}(S_a) = \mathrm{Aut}(S_a/\mathbb{U})$.

Next we establish the independence of the fibres. The point is that the new structure in $C_A(\mathcal{G})$ is just $\mathcal{G}'$ which has no morphisms between different fibres S_a as the $\mathcal{G}'_a$ are its connected components. But we give some details. Since S is the only new sort of $C_A(\mathcal{G})$, and the fibres S_a are $\mathbb{U}$-definable, the map $\mathrm{Aut}\big(C_A(\mathcal{G})/\mathbb{U}\big) \to \prod_{a \in A} \mathrm{Aut}(S_a/\mathbb{U})$ given by $\sigma \mapsto (\sigma|_{S_a} : a \in A)$ is an injective homomorphism. To prove surjectivity we start with $\sigma_a \in \mathrm{Aut}(S_a/\mathbb{U})$ for each $a \in A$, and define $\sigma := \mathrm{id}_{\mathbb{U}} \cup \bigcup_{a \in A} \sigma_a$. We need to show that σ is an automorphism of $C_A(\mathcal{G})$. It suffices to show that σ extends to an automorphism of $\mathbb{U}' = (\mathbb{U}, M_*, S, \dots)$. Given $m \in M_*$ we define $\sigma(m)$ as follows: if $f_m \in \mathrm{Hom}_{\mathcal{G}'_a}(S_a, O_i)$ for some $i \in I_a$ then we define $\sigma(m) \in M_*$ to be such that $f_{\sigma(m)} = f_m \sigma_a^{-1}$. Note that this makes sense as $\sigma_a \in \mathrm{Aut}(S_a/\mathbb{U}) = \mathrm{Aut}_{\mathcal{G}'_a}(S_a)$ and hence $f_m \sigma_a^{-1} \in \mathrm{Hom}_{\mathcal{G}'_a}(S_a, O_i)$. If instead $f_m \in \mathrm{Hom}_{\mathcal{G}'_a}(O_i, S_a)$ then $\sigma(m)$ is such that $f_{\sigma(m)} = \sigma_a f_m$. Finally, if $f_m \in \mathrm{Aut}_{\mathcal{G}'_a}(S_a)$ then define $\sigma(m)$ so that $f_{\sigma(m)} = \sigma_a f_m \sigma_a^{-1}$. This extends σ to a permutation of all the sorts of $\mathbb{U}'$. The only new basic relation in $\mathbb{U}'$, besides $S \to A$ and $M_* \to A$ which σ clearly preserves, is the ternary relation that exhibits elements of M_* as encoding definable bijections involving the fibres of $S \to A$. It is easy to check that we have defined σ so that this relation is preserved.

Finally, we need to check that the language of $C_A(\mathcal{G})$, say L', is finitely generated over L. This amounts to a relativisation of the proof Proposition 3.6, which we can

effect using the independence of the fibres. Since the fibres of $S \to A$ are internal to $\mathbb{U}$ which eliminates imaginaries, by compactness there exists a 0-definable set B in $C_A(\mathcal{G})$ with a surjective 0-definable map $B \to A$, and a 0-definable map $F : S \times_A B \to X$ over B, for some sort X of $\mathbb{U}$, that is injective on the fibres over B. So for $b \in B_a$, $F_b : S_a \to X$ is a b-definable embedding of S_a into $\mathbb{U}$. There exists an L'-formula $\theta(v, w)$ such that $\models \theta(b, c)$ if and only if $b, c \in B_a$ for some $a \in A$, $F_b(S_a) = F_c(S_a)$ as subsets of X, and $F_c^{-1} \circ F_b : S_a \to S_a$ is in $\text{Aut}(S_a/\mathbb{U})$. This is because $\text{Aut}(S_a/\mathbb{U})$ and its action on S_a is a-definable; it is $\text{Aut}_{\mathcal{G}'_a}(S_a)$. By stable embeddedess (Lemma 5.4) $\text{Aut}(S_a/\mathbb{U})$ is precisely the set of bijections $f : S_a \to S_a$ such that $f \cup \text{id}_{\mathbb{U}}$ is a partial L'-elementary map. It follows by compactness that there must be a finite sublanguage $\Sigma \subseteq L'$ such that whenever $b, c \in B_a$ and $F_b(S_a) = F_c(S_a)$, then $F_c^{-1} \circ F_b \cup \text{id}_{\mathbb{U}}$ is a partial L'-elementary map if and only if it is a partial Σ-elementary map. Extending Σ we may also assume that F and B are defined by Σ-formulas. Set $L'_\circ := L \cup \Sigma$. We show that $L'_\circ$ witnesses the finite-generatedness of L' over L. As in Proposition 3.6, this reduces to showing that $\text{Aut}(\mathbb{U}'_\circ/\mathbb{U}) = \text{Aut}(\mathbb{U}'/\mathbb{U})$, where $\mathbb{U}'_\circ$ is the reduct of $\mathbb{U}'$ to $L'_\circ$. Suppose $\sigma \in \text{Aut}(\mathbb{U}'_\circ/\mathbb{U})$. By the independence of fibres, it suffices to show that $\sigma|_{S_a} \in \text{Aut}(S_a/\mathbb{U})$ for each $a \in A$. But fixing any $b \in B_a$, as we have seen a few times, we get $\sigma|_{S_a} = F_{\sigma(b)}^{-1} \circ F_b$. In particular, $F_{\sigma(b)}^{-1} \circ F_b \cup \text{id}_{\mathbb{U}}$ is a partial $L'_\circ$-elementary map, and hence a partial L'-elementary map by choice of $\Sigma \subseteq L'_\circ$. So, $\sigma|_{S_a} \in \text{Aut}(S_a/\mathbb{U})$ as desired. $\square$

When A is a singleton $C_A(\mathcal{G})$ coincides with the internal cove $C(\mathcal{G})$ of §4.4. Even when A is infinite, $C_A(\mathcal{G})$ may still be an internal cover. For example, if $\mathcal{G}$ is such that between any two objects there is at most one morphism, then $\mathcal{G}$ is essentially the 0-definable equivalence relation $\text{Iso}_{\mathcal{G}}$, and $C_A(\mathcal{G})$ is part of $\mathbb{U}^{\text{eq}}$. Internality of $C_A(\mathcal{G})$ corresponds to this happening for all but finitely many $a \in A$.

Proposition 5.7. *The 1-analysable cover $C_A(\mathcal{G})$ is an internal cover if and only if for all but finitely many $a \in A$ the automorphism group of any (equivalently some) object of $\mathcal{G}_a$ is trivial.*
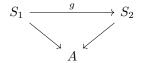
Proof. For any cover $\mathbb{U}' = (\mathbb{U}, S, \dots)$, in a finitely generated language, internality is equivalent to the cardinality of $\text{Aut}(\mathbb{U}'/\mathbb{U})$ being bounded by $|\mathbb{U}|$. This characterisation seems to be well known; for example, it is suggested without proof in [5, Remark 1.2]. The left-to-right direction is because internality implies that $\text{Aut}(\mathbb{U}'/\mathbb{U})$ is interpretable in $\mathbb{U}'$ and hence cannot have cardinality greater than that of the universe. For the converse one observes that if S is not internal to $\mathbb{U}$ then for any partial elementary map f with small domain $A \subseteq S$, since $S \not\subseteq \text{dcl}(\mathbb{U}, A)$, $f \cup \text{id}_{\mathbb{U}}$ can be extended in two distinct ways to a partial elementary map on $A \cup \{a\}$ for some $a \in S$. We can thus build, back-and-forthly, a binary tree of partial elementary maps whose paths give rise to $2^{|\mathbb{U}|}$-many elements of $\text{Aut}(\mathbb{U}'/\mathbb{U})$.

Now, by Theorem 5.6 we have that

$$\text{Aut}\left(\mathbb{C}_A(\mathcal{G})/\mathbb{U}\right) = \prod_{a \in A} \text{Aut}(S_a/\mathbb{U}) = \prod_{a \in A} \text{Aut}_{\mathcal{G}'_a}(S_a).$$

The set of $a \in A$ such that $\text{Aut}_{\mathcal{G}'_a}(S_a)$ is not trivial is definable, and hence is either finite or of the cardinality of the universe. In the former case $|\text{Aut}\left(\mathbb{C}_A(\mathcal{G})/\mathbb{U}\right)| \leq |\mathbb{U}|$ and in the latter $|\text{Aut}\left(\mathbb{C}_A(\mathcal{G})/\mathbb{U}\right)| = 2^{|\mathbb{U}|}$. $\square$

Theorems 5.5 and 5.6 relativise Hrushovki's correspondence between definable connected groupoids and internal generalised imaginary sorts. In fact the C_A and G_A given by these theorems are functorial with respect to the following natural notions of morphism: Given definable families of connected groupoids, $\mathcal{G}_1 \to A$ and $\mathcal{G}_2 \to A$, a *morphism* is simply a definable family of morphisms of definable connected groupoids $h_a : \mathcal{G}_{1a} \to \mathcal{G}_{2a}$ parameterised by A. Given 1-analysable covers $\mathbb{U}_1 = (\mathbb{U}, S_1, \dots)$ and $\mathbb{U}_2 = (\mathbb{U}, S_2, \dots)$ over A, a *morphism* between them is just a morphism of covers $g : S_1 \to S_2$ such that

$$S_1 \xrightarrow{\quad g \quad} S_2$$
$$\searrow \qquad \swarrow$$
$$A$$

commutes. The arguments of §4 relativise so that C_A and G_A are functors witnessing the equivalence of the following two categories:

- finitely faithful 0-definable connected groupoids over A, and
- 1-analysable covers over A with independent fibres.

We leave it to the reader to verify the details.

For a 1-analysable cover $S \to A$ to have independent fibres is a rather restrictive condition. One can think of it as asking that the cover $\mathbb{U}' = (\mathbb{U}, S, \dots)$ be as far from being internal to $\mathbb{U}$ as is consistent with the fibres of $S \to A$ being $\mathbb{U}$-internal. For example, in differentially closed fields of characteristic zero, the best known non-internal 1-analysable cover of the field of constants C is the subgroup S of the multiplicative group defined by the second order equation $\delta\left(\frac{\delta x}{x}\right) = 0$, with $S \to C$ being given by the logarithmic derivative $x \mapsto \frac{\delta x}{x}$. But the fibres of $S \to C$ are not independent: S admits a definable group structure (field multiplication) that intertwines the fibres. Ruizhang Jin has, however, communicated to us the following positive example in differentially closed fields, details of which will appear in his PhD thesis.

Example 5.8 (Jin). Suppose $\mathbb{M} = (K, 0, 1, +, -, \times, \delta)$ is a saturated differentially closed field in characteristic zero, $\mathbb{U} = (C, 0, 1, +, -, \times)$ is its field of constants, and $t \in K$ is a fixed element with $\delta t = 1$ that we will use as a parameter. Consider the t-definable set,

$$S := \left\{ x \in K : \frac{\delta x}{x} = \frac{1}{(t+c)^2} \text{ for some } c \in \mathbb{U} \right\}$$

and the t-definable map $\pi : S \to C$ given by $\pi(x) = \delta\left(\frac{x}{2\delta x}\right) - t$. Then the fibre above $c \in C$ is $S_c = \left\{ x : \frac{\delta x}{x} = \frac{1}{(t+c)^2} \right\}$, which is a multiplicative translate of the multiplicative group of constants, and hence is internal to $\mathbb{U}$. But, as Jin has worked out, any finite set of elements of S coming from different fibres will be acl-independent over $C(t)$. This can be used to show that the t-definable structure induced on (C, S) is a 1-analysable cover over C with independent fibres.

The question remains as to what definable object intrinsic to $\mathbb{U}$ captures the data of a 1-analysable cover $\mathbb{U}' = (\mathbb{U}, S, \dots)$ over A whose fibres may not be independent. Probably one should make the additional assumption that $(\mathbb{U}, S_a)$ is stably embedded in $\mathbb{U}'$ for each $a \in A$. Hrushovski [5] suggests "definable simplicial groupoids". It would be of interest to realise this suggestion.

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