Not all F-sparse sets are F-sets

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We show that the converse to [1, Theorem 7.4] fails.

Consider the additive group $\mathbb{Z}^2$, and consider $F\colon \mathbb{Z}^2 \to \mathbb{Z}^2$ given by $\begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} 5x \\ 5y \end{pmatrix}$. Let

$$\Sigma = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : x, y \in \{ -4, \dots, 4 \} \right\}$$

Remark 1. If Σ' is a spanning set for (Γ', F') then $\Sigma' \times \Sigma'$ is a spanning set for $((\Gamma')^2, F' \times F')$.

Since $\{ -4, \dots, 4 \}$ is a spanning set for $(\mathbb{Z}, F')$ with $F'(x) = 5x$, it then follows that Σ is a spanning set for $(\mathbb{Z}^2, F)$. Let $A \subseteq \mathbb{Z}^2$ be $\left\{ \begin{pmatrix} 5^n \\ 5^m \end{pmatrix} : n < m < \omega \right\}$. Then A is F-sparse: it is $[L]_F$ where

$$L = \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}^* \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix}^* \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

which is sparse by [1, Proposition 7.1].

Note that $\mathrm{Th}(\mathbb{Z}^2, 0, +, A)$ isn't stable: if we let $a_i = \begin{pmatrix} 5^i \\ 0 \end{pmatrix}$ and $b_j = \begin{pmatrix} 0 \\ 5^j \end{pmatrix}$ then $a_i + b_j \in A$ if and only if $i < j$, so $A(x + y)$ has the order property. So by [2, Theorem A] we should expect that A not have finite symmetric difference from an F-set (since expanding by the latter yields a stable theory); we prove this directly.

Suppose for a contradiction that $B \subseteq \mathbb{Z}^2$ were an F-set with $A \triangle B$ finite. Suppose

$$B \supseteq \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} + \sum_{i=1}^{n} C\left(\begin{pmatrix} x_i \\ y_i \end{pmatrix} ; \delta_i \right) + \Delta$$

where $x_i, y_i \in \mathbb{Z}$ and $\Delta \leq \mathbb{Z}^2$ and $\begin{pmatrix} x_i \\ y_i \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ when $i > 0$.

Claim 1. $\Delta = 0$.

Proof. Otherwise there is $\begin{pmatrix} x \\ y \end{pmatrix} \in \Delta$ such that $x < 0$ or $y < 0$. But then

$$\sum_{i=0}^{n} \begin{pmatrix} x_i \\ y_i \end{pmatrix} + k \begin{pmatrix} x \\ y \end{pmatrix} \in B$$

for all $k > 0$; so B contains infinitely many pairs with a negative coordinate, and thus so does A, a contradiction. □ Claim 1

Claim 2. $y_i \neq 0$ *for all* $i > 0$.

Proof. Suppose some $y_i = 0$ with $i > 0$. Then

$$\frac{5^{\delta_i (k+1)} - 1}{5^{\delta_i} - 1} \begin{pmatrix} x_i \\ 0 \end{pmatrix} + \sum_{j \neq i} \begin{pmatrix} x_j \\ y_j \end{pmatrix} = \begin{pmatrix} x_i \\ y_i \end{pmatrix} + 5^{\delta_i} \begin{pmatrix} x_i \\ y_i \end{pmatrix} + \cdots + 5^{\delta_i k} \begin{pmatrix} x_i \\ y_i \end{pmatrix} + \sum_{j \neq i} \begin{pmatrix} x_j \\ y_j \end{pmatrix} \in B$$

for all $k < \omega$. So since $x_i \neq 0$ (by hypothesis) we get that B contains a family of pairs with unbounded first component and bounded second component, and thus so does A, contradicting the definition of A. □ Claim 2

Claim 3. $n < 2$.

Proof. Suppose otherwise. Since $y_1 \neq 0$ (by previous claim) and

$$\frac{(5^{\delta_1(\ell+1)} - 1)}{5^{\delta_1} - 1} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \sum_{j \neq 1} \begin{pmatrix} x_j \\ y_j \end{pmatrix} \in B$$

for all $\ell < \omega$ we get that the set of

$$a_\ell := \frac{5^{\delta_1(\ell+1)} - 1}{5^{\delta_1} - 1} y_1 + \sum_{j \neq 1} y_j$$

is an infinite set cofinitely many of whose elements are powers of 5. So for some cofinite $S \subseteq \omega$ we get for $\ell \in S$ that a_ℓ is a power of 5 strictly larger than $\max\left(\frac{5^{\delta_2+1}}{4} y_2, 5\right)$. But we also have

$$\frac{(5^{\delta_1(\ell+1)} - 1)}{5^{\delta_1} - 1} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} + 5^{\delta_2} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} + \sum_{j \neq 1,2} \begin{pmatrix} x_j \\ y_j \end{pmatrix} \in B$$

for all ℓ; so $a_\ell + 5^{\delta_2} y_2$ is a power of 5 for ℓ in some cofinite $T \subseteq \omega$. But then if $\ell \in S \cap T$ then a_ℓ and $a_\ell + 5^{\delta_2} y_2$ are both powers of 5. But the closest power of 5 to a_ℓ is $\frac{1}{5} a_\ell$, and $y_2 \neq 0$; so $|5^{\delta_2} y_2| \geq \frac{4}{5} a_\ell$ and $a_\ell \leq \frac{5^{\delta_2+1}}{4} y_2$, contradicting our choice of S. □ Claim 3

Hence B is a finite union of sets of the form

$$\left\{ \begin{pmatrix} u \\ v \end{pmatrix} + \frac{5^{\delta \ell+1} - 1}{5^\delta - 1} \begin{pmatrix} x \\ y \end{pmatrix} : \ell < \omega \right\}$$

But given a set C of this form, if $x \neq 0$ then there is a bound on $\frac{z}{w}$ for $\begin{pmatrix} w \\ z \end{pmatrix} \in C$ with $w \neq 0$; and if $x = 0$ then whenever $\begin{pmatrix} w \\ z \end{pmatrix} \in C$ we have $w = u$. So there is finite $S \subseteq \omega$ such that there is a bound on $\frac{z}{w}$ for $\begin{pmatrix} w \\ z \end{pmatrix} \in B$ with $w \notin S$; hence the same holds true of A. But this contradicts the definition of A; so no such B exists. □

One final remark: there are even subsets of $(\mathbb{Z}, F')$ where $F'(x) = 5x$ that are F'-sparse but don't have finite symmetric difference from an F'-set. Indeed, let $\Sigma' = \{-4, \ldots, 4\}$ be the standard spanning set for $(\mathbb{Z}, F')$ and consider $L' = \left\{ a_0 b_0 a_1 b_1 \cdots a_{n-1} b_{n-1} : \begin{pmatrix} a_0 \\ b_0 \end{pmatrix} \cdots \begin{pmatrix} a_{n-1} \\ b_{n-1} \end{pmatrix} \in L \right\} \subseteq (\Sigma')^*$. Roughly speaking, we are encoding $\mathbb{Z}^2$ in $\mathbb{Z}$ by interleaving the digits, and simply looking at our original L. Then $A' = [L']_{F'} = [(00)^*(10)(00)^*(01)]_{F'}$ is F'-sparse. But $A'(x+y)$ again isn't stable: if $a_i = 5^{2i+1}$ and $b_j = 5^{2j}$, then $a_i + b_j \in A'$ if and only if $i < j$. So again by [2, Theorem A] we get that A' can't have finite symmetric difference from any F'-set.

References

[1] Jason Bell and Rahim Moosa. "F-sets and finite automata". In: *Journal de théorie des nombres de Bordeaux* 31.1 (2019), pp. 101–130 (cit. on p. 1).

[2] Rahim Moosa and Thomas Scanlon. "F-structures and integral points on semiabelian varieties over finite fields". In: *American Journal of Mathematics* 126.3 (2004), pp. 473–522 (cit. on pp. 1, 2).