MODEL THEORY OF FIELDS WITH FREE OPERATORS IN CHARACTERISTIC ZERO

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Abstract. Generalising and unifying the known theorems for difference and differential fields, it is shown that for every finite free algebra scheme $\mathcal{D}$ over a field A of characteristic zero, the theory of $\mathcal{D}$-fields has a model companion $\mathcal{D}$-CF_0 which is simple and satisfies the Zilber dichotomy for finite-dimensional minimal types.

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1. Introduction

The theories of differential and difference fields instantiate some of the most sophisticated ideas and theorems in model theoretic stability theory. For example, the theory of differentially closed fields of characteristic zero, DCF_0, is an ω-stable theory for which the full panoply of geometric stability theory applies from the existence and uniqueness of prime models (and, hence, of differential closures) to the theory of liaison groups (and, thus, a very general differential Galois theory) to the Zilber trichotomy for strongly minimal sets (from which strong theorems about function field arithmetic have been deduced). Likewise, the model companion of the theory of difference fields, ACFA, is supersimple and admits an analogous theory of internal automorphism groups and satisfies a version of the Zilber dichotomy for its minimal types. Beyond the formal analogies and parallel theorems, the proofs of the basic results in the model theory of differential and difference fields follow similar though not identical lines. In this paper we formalise the sense in which these theories are specialisations of a common theory of fields with operators and how the theories may be developed in one fell swoop. On the other hand, features which emerge from the general theory explain how the theories of differential and difference fields diverge.

By definition a derivation on a commutative ring R is an additive map $\partial : R \to R$ which satisfies the Leibniz rule $\partial(xy) = x\partial(y) + y\partial(x)$. Equivalently, the function $e : R \to R[\epsilon]/(\epsilon^2)$ given by $x \mapsto x + \partial(x)\epsilon$ is a homomorphism of rings. An endomorphism $\sigma : R \to R$ of a ring is simply a ring homomorphism from the ring R back to itself, but at the risk of complicating the definition, we may also say that a function $\sigma : R \to R$ is an endomorphism if the function $e : R \to R \times R$ given by $x \mapsto (x, \sigma(x))$ is a homomorphism of rings. With each of the latter presentations we see differential (respectively, difference) ring as a $\mathcal{D}$-ring in the sense introduced in [13].

As the details of the $\mathcal{D}$-ring formalism along with many examples are presented in Chapter 3, we limit ourselves to a loose discussion here. For each fixed ring scheme $\mathcal{D}$ (possibly over some base ring A) satisfying some additional requirements we have a theory of $\mathcal{D}$-fields. In particular, we require that the underlying additive group scheme of $\mathcal{D}$ be some power of the additive group scheme so that for any A-algebra R, $\mathcal{D}(R) = (R^n, +, \boxtimes)$ where the multiplication $\boxtimes$ is given by some bilinear form defined over A. We require that $\mathcal{D}$ comes equipped with a functorial projection map to the standard ring scheme and that for the sake of concreteness, read relative to coordinates this projection map be given by projection onto the first coordinate. A $\mathcal{D}$-ring is then a pair (R, e) consisting of an A-algebra R and a map of A-algebras $e : R \to \mathcal{D}(R)$ which is a section of the projection. In the motivating examples, $\mathcal{D}(R) = R[\epsilon]/(\epsilon^2) = (R^2, +, \boxtimes)$ where $(x_1, x_2) \boxtimes (y_1, y_2) = (x_1y_1, x_1y_2 + x_2y_1)$ gives rise to differential rings and $\mathcal{D}(R) = R \times R$ with coordinatewise ring operations gives rise to difference rings.

In general, using the coordinatization of $\mathcal{D}(R)$, the data of a $\mathcal{D}$-ring (R, e) is equivalent to that of a ring R given together with a sequence $\partial_0, \dots, \partial_{n-1}$ of operators $\partial_i : R \to R$ for which the map $e : R \to \mathcal{D}(R)$ is given in coordinates by $x \mapsto (\partial_0(x), \dots, \partial_{n-1}(x))$. The requirements on such a sequence of operators that they define a $\mathcal{D}$-ring structure may be expressed by certain universal axioms. For example, to say that the map $e : R \to \mathcal{D}(R)$ is a section of the projection is just to say that $(\forall x \in R)\partial_0(x) = x$ and the requirement that e convert multiplication in R to the multiplication of $\mathcal{D}(R)$ may be expressed by insisting that certain polynomial relations hold amongst $\partial_0(x), \ldots, \partial_{n-1}(x); \partial_0(y), \ldots, \partial_{n-1}(y); \partial_0(xy), \ldots, \partial_{n-1}(xy)$. In this way, the class of $\mathcal{D}$-fields is easily seen to be first order in the language of rings augmented by unary function symbols for the operators $\partial_0, \ldots, \partial_{n-1}$. On the other hand, the interpretation of the operators as components of a ring homomorphism permits us to apply ideas from commutative algebra and algebraic geometry to analyze these theories.

Our first main theorem is that for any ring scheme $\mathcal{D}$ (meeting the requirements set out in Chapter 3), the theory of $\mathcal{D}$-fields of characteristic zero has a model companion, which we denote by $\mathcal{D}\text{-CF}_0$ and call the theory of $\mathcal{D}$-closed fields. Our axiomatization of $\mathcal{D}\text{-CF}_0$ follows the geometric style which first appeared in the Chatzidakis-Hrushovski axioms for ACFA [3] and was then extended to differential fields by Pierce and Pillay [15]. Moreover, the proofs will be familiar to anyone who has worked through the corresponding results for difference and differential fields. Following the known proofs for difference and differential fields, we establish a quantifier simplification theorem and show that $\mathcal{D}\text{-CF}_0$ is always simple.

As noted above, the theory DCF_0 is the quintessential ω-stable theory, but ACFA, the model companion of the theory of difference fields is not even stable. At a technical level, the instability of ACFA may be traced to the failure of quantifier elimination which, algebraically, is due to the non-uniqueness (up to isomorphism) of the extension of an automorphism of a field to an automorphism of its algebraic closure. We show that this phenomenon, namely that instability is tied to the nonuniqueness of extensions of automorphisms, pervades the theory of $\mathcal{D}$-fields. That is, for each $\mathcal{D}$ there is a finite list of associated endomorphisms expressible as linear combinations of the basic operators. Since we require that ∂_0 is the identity map, one of these associated endomorphisms is always the identity map. If there are any others, then the theory of $\mathcal{D}\text{-CF}_0$ suffers from instability and the failure of quantifier elimination just as does ACFA. On the other hand, if there are no other associated endomorphisms, then $\mathcal{D}\text{-CF}_0$ is stable.

The deepest of the fine structural theorems for types in DCF_0 and in ACFA is the Zilber dichotomy for minimal types, first established by Sokolović and Hrushovski for DCF_0 using Zariski geometries [9], for ACFA_0 by Chatzidakis and Hrushovski through a study of ramification [3], and by Chatzidakis, Hrushovski and Peterzil for ACFA in all characteristics using the theory of limit types and a refined form of the theory of Zariski geometries [4]. Subsequently, Pillay and Ziegler established a stronger form of the trichotomy theorem in characteristic zero [16] by adapting jet space arguments Campana and Fujiki used to study complex manifolds [2, 6]. Here we implement the Pillay-Ziegler strategy for $\mathcal{D}\text{-CF}_0$ by using the theory of $\mathcal{D}$-jet spaces from [14]. In particular, we show that finite dimensional types in $\mathcal{D}\text{-CF}_0$ satisfy the canonical base property.

The model companion of the theory of difference fields of characteristic zero with n automorphisms appears as $\mathcal{D}\text{-CF}_0$ where $\mathcal{D}(R) := R^{1+n}$ in contradistinction to the well-known fact that the theory of difference fields with n (> 1) commuting automorphisms does not have a model companion. On the other hand, if $(\mathbb{U}, \partial_0, \partial_1, \ldots, \partial_n) \models \mathcal{D}\text{-CF}_0$ is sufficiently saturated, then the type definable field obtained as the intersection of the fixed fields of all the elements of the commutator group of the group generated by $\partial_1, \ldots, \partial_n$ has Lascar rank ω^n and may be regarded as a universal domain for difference fields with n commuting automorphisms. (See Section 1.2 of [8] for a discussion of these issues.) Likewise, models of the theory $\mathrm{DCF}_{0,n}$ of differentially closed fields with n commuting derivations may be realized as type definable fields in models of $\mathcal{D}$-CF_0 where $\mathcal{D}(R) = R[\epsilon_1, \ldots, \epsilon_n]/(\epsilon_1, \ldots, \epsilon_n)^2$.

While omitting commutation allows for model companions in characteristic zero, it complicates matters in positive characteristic. Under a natural algebraic hypothesis on p and $\mathcal{D}$, namely that there be some $\epsilon \in \mathcal{D}(A)$ which is nilpotent but for which $\epsilon^p \neq 0$, we observe with Proposition 7.2 that no model companion of the theory of $\mathcal{D}$-fields of characteristic p exists. This proposition is consonant with the known examples of ACFA_p and DCF_p where no such ϵ exists. However, it implies that the theory of (not necessarily iterative) Hasse-Schmidt differential fields of positive characteristic does not have a model companion, which is at odds with the iterative theory $\mathrm{SCH}_{p,e}$ considered by Ziegler [18]. While the theory of iterative $\mathcal{D}$-fields developed in [14] was intended as an abstraction of the theory of iterative Hasse-Schmidt differential fields, we have not yet understood the extent to which the theorems around $\mathrm{SCH}_{p,e}$ generalise to iterative $\mathcal{D}$-fields. As such, we leave open the problems of which theories of iterative $\mathcal{D}$-fields and which theories of positive characteristic $\mathcal{D}$-fields have model companions.

This paper is organized as follows. We begin in Chapter 2 with some remarks about our conventions. With Chapter 3 we recall the formalism of $\mathcal{D}$-rings in detail and present several examples. In Chapter 4 we give axioms for the theory $\mathcal{D}$-CF_0 and prove that it is in fact the model companion of the theory of $\mathcal{D}$-fields of characteristic zero. In Chapter 5 we establish the essential model theoretic properties of $\mathcal{D}$-closed fields. In Chapter 6 we give a proof of the Zilber dichotomy for minimal types of finite dimension. We conclude with an appendix in which we show that the theory of $\mathcal{D}$-fields does not have a model companion for most choices of $\mathcal{D}$ in positive characteristic, and also explain how a convenient set of assumptions made early in the paper can be removed.

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2. Notation and conventions

All rings are commutative and unitary. As a general rule, we follow standard conventions in model theory and differential algebra and introduce unfamiliar notation as needed. We do move between scheme theory and Weil-style algebraic geometry. For the most part, the theory of prolongation and jet spaces must be developed scheme theoretically as we make essential use of nonreduced bases. However, in the applications to the first order theories of fields, as is common in model theoretic algebra, we sometimes use Weil-style language. Let us note some of these conventions. If X is some variety over a field K, L is an extension field of K, and $a \in X(L)$ is an L-rational point, then $\mathrm{loc}(a/K)$, the locus of a over K, is the intersection of all closed K-subvarieties $Y \subseteq X$ with $a \in Y(L)$. If X is affine with coordinate ring $\mathcal{O}_X$, then we define $I(a/K) := \{f \in \mathcal{O}_X \ : \ f(a) = 0\}$ to be the ideal of a over K. Generalizing somewhat, for $Y \subseteq X_L$ a subvariety of the base change of X to L, we define $I(Y/K) := \{f \in \mathcal{O}_X \ : \ f \text{ vanishes on } Y\}$. Note that the locus of a over K is the variety defined by the ideal of a over K. We say that

a is a generic point of X over K (or is K-generic in X) if $\mathrm{loc}(a/K) = X$. Scheme theoretically, one would say that $I(a/K)$ is the generic point of X, but these two points of view will not appear in the same section.

We sometimes identify a scheme X over a ring A with the functor from A-algebras to sets which takes an A-algebra R to $X(R)$, the set of R-points on X. If $f : X \to Y$ is a morphism of schemes over A then we will use $f^R : X(R) \to Y(R)$ to denote the corresponding map on sets of points.

3. $\mathcal{D}$-RINGS

Throughout this paper we will fix the following data, sometimes making further assumptions about them:

A. a base ring A
B. a finite free A-algebra $\mathcal{D}(A)$; that is, $\mathcal{D}(A)$ is an A-algebra which as an A-module is free of finite rank,
C. an A-algebra homomorphism $\pi^A : \mathcal{D}(A) \to A$, and
D. an A-basis $(\epsilon_0, \dots, \epsilon_{\ell-1})$ for $\mathcal{D}(A)$ such that $\pi^A(\epsilon_0) = 1$ and $\pi^A(\epsilon_i) = 0$ for all $i = 1, \dots, \ell - 1$.

An equivalent scheme-theoretic way to describe these data is as a *finite free* $\mathbb{S}$-*algebra scheme over* A *with basis* in the sense of [13] and [14]. Here $\mathbb{S}$ denotes the ring scheme which when evaluated at any A-algebra R is just the ring R itself. That is, $\mathbb{S}$ is simply the affine line $\mathrm{Spec}\,\big(A[x]\big)$ endowed with the usual ring scheme structure. Instead of B, C, D as above we could consider the basic data as being

B′. an $\mathbb{S}$-algebra scheme $\mathcal{D}$ over A; that is, a ring scheme $\mathcal{D}$ over A together with a ring scheme morphism $s : \mathbb{S} \to \mathcal{D}$ over A,
C′. a morphism of $\mathbb{S}$-algebra schemes $\pi : \mathcal{D} \to \mathbb{S}$ over A, and
D′. an $\mathbb{S}$-linear isomorphism $\psi : \mathcal{D} \to \mathbb{S}^\ell$ over A such that π is ψ composed with the first co-ordinate projection on $\mathbb{S}^\ell$.

Indeed, as is explained on page 14 of [14], we obtain the second presentation from the first as follows: given any A-algebra R define $\mathcal{D}(R) = R \otimes_A \mathcal{D}(A)$, $\pi^R = \mathrm{id}_R \otimes \pi^A$, and $\psi^R = \mathrm{id}_R \otimes \psi^A$ where $\psi^A : \mathcal{D}(A) \to A^\ell$ is the A-linear isomorphism induced by the choice of basis $(\epsilon_0, \dots, \epsilon_{\ell-1})$. To go in the other direction is clear, one just evaluates all the scheme-theoretic data on the ring A. A key point is that given $(\mathcal{D}, \pi, \psi)$, for any A-algebra R there is a canonical isomorphism induced by ψ between $\mathcal{D}(R)$ and $R \otimes_A \mathcal{D}(A)$. While the first presentation of the data is more immediately accessible, it is the second scheme-theoretic one that is more efficient and that we will use.

Remark 3.1. The assumption in D′ that π is ψ composed with the first co-ordinate projection on $\mathbb{S}^\ell$ is new in that it was not made in Definition 2.2 of [14]. However, it can always be made to hold through a change of basis.

The multiplicative structure on $\mathcal{D}(A)$, and hence on $\mathcal{D}(R)$ for any A-algebra R, can be described in terms of the basis by writing

$$\epsilon_i\epsilon_j = \sum_{k=0}^{\ell-1} a_{i,j,k}\epsilon_k \tag{1}$$

$$1_{\mathcal{D}(A)} = \sum_{k=0}^{\ell-1} c_k\epsilon_k \tag{2}$$

where the $a_{i,j,k}$'s and c_k's are elements of A. Note that $a_{0,0,0} = 1$ and $c_0 = 1$.

Definition 3.2 ($\mathcal{D}$-rings)**.** By a *$\mathcal{D}$-ring* we will mean an A-algebra R together with a sequence of operators $\partial := (\partial_1, \dots, \partial_{\ell-1})$ on R such that the map $e : R \to \mathcal{D}(R)$ given by

$$e(r) := r\epsilon_0 + \partial_1(r)\epsilon_1 + \cdots + \partial_{\ell-1}(r)\epsilon_{\ell-1}$$

is an A-algebra homomorphism.

Via the above identity we can move back and forth between thinking of a $\mathcal{D}$-ring as (R, ∂) or as (R, e). It should be remarked that this is not exactly consistent with [13]. In that paper, a "$\mathcal{D}$-ring" was defined to be simply an A-algebra R together with an A-algebra homomorphism $e : R \to \mathcal{D}(R)$. Hence, under the correspondence $(R, \partial) \mapsto (R, e)$, the $\mathcal{D}$-rings of the current paper are precisely the "$\mathcal{D}$-rings" of [13] with the additional assumption that e is a section to $\pi^R : \mathcal{D}(R) \to R$. (Note that this latter assumption already appears in Definition 2.4 of [14].)

The class of $\mathcal{D}$-rings is axiomatisable in the language

$$\mathcal{L}_{\mathcal{D}} := \{0, 1, +, -, \times, (\lambda_a)_{a\in A}, \partial_1, \dots, \partial_{\ell-1}\}$$

where λ_a is scalar multiplication by $a \in A$. Indeed, the class of A-algebras is cleary axiomatisable, and the A-linearity of $e : R \to \mathcal{D}(R)$, which is equivalent to $\partial_1, \dots, \partial_{\ell-1}$ being A-linear operators on R, is also axiomatisable. Finally, that e is in addition a ring homomorphism corresponds to the satisfaction of certain A-linear functional equations on the operators. Indeed, using (1) above, we see that the multiplicativity of e is equivalent to

$$\partial_k(xy) = \sum_{i=0}^{\ell-1}\sum_{j=0}^{\ell-1} a_{i,j,k}\partial_i(x)\partial_j(y) \quad \text{for all } x, y, \text{ and} \tag{3}$$

$$\partial_k(1_R) = c_k \tag{4}$$

for all $k = 1, \dots, \ell - 1$.

Example 3.3 (Prime $\mathcal{D}$-ring)**.** For any $(A, \mathcal{D}, \pi, \psi)$ there is a unique $\mathcal{D}$-ring structure on A, namely where the $\partial_i : A \to A$ are A-linear and satisfy

$$1 = \epsilon_0 + \sum_{i=1}^{\ell-1} \partial_i(1)\epsilon_i$$

This corresponds to the case when e is just $s^A : A \to \mathcal{D}(A)$, the given A-algebra structure on $\mathcal{D}(A)$.

Example 3.4 (Fibred products)**.** We can always combine examples. Given $(\mathcal{D}, \pi, \psi)$ and $(\mathcal{D}', \pi', \psi')$ we can consider the fibred product of $\pi : \mathcal{D} \to \mathbb{S}$ and $\pi' : \mathcal{D}' \to \mathbb{S}$ over $\mathbb{S}$, which we denote by $\mathcal{D} \times_{\mathbb{S}} \mathcal{D}'$. We then have the natural morphisms $\pi \times \pi' : \mathcal{D} \times_{\mathbb{S}} \mathcal{D}' \to \mathbb{S} \times_{\mathbb{S}} \mathbb{S} = \mathbb{S}$, and $\psi \times \psi' : \mathcal{D} \times_{\mathbb{S}} \mathcal{D}' \to \mathbb{S}^\ell \times_{\mathbb{S}} \mathbb{S}^{\ell'} = \mathbb{S}^{\ell+\ell'-1}$.

The $\mathcal{D} \times_{\mathbb{S}} \mathcal{D}'$-rings will be precisely those of the form (R, ∂, ∂') where (R, ∂) is a $\mathcal{D}$-ring and (R, ∂') is a $\mathcal{D}'$-ring. Note that the theory of $\mathcal{D} \times_{\mathbb{S}} \mathcal{D}'$-rings does not ask ∂ and ∂' to commute.

Example 3.5 (Tensor products)**.** Here is another way to combine examples. Given $(\mathcal{D}, \pi, \psi)$ and $(\mathcal{D}', \pi', \psi')$ we can consider the tensor product of algebra schemes $s : \mathbb{S} \to \mathcal{D}$ and $s' : \mathbb{S} \to \mathcal{D}'$ over $\mathbb{S}$, $\mathcal{D} \otimes_{\mathbb{S}} \mathcal{D}'$, equipped with $\pi \otimes \pi' : \mathcal{D} \otimes_{\mathbb{S}} \mathcal{D}' \to \mathbb{S}$, and $\psi \otimes \psi' : \mathcal{D} \otimes_{\mathbb{S}} \mathcal{D}' \to \mathbb{S}^{\ell} \otimes_{\mathbb{S}} \mathbb{S}^{\ell'} = \mathbb{S}^{\ell\ell'}$. If R has both a $\mathcal{D}$-ring structure (R, ∂) and a $\mathcal{D}'$-ring structure (R, ∂'), then it has the natural $(\mathcal{D} \otimes_{\mathbb{S}} \mathcal{D}')$-structure (R, D) where $D_{i+j\ell} = \partial'_j \circ \partial_i$. This comes from the fact that

$$\begin{aligned}
(\mathcal{D} \otimes_{\mathbb{S}} \mathcal{D}')(R) &= R \otimes_A \big(\mathcal{D}(A) \otimes_A \mathcal{D}'(A)\big) \\
&= \big(R \otimes_A \mathcal{D}(A)\big) \otimes_A \mathcal{D}'(A) \\
&= \mathcal{D}(R) \otimes_A \mathcal{D}'(A) \\
&= \mathcal{D}'\big(\mathcal{D}(R)\big)
\end{aligned}$$

But not every $(\mathcal{D} \otimes_{\mathbb{S}} \mathcal{D}')$-structure on R is of this form. For example, one also has $(R, \widetilde{D})$ with $\widetilde{D}_{i+j\ell} = \partial_i \circ \partial'_j$ that arises from regarding $(\mathcal{D} \otimes_{\mathbb{S}} \mathcal{D}')$ as $\mathcal{D} \circ \mathcal{D}'$ instead. The identities imposed by being a $(\mathcal{D} \otimes_{\mathbb{S}} \mathcal{D}')$-ring are just the generalised Leibniz rules satisfied by compositions of the components of $\mathcal{D}$-ring and $\mathcal{D}'$-ring structures.

Example 3.6. We list here some of the main motivating examples. Characteristic 0 or p specialisations of these examples are obtained by letting A be $\mathbb{Q}$ or $\mathbb{F}_p$, respectively. In what follows R ranges over all A-algebras.

(a) *Differential rings.* Let $\mathcal{D}(R) = R[\eta]/(\eta^2)$ with the natural R-algebra structure, $\pi^R : R[\eta]/(\eta^2) \to R$ be the quotient map, and $(1, \eta)$ the R-basis. Then a $\mathcal{D}$-ring is precisely an A-algebra equipped with a derivation over A.

(b) *Truncated higher derivations.* Generalising the above example, let

$$\mathcal{D}(R) = R[\eta]/(\eta^{n+1})$$

with the natural R-algebra structure, $\pi^R : R[\eta]/(\eta^{n+1}) \to R$ the quotient map, and take as an R-basis $(1, \eta, \ldots, \eta^n)$. Then a $\mathcal{D}$-ring is precisely an A-algebra equipped with a *higher derivation of length* n *over* A in the sense of [11]; that is, a sequence of A-linear maps $(\partial_0 = \mathrm{id}, \partial_1, \ldots, \partial_m)$ such that $\partial_i(xy) = \sum_{r+s=i} \partial_r(x)\partial_s(y)$.

It is worth pointing out here that even in characteristic zero (so when $A = \mathbb{Q}$ for example) this is a proper generalisation of differential rings. It is true that ∂_1 is a derivation, and *if* we had imposed the usual iterativity condition then we would have $\partial_i = \dfrac{\partial_1^i}{i!}$. But the point is that being a $\mathcal{D}$-ring does not impose iterativity, the operators are in this sense "free".

(c) *Difference rings.* Let $\mathcal{D}(R) = R^2$ with the product R-algebra structure, π^R the projection onto the first co-ordinate, and (ϵ_0, ϵ_1) the standard basis. Then a $\mathcal{D}$-ring is precisely an A-algebra equipped with an A-endomorphism.

(d) *Partial higher differential-difference rings.* Taking fibred products as in Example 3.4, we can combine the above examples. That is, suppose we are given positive integers $m_1, n_1, \ldots, n_{m_1}$, and m_2. For an appropriate choice of $(\mathcal{D}, \pi, \psi)$ the $\mathcal{D}$-rings will be precisely the A-algebras equipped with m_1 higher A-derivations (of length $n_1, \ldots, n_{m_1}$ respectively) and m_2

A-endomorphisms. Note that being a $\mathcal{D}$-ring will not impose that the various operations commute.

(e) *D-rings.* Fix $c \in A$ and let $\mathcal{D}(R) := R^2$ as an R-module and define multiplication by

$$(x_1, y_1) \cdot (x_2, y_2) := (x_1x_2, x_1y_2 + y_1x_2 + y_1y_2c).$$

A $\mathcal{D}$-ring is then an A-algebra R equipped with an A-linear map $D : R \to R$ satisfying the twisted Leibniz rule $D(xy) = xD(y)+D(x)y+D(x)D(y)c$. If we define $\sigma : R \to R$ by $\sigma(x) := x + D(x)c$, then σ is a ring endomorphism of R. If $c = 0$ then D is a derivation on R, if c is invertible in A then D may be computed from σ by the rule $D(x) = c^{-1}(\sigma(x) - x)$. Such structures were considered by the second author in [17].

Example 3.7. In order to exhibit the variety of operators that can be put into this formalism, let us describe two *ad hoc* examples.

(a) *Derivations of an endomorphism.* Consider $\mathcal{D}(R) = R \times R[\eta]/(\eta^2)$ with basis $\{(1, 0),(0, 1),(0, \eta)\}$ and π the projection onto the first co-ordinate. A $\mathcal{D}$-ring is then an A-algebra equipped with an A-endomorphism σ and an A-linear map δ satisfying the σ-twisted Leibniz rule

$$\delta(xy) = \sigma(x)\delta(y) + \delta(x)\sigma(y)$$

(b) Suppose $\mathcal{D}(R) = R[\eta_1, \eta_2]/(\eta_1^2, \eta_2^2)$, with basis $\{1, \eta_1, \eta_2, \eta_1\eta_2\}$ and π the quotient modulo (η_1, η_2) map. A $\mathcal{D}$-ring is an A-algebra R equipped with three operators, $\partial_1, \partial_2, \partial_3$, such that ∂_1 and ∂_2 are derivations and ∂_3 is an A-linear map satisfying

$$\partial_3(xy) = x\partial_3(y) + y\partial_3(x) + \partial_1(x)\partial_2(y) + \partial_2(x)\partial_1(y)$$

For example, if ∂_1, ∂_2 are *arbitrary* derivations on R and $\partial_3 := \partial_1 \circ \partial_2$ then $(R, \partial_1, \partial_2, \partial_3)$ is a $\mathcal{D}$-ring, though not every $\mathcal{D}$-ring is of this form. This example is obtained by the tensor product construction of Example 3.5.

This formalism of $\mathcal{D}$-rings is rather general. We leave to future work the systematical classification of the operators on A-algebras which it covers.

Remark 3.8. It may at this point be worth explaining in what sense we consider these operators to be "free". The theory of $\mathcal{D}$-rings does not impose any non-trivial functional equations on the operators $\partial_1, \dots, \partial_{\ell-1}$. Of course in a particular $\mathcal{D}$-ring some such identity may happen to hold – the ∂_i's may commute, for example, or ∂_3 may be the composition of ∂_1 and ∂_2 – but for no $\mathcal{D}$ can this be a consequence of the theory of $\mathcal{D}$-rings. One way to verify this claim is to note that the *$\mathcal{D}$-polynomial ring* $A\{x\}$, namely the polynomial ring over A in variables ξx where ξ ranges over the set of all finite words on $\{\partial_1, \dots, \partial_{\ell-1}\}$, is naturally equipped with the $\mathcal{D}$-structure $e(\xi x) = (\xi x)\epsilon_0 + (\partial_1\xi x)\epsilon_1 \cdots + (\partial_{\ell-1}\xi x)\epsilon_{\ell-1}$. Clearly, no non-trivial functional equations among the operators can hold on this $\mathcal{D}$-ring. In fact, one can use Lemma 4.9 below to show that the $\mathcal{D}$-structure on $A\{x\}$ extends to the fraction field, so that this claim is also true of the theory of $\mathcal{D}$-fields.

In the next chapter we will show that the theory of $\mathcal{D}$-fields in characteristic zero has a model companion. In order to describe the axioms of this model companion we will make use of the *abstract prolongations* introduced and discussed in §4 of [13]. Let us briefly recall them here.

Given a $\mathcal{D}$-ring (R,∂) and an algebraic scheme X over R, the prolongation[1] of X, denoted by $\tau(X,\mathcal{D},e)$ or just τX for short, is itself a scheme over R with the characteristic property that its R-points can be canonically identified with $X(\mathcal{D}(R))$ where X is regarded as a scheme over $\mathcal{D}(R)$ via the base change coming from $e : R \to \mathcal{D}(R)$. Via this identification, note that e induces a map $\nabla : X(R) \to \tau(X,\mathcal{D},e)(R)$.

In terms of equations, if $X \subset \mathbb{A}^n_R$ is the affine scheme $\operatorname{Spec}\big(R[x]/I\big)$ where $x = (x_1,\dots,x_n)$ is really an n-tuple of indeterminates then τX will be the closed subscheme of $\mathbb{A}^{n\ell}_R$ given by $\operatorname{Spec}\big(R[x^{(0)},x^{(1)},\dots,x^{(\ell-1)}]/I'\big)$ where I' is obtained as follows: For each $P(x) \in I$ let $P^e(x) \in \mathcal{D}(R)[x]$ be the polynomial obtained by applying e to the coefficients of P, and compute

$$P^e\Big(\sum_{j=0}^{\ell-1} x^{(j)}\epsilon_j\Big) \quad = \quad \sum_{j=0}^{\ell-1} P^{(j)}(x^{(0)},x^{(1)},\dots,x^{(\ell-1)})\epsilon_j$$

in the polynomial ring $\mathcal{D}(R)[x^{(0)},x^{(1)},\dots,x^{(\ell-1)}] = \displaystyle\bigoplus_{i=0}^{\ell-1} R[x^{(0)},x^{(1)},\dots,x^{(\ell-1)}]\cdot\epsilon_i$.

Then I' is the ideal of $R[x^{(0)},x^{(1)},\dots,x^{(\ell-1)}]$ generated by $P^{(0)},\dots,P^{(\ell-1)}$ as P ranges in I. Note that the $P^{(i)}$'s are computed using (1) above. With respect to these co-ordinates, the map $\nabla : X(R) \to \tau(X,\mathcal{D},e)(R)$ is given by $\nabla(a) = \big(a,\partial_1(a),\dots,\partial_{\ell-1}(a)\big)$.

4. Existentially closed $\mathcal{D}$-fields

We aim to show that the theory of $\mathcal{D}$-fields of characteristic zero admits a model companion when the base ring A enjoys some additional properties that are spelled out with the following assumptions. In fact, as is explained in the appendix, these assumptions are not necessary. Nevertheless, for the sake of significant ease of notation, and in order to better fix ideas, we will impose the following:

Assumptions 4.1. *The following assumptions will be in place throughout the rest of the paper, unless explicitly stated otherwise:*

(i) *The ring A is a field.*
(ii) *Writing $\mathcal{D}(A) = \prod_{i=0}^{t} B_i$, where the B_i are local finite A-algebras, the residue field of each B_i, which is necessarily a finite extension of A, is in fact A itself.*

Note that the B_i's are unique up to isomorphism and reordering of the indices. They can be obtained by running through the finitely many maximal ideals of $\mathcal{D}(A)$ and quotienting out by a sufficiently high power of them.

All of the motivating examples described in 3.6, specialised to the case when $A = \mathbb{Q}$ or $\mathbb{F}_p$, satisfy Assumptions 4.1. The following is an example where 4.1(ii) is not satisfied.

Example 4.2. Let $A = \mathbb{Q}$ and $\mathcal{D}(\mathbb{Q}) = \mathbb{Q} \times \mathbb{Q}[x]/(x^2-2)$, with standard basis $\{(1,0),(0,1),(0,x)\}$. Then, in the notation of 4.1(ii) we have $t = 1$, $B_0 = \mathbb{Q}$, $B_1 = Q[x]/(x^2-2)$, and the residue field of B_1 is B_1 itself. So 4.1(ii) is not satisfied. The $\mathcal{D}$-rings in this case are precisely the $\mathbb{Q}$-algebras R equipped with linear operators ∂_1,∂_2 such that

[1]The prolongation does not always exist; however it does exist for any quasi-projective scheme. See the discussion after Definition 4.1 of [13] for details.

- $\partial_1(ab) = \partial_1(a)\partial_1(b) + 2\partial_2(a)\partial_2(b)$
- $\partial_2(ab) = \partial_1(a)\partial_2(b) + \partial_2(a)\partial_1(b)$

Note that if $(K, \partial_1, \partial_2)$ is a $\mathcal{D}$-field with $\sqrt{2} \in K$, then ∂_1, ∂_2 are interdefinable with the pair of endomorphisms $\partial_1 + \sqrt{2}\partial_2$ and $\partial_1 - \sqrt{2}\partial_2$ of K. This gives a hint as to how we should handle the situation when 4.1(ii) fails, see §7.2 below.

4.1. **The associated endomorphisms.** Our axioms for the model companion must take into account certain definable endomorphisms that are induced by the $\mathcal{D}$-operators given the above decomposition of $\mathcal{D}(A)$ into local artinian A-algebras.

First some notation. Fixing A-bases for $B_0, \ldots, B_t$, we get

- finite free local $\mathbb{S}$-algebra schemes with bases, $(\mathcal{D}_i, \psi_i)$ for $i = 0, \ldots, t$, such that $\mathcal{D}_i(A) = B_i$,
- $\mathbb{S}$-algebra homomorphisms $\theta_i : \mathcal{D} \to \mathcal{D}_i$ corresponding to $\mathcal{D} = \prod_{i=0}^{t} \mathcal{D}_i$,
- $\mathbb{S}$-algebra homomorphisms $\rho_i : \mathcal{D}_i \to \mathbb{S}$ which when evaluated at A are the residue maps $B_i \to A$, and
- $\pi_i := \rho_i \circ \theta_i : \mathcal{D} \to \mathbb{S}$.

Note that one of the maximal ideals of $\mathcal{D}(A)$, say the one corresponding to B_0, is the kernel of our A-algebra homomorphism $\pi^A : \mathcal{D}(A) \to A$. In particular, $\pi_0 = \pi$.

Now suppose (R, ∂) is a $\mathcal{D}$-ring. For each $i = 0, \ldots, t$, we have the A-algebra endomorphism $\sigma_i := \pi_i^R \circ e : R \to R$. Since $\pi_0 = \pi$, $\sigma_0 = \mathrm{id}$. The others, $\sigma_1, \ldots, \sigma_t$, will be possibly non-trivial endomorphisms of R. As the π_i are $\mathbb{S}$-linear morphisms over A, the σ_i are A-linear combinations of the operators $\partial_1, \ldots, \partial_{\ell-1}$. In particular, these are 0-definable in (R, ∂). We call them the *associated endomorphisms* and $(R, \sigma_1, \ldots, \sigma_t)$ the *associated difference ring.*

Example 4.3. In the partial difference-differential case of Example 3.6(d), that is, of (R, ∂, σ) where ∂ is a tuple of m_1 (higher truncated) derivations on R and σ is a tuple of m_2 endomorphisms of R, the associated difference ring is, as expected, (R, σ). In the D-rings of Example 3.6(e), the associated endomorphism is the map $\sigma(x) := x + D(x)c$.

Definition 4.4. An *inversive* $\mathcal{D}$-ring is one for which the associated endomorphisms are surjective. That is, a $\mathcal{D}$-ring (R, ∂) is inversive just in case the associated difference ring (R, σ) is inversive.

As a consequence of Lemmas 4.9 and 4.11 below, we will see that if R is an integral domain of characteristic zero whose associated endomorphisms are injective, then (R, ∂) embeds into an inversive $\mathcal{D}$-field.

Suppose now that (K, ∂) is an inversive $\mathcal{D}$-field. The associated endomorphisms are thus automorphisms of K. Given $B \subseteq K$, the *inversive closure of B in K*, denoted by $\langle B \rangle$, is the smallest inversive $\mathcal{D}$-subfield of K containing B. That is, it is the intersection of all inversive $\mathcal{D}$-subfields containing B. Because σ and ∂ may not commute, it is not the case that $\langle B \rangle$ is simply the inversive difference field generated by the $\mathcal{D}$-subring generated by B. Rather, $\langle B \rangle$ is the fraction field of $\bigcup_{i<\omega} R_i$ where $R_{-1} = B$ and R_i is the $\mathcal{D}$-subring generated by $\{\sigma_j^{-1}(a) : a \in R_{i-1}, j = 1, \ldots, t\}$.

If k is an inversive $\mathcal{D}$-subfield of K and $a = (a_1, \ldots, a_n)$ is a tuple from K, then $k\langle a \rangle$ is used to abbreviate $\langle k \cup \{a_1, \ldots, a_n\} \rangle$. If we denote by Θa the (infinite) tuple whose co-ordinates are of the form θa_i as θ ranges over all finite words on the set $\{\partial_1, \ldots, \partial_{\ell-1}, \sigma_1^{-1}, \ldots, \sigma_t^{-1}\}$, then $k\langle a \rangle = k(\Theta a)$.

Remark 4.5. It is well known that if (L,σ) is a difference field with respect to a single endomorphism, then the isomorphism type of the inversive closure of (L,σ) does not depend on the ambient inversive difference field in which it is computed. However, in general, the inversive closure defined above does depend on the ambient inversive $\mathcal{D}$-field in which the closure is taken. This can already be seen in the case of difference fields with respect to more than one endomorphism.

One can find a finitely generated monoid M, generated by elements $a_1,\dots,a_n$, which admits two different embeddings $\rho : M \to H$ and $\pi : M \to G$ to groups H and G for which the image of M generates H and G but H and G are not isomorphic. For example, one may take M to be the free monoid on two generators, H to be the free group on two generators, and H a group of matrices generated by $\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} c & d \\ 0 & 1 \end{pmatrix}$ with a,b,c,d sufficiently general. Then for any field k the monoid ring $k[M]$ and group rings $k[H]$ and $k[G]$ are naturally difference rings with respect to n endomorphisms $\sigma_1,\dots,\sigma_n$ via $\sigma_i(g) := a_i g$ (where we identify a_i with $\rho(a_i)$ or $\pi(a_i)$ in H or G). It is easy to see that these difference ring structures extend to the fields of fractions and that with respect to the embeddings induced by ρ and π the difference fields $k(H)$ and $k(G)$ are non-isomorphic inversive closures of $k(M)$.

4.2. **The model companion.** Our model companion for $\mathcal{D}$-fields will include a "geometric" axiom in the spirit of Chatzidakis-Hrushovski [3] or Pierce-Pillay [15]. To state these we need some further notation. Suppose X is an algebraic scheme over a $\mathcal{D}$-ring R. For each $i=0,\dots,t$, since $\sigma_i = \pi_i^R \circ e$, the morphism $\pi_i : \mathcal{D} \to \mathbb{S}$ induces[2] a surjective morphism on the prolongations $\widehat{\pi}_i : \tau(X,\mathcal{D},e) \to \tau(X,\mathbb{S},\sigma_i)$. For ease of notation we will set $X^{\sigma_i} := \tau(X,\mathbb{S},\sigma_i)$. Note that X^{σ_i} is nothing other than X base changed via $\sigma_i : R \to R$, and that in terms of equations it is obtained by applying σ_i to the coefficients of the defining polynomials.

Theorem 4.6. *With Assumptions 4.1 in place, let $\mathcal{K}$ denote the class of $\mathcal{D}$-rings (R,∂) such that R is an integral domain of characteristic zero and the associated endomorphisms are injective. Then $(K,\partial) \in \mathcal{K}$ is existentially closed if and only if*

I. *K is an algebraically closed field,*
II. *(K,∂) is inversive, and*
III. *if X is an irreducible affine variety over K and $Y \subseteq \tau(X,\mathcal{D},e)$ is an irreducible subvariety over K such that $\widehat{\pi}_i(Y)$ is Zariski dense in X^{σ_i} for all $i=0,\dots,t$, then there exists $a \in X(K)$ with $\nabla(a) \in Y(K)$.*

Note that $\mathcal{K}$ is a universally axiomatisable class. (This uses the fact that the associated endomorphisms of a $\mathcal{D}$-ring are A-linear combinations of the operators, and hence definable.) Moreover, the characterisation of existentially closed models given in the theorem is also first-order. Indeed, the only one that is not obviously elementary is condition III, but since irreducibility and Zariski-density are parametrically definable in algebraically closed fields, the only thing to check is that if X varies in an algebraic family then so do the $\widehat{\pi}_i : \tau X \to X^{\sigma_i}$. That τX and X^{σ_i} vary uniformly in families follows from Proposition 4.7(b) of [13], but can also be verified directly by looking at the equations that define the prolongations. That $\widehat{\pi}_i$ also varies algebraically follows from the construction of these morphisms in §4.1

[2]See §4.1 of [13] for the construction.

of [13]; see in particular Proposition 4.8(b) of that paper. The following is therefore an immediate corollary of Theorem 4.6:

Corollary 4.7. *Under assumptions 4.1 the theory of $\mathcal{D}$-fields of characteristic zero admits a model companion. We denote the model companion by $\mathcal{D}$-CF_0, and we call its models* $\mathcal{D}$-closed fields*.*

We now work toward a proof of Theorem 4.6.

To prove properties I through III of Theorem 4.6 for every existentially closed model in $\mathcal{K}$ is to prove various extension lemmas about $\mathcal{K}$. In order to facilitate this we introduce the following auxiliary class.

Definition 4.8 (The class $\mathcal{M}$)**.** The class $\mathcal{M}$ is defined to be the class of triples (R, S, ∂) where $R \subseteq S$ are integral A-algebras of characteristic zero and $\partial = (\partial_1, \dots, \partial_{\ell-1})$ is a sequence of maps from R to S such that $e : R \to \mathcal{D}(S)$ given by $e(r) := r\epsilon_0 + \partial_1(r)\epsilon_1 + \cdots + \partial_{\ell-1}(r)\epsilon_{\ell-1}$ has the following properties:

(i) e is an A-algebra homomorphism,

(ii) for each $i = 1, \dots, t$, $\sigma_i := \pi_i^S \circ e : R \to S$ is injective.

Note that $\sigma_0 := \pi_0^S \circ e = \pi^S \circ e$ is then the inclusion map.

So $(R, \partial) \in \mathcal{K}$ if and only if $(R, R, \partial) \in \mathcal{M}$.

The following lemma will imply that every existentially closed member of $\mathcal{K}$ is a field. It is here that we require the associated endomorphisms to be injective.

Lemma 4.9. *Suppose $(R, L, \partial) \in \mathcal{M}$ with L a field. Then we can (uniquely) extend ∂ to the fraction field F of R so that $(F, L, \partial) \in \mathcal{M}$.*

Proof. Let us first extend $e : R \to \mathcal{D}(L)$ to an A-algebra homomorphism from F to $\mathcal{D}(L)$. By the universal property of localisation it suffices (and is necessary) to show that e takes nonzero elements of R to units in $\mathcal{D}(L)$. Note that an element $x \in \mathcal{D}(L)$ is a unit if and only if each of its projections $\theta_i^L(x)$ is a unit in the local L-algebra $\mathcal{D}_i(L)$, which in turn is equivalent to the residue of $\theta_i^L(x)$, namely $\pi_i^L(x) \in L$, being nonzero. But for all $i = 1, \dots, t$, $\pi_i^L \circ e$ is injective on R by assumption. So $e(a)$ is a unit in $\mathcal{D}(L)$ for nonzero $a \in R$.

We thus have an extension $e : F \to \mathcal{D}(L)$. The injectivity of $\sigma_1, \dots, \sigma_t$ is immediate as these are A-algebra homomorphisms between fields. Moreover, $\pi \circ e$ is the identity on F as it is extends the identity on R. So, letting ∂ be the operators corresponding to $e : F \to \mathcal{D}(L)$, we have that $(F, L, \partial) \in \mathcal{M}$. $\square$

The next lemma shows that existentially closed models are algebraically closed.

Lemma 4.10. *Suppose $(F, L, \partial) \in \mathcal{M}$ where F and L are fields and L is algebraically closed. Then we can extend ∂ to $F^{\mathrm{alg}} \subseteq L$ so that $(F^{\mathrm{alg}}, L, \partial) \in \mathcal{M}$.*

In fact, more is true. If σ is the tuple of embeddings $F \to L$ associated to ∂, and σ' is any extension of σ to F^{alg}, then there is exactly one extension ∂' of ∂ to F^{alg} with associated embeddings σ'.

Proof. By iteration it suffices to prove, for any given $a \in F^{\mathrm{alg}}$, that we can extend e to $F(a)$ in such a way that $\pi^L \circ e$ is still the identity on $F(a)$. Let $P(x) \in F[x]$ be the minimal polynomial of a over F. Fixing $i = 1, \dots, t$, we let $c_i \in L$ be a root for $P^{\sigma_i}(x) \in L[x]$, where $\sigma_i : F \to L$ is the field embedding $\pi_i^L \circ e : F \to L$. We cover the $i = 0$ case by letting $c_0 := a$; note that as $P^{\sigma_0}(x) = P(x)$, c_0 is a root of $P^{\sigma_0}(x)$. Now, let $e_i : F \to \mathcal{D}_i(L)$ be $\theta_i^L \circ e$. Note that by construction

$P^{\sigma_i}(x) \in L[x]$ is the reduction of $P^{e_i}(x) \in \mathcal{D}_i(L)[x]$ modulo the maximal ideal of the local artinian ring $\mathcal{D}_i(L)$. Since $P^{\sigma_i}(x)$ is separable (we are in characteristic zero), Hensel's Lemma allows us to lift c_i to a root b_i of $P^{e_i}(x)$ in $\mathcal{D}_i(L)$. In fact there is a unique such lifting as $F \hookrightarrow F(a)$ is étale. Then $b = (b_0, \ldots, b_t) \in \mathcal{D}(L)$ is a root of $P^e(x) \in \mathcal{D}(L)[x]$, and we can extend e to $F(a)$ by sending a to b. By construction $\pi^L e(a) = \pi_0^L e(a) = c_0 = a$, so that $\pi^L \circ e = \mathrm{id}_{F(a)}$.

In the above argument any choice of roots $c_1, \ldots, c_t$ works, and once that choice is made there is a unique possibility for b. This leads to the "in fact" clause of the lemma. □

The next lemma will imply that existentially closed models are inversive.

Lemma 4.11. *Suppose $(F, L, \partial) \in \mathcal{M}$ where F and L are fields. Then there exists an extension L' of L and an extension of ∂ to an inversive $\mathcal{D}$-field structure on L'.*

Proof. Since the $\sigma_i : F \to L$ are partial isomorphisms of subfields of L, we can extend them to automorphisms $\sigma_1', \ldots, \sigma_t'$ of some algebraically closed $L' \supseteq L$. Now fix a transcendence basis B for L' over F. For each $b \in B$ let $b_0 \in \mathcal{D}_0(L')$ lift b, and let $b_i \in \mathcal{D}_i(L')$ lift $\sigma_i'(b)$ for $i = 1, \ldots, t$. Then define $e(b)$ to be $(b_0, \ldots, b_t) \in \mathcal{D}(L')$. This gives us an extension $F[B] \to \mathcal{D}\big(L'\big)$ of e such that $\pi^{L'} \circ e,\ \pi_1^{L'} \circ e, \ldots, \pi_t^{L'} \circ e$ agree with $\mathrm{id}, \sigma_1', \ldots, \sigma_t'$, respectively. By Lemma 4.9 we can extend e to $F(B) \to \mathcal{D}(L')$ preserving this property. By Lemma 4.10 we can extend e to a $\mathcal{D}$-structure on $L' = F(B)^{\mathrm{alg}}$ in such a way that the associated endomorphisms remain the automorphisms $\sigma_1', \ldots, \sigma_t'$. □

Proof of Theorem 4.6. Suppose $(K, \partial) \in \mathcal{K}$ is existentially closed. By Lemmas 4.9, 4.10, and 4.11 we know that K is an algebraically closed field and that $\sigma_1, \ldots, \sigma_t$ are automorphisms of K. It remains to check condition III. Let $X \subseteq \mathbb{A}_K^n$ and $Y \subseteq \tau X$ be as in that condition. Let L be an algebraically closed field extending K and let $b \in Y(L)$ be a K-generic point of Y. Let $a := \hat{\pi}(b) \in X(L)$. Our goal is to extend ∂ to a $\mathcal{D}$-field structure on some extension of L in such a way that $\nabla(a) = b$. This will suffice, because then by existential closedness there must exist a K-point of X with the property that its image under ∇ is a K-point of Y.

As described in §4 of [13], $\tau X(L)$ can be canonically identified with the $\mathcal{D}(L)$-points of the affine scheme over $\mathcal{D}(K)$ obtained from X by applying e to the coefficients of the defining polynomials. Let b' be the n-tuple from $\mathcal{D}(L)$ that corresponds to $b \in \tau X(L)$ under this identification. So $P^e(b') = 0$ for all $P(x) \in I(X/K)$. Since $\hat{\pi}(Y) = \hat{\pi}_0(Y)$ is Zariski dense in $X^{\sigma_0} = X$ and b is K-generic in Y, we have that $a = \hat{\pi}(b)$ is K-generic in X. So $I(X) = I(a/K)$. We thus have that $P^e(b') = 0$ for all $P(x) \in I(a/K)$. That is, we can extend $e : K \to \mathcal{D}(L)$ to an A-algebra homomorphism $e : K[a] \to \mathcal{D}(L)$ by $e(a) = b'$. The fact that $\hat{\pi}(b) = a$ implies that $\pi^L(b') = a$ so that $\pi^L \circ e = \mathrm{id}_{K[a]}$. For each $i = 1, \ldots, t$, the fact that $\hat{\pi}_i(Y)$ is Zariski dense in X^{σ_i} implies that $\hat{\pi}_i(b)$ is K-generic in X^{σ_i}, and hence for any $P(x) \in K[x]$ on which a does not vanish, $\pi_i^L e\big(P(a)\big) = P^{\sigma_i}\big(\pi_i^L(b')\big) \neq 0$. That is, $\pi_i^L \circ e : K[a] \to L$ is injective for each $i = 1, \ldots, t$. So, letting ∂ be the corresponding operators, we have $\big(K[a], L, \partial\big) \in \mathcal{M}$. By Lemma 4.9 this extends to $\big(K(a), L, \partial\big) \in \mathcal{M}$. By Lemma 4.11, there is an extension L' of L such that ∂ extends to a $\mathcal{D}$-field structure on L'. The fact that $e(a) = b'$ implies that $\nabla(a) = b$, as desired.

Now for the converse. Suppose (K,∂) is a $\mathcal{D}$-field satisfying I through III. To show that (K,∂) is existentially closed in $\mathcal{K}$ it suffices to consider a conjunction of atomic $\mathcal{L}_{\mathcal{D},K}$-formulae that is realised in some extension of (K,∂) in $\mathcal{K}$, and show that it is already realised in (K,∂). Indeed, all inequations of the form $t(x_1,\dots,x_m)\neq 0$ that might appear can be replaced by $t(x_1,\dots,x_m)y-1=0$ where y is a new variable. We can also assume, by Lemmas 4.9 and 4.10, that the extension in which we have a realisation is an algebraically closed $\mathcal{D}$-field.

Let $\phi(x)$ be a conjunction of atomic $\mathcal{L}_{\mathcal{D},K}$-formulae where $x=(x_1,\dots,x_m)$ is an m-tuple of variables, let (L,∂) be an algebraically closed $\mathcal{D}$-field extension of (K,∂), and let $c_0\in L^m$ realise $\phi(x)$. Let Ξ be the set of all finite words on $\{\partial_1,\dots,\partial_{\ell-1}\}$, and for each $r\geq 0$ let Ξ_r be those words of length at most r. Fix an enumeration of Ξ so that Ξ_r is an initial segment of Ξ_{r+1} for all $r\geq 0$. Define $\nabla_r: L\to L^{n_r}$ by $b\mapsto\big(\xi(b):\xi\in\Xi_r\big)$, where $n_r:=|\Xi_r|$. Then for some $r\geq 0$, $\phi(x)^L=\{b\in L^m:\nabla_r(b)\in Z\}$ where $Z\subseteq L^{mn_r}$ is a Zariski-closed set over K. Note that if $r=0$ then $\phi(x)$ is equivalent to a formula over K in the language of rings with a realisation in an extension, and so, as K is algebraically closed, $\phi(x)$ is realised in K. We may thus assume that $r>0$. Let

$$\begin{aligned} c &:= \nabla_{r-1}(c_0)\in L^{mn_{r-1}}\\ X &:= \operatorname{loc}(c/K)\subseteq L^{mn_{r-1}}\\ Y &:= \operatorname{loc}(\nabla c/K)\subseteq \tau X(L)\subseteq L^{\ell mn_{r-1}}\end{aligned}$$

Note that for each $i=0,\dots,t$, $\hat{\pi}_i(\nabla c)=\sigma_i(c)\in X^{\sigma_i}(L)$. Since ∇c is K-generic in Y and $\sigma_i(c)$ is K-generic in X^{σ_i} (as σ_i restricts to an automorphism of K by assumption), it follows that $\hat{\pi}_i(Y)$ is Zariski dense in X^{σ_i}. Hence, by III, there exists $a\in X(K)$ such that $\nabla a\in Y(K)$. Let a_0 be the first m co-ordinates of a. It remains to verify that a_0 satisfies $\phi(x)$; that is, that $\nabla_r(a_0)\in Z(K)$.

First of all, we note that $\nabla_{r-1}(a_0)=a$. Indeed, we show by induction on the length of $\xi\in\Xi_{r-1}$ that $\xi(a_0)=a_\xi$, where $a=(a_\xi:\xi\in\Xi_{r-1})$. For $\xi=\mathrm{id}$ this is clear by choice of a_0. Now suppose $\xi=\partial_i\xi'$. Since $\nabla_{r-1}(c_0)=c$, we know that $\partial_i c_{\xi'}=c_\xi$. Because ∇a is in the K-locus of ∇c, we have $\partial_i a_{\xi'}=a_\xi$ also. But by the inductive hypothesis, $\partial_i a_{\xi'}=\partial_i\xi'(a_0)=\xi(a_0)$, so that $\xi(a_0)=a_\xi$ as desired.

Finally, since c_0 is a realisation of $\phi(x)$, we know that $\nabla_r c_0\in Z$. The latter can be seen as an algebraic fact about $\nabla\nabla_{r-1}c_0$. Since $\nabla\nabla_{r-1}a_0=\nabla a$ is in the K-locus of $\nabla\nabla_{r-1}c_0=\nabla c$, it follows that $\nabla_r a_0\in Z$, as desired.

This completes the proof of Theorem 4.6. □

Theorem 4.6 specialised to the various examples, say in 3.6, will yield model companions for a variety of theories of fields with operators. In the classical examples one recovers the known "geometric" axiomatisations.

The following proposition says that the difference field associated to a model of $\mathcal{D}$-CF_0 is itself difference-closed.

Proposition 4.12. *If $(K,\partial)\models\mathcal{D}$-$\mathrm{CF}_0$ and (K,σ) is the associated difference field, then $(K,\sigma)\models\mathrm{ACFA}_{0,t}$, that is, it is an existentially closed model of the theory of fields of characteristic zero equipped with t (not necessarily commuting) automorphisms.*

Proof. The axioms for $\mathrm{ACFA}_{0,t}$ appear in §1.2 of [8]. They say that K should be algebraically closed, $\sigma_1,\dots,\sigma_t$ should be automorphisms of K, and, the only one that requires checking, *if X is an irreducible affine variety over K and $Y\subseteq X\times X^{\sigma_1}\times$*

$\cdots \times X^{\sigma_t}$ *is an irreducible subvariety over* K *whose projections onto each factor are Zariski-dense, then there should exist* $a \in X(K)$ *with* $\big(a, \sigma_1(a), \ldots, \sigma_t(a)\big) \in Y(K)$. To check this, consider the pull-back Y' of Y under $\tau(X, \mathcal{D}, e) \to X \times X^{\sigma_1} \times \cdots \times X^{\sigma_t}$, and apply axiom III of Theorem 4.6 to an irreducible component of Y' that projects dominantly onto Y (there will be one). □

As a consequence of 4.12 we see that the theory of $\mathcal{D}$-fields imposes no non-trivial functional equations on the associated endomorphisms.

5. Basic model theory of $\mathcal{D}$-CF$_0$

We begin now to investigate the model theory of $\mathcal{D}$-CF$_0$ using the study of existentially closed difference fields as it appears in §1 of [3] as a template. Assumptions 4.1 and the notation of the previous chapter remain in place.

5.1. **Completions.** We aim to describe the completions of $\mathcal{D}$-CF$_0$.

Lemma 5.1. *Suppose* (K, ∂) *and* (L, γ) *are inversive* $\mathcal{D}$*-fields extending an inversive* $\mathcal{D}$*-field* (F, ∂) *with* K *and* L *linearly disjoint over* F *(inside some fixed common field extension). Then we can simultaneously extend* (K, ∂) *and* (L, γ) *uniquely to a* $\mathcal{D}$*-field structure on the compositum* KL.

Proof. It follows from linear disjointedness that $R := K \otimes_F L$ is an integral domain whose fraction field is the compositum KL. Here we identify K with $K \otimes 1 \subset R$, L with $1 \otimes L \subset R$ and F with $1 \otimes F = F \otimes 1 \subseteq K \cap L$. It suffices to find a common extension of ∂ and γ to R. Indeed, by Lemma 4.9 we can then further extend R uniquely to a $\mathcal{D}$-field structure on the fraction field KL.

Let $e_1 : K \to \mathcal{D}(K) \subseteq \mathcal{D}(R)$ and $e_2 : L \to \mathcal{D}(L) \subseteq \mathcal{D}(R)$ be the corresponding A-algebra homomorphisms. Since these agree on F we have the induced map $e : R \to \mathcal{D}(R)$ determined by $e(a \otimes b) := e_1(a)e_2(b)$, which is easily seen to be an A-algebra homomorphism that extends both e_1 and e_2. For $i = 0, \ldots, t$,

$$\begin{aligned} \pi_i^R e(a \otimes b) &= \big(\pi_i^R e_1(a)\big)\big(\pi_i^R e_2(b)\big) \\ &= \big(\pi_i^K e_1(a)\big)\big(\pi_i^L e_2(b)\big) \\ &= \big(\sigma_i(a) \otimes 1\big)\big(1 \otimes \tau_i(b)\big) \\ &= \sigma_i(a) \otimes \tau_i(b) \end{aligned}$$

where the $\sigma_i s$ and τ_i's are the associated automorphisms of K and L respectively. Applying this to $i = 0$ we see that $\pi^R \circ e = \pi_0^R \circ e$ is the identity on R; hence (R, e) is a $\mathcal{D}$-ring. For $i \geq 1$, since σ_i and τ_i extend an automorphism of F (by the inversiveness assumption) and K is linearly disjoint from L over F, $a \otimes b \mapsto \sigma_i(a) \otimes \tau_i(b)$ determines an automorphism of $R = K \otimes_F L$. Hence $\pi_i^R \circ e$ is an automorphism of R for $i = 1, \ldots, t$. So (R, e) is in the class $\mathcal{K}$, and extends (K, e_1) and (L, e_2), as desired. □

Proposition 5.2. *If* (K, ∂) *and* (L, γ) *are models of* $\mathcal{D}$-CF$_0$ *with a common algebraically closed inversive* $\mathcal{D}$*-subfield* F*, then* $(K, \partial) \equiv_F (L, \gamma)$.

Proof. Working in a sufficiently saturated algebraically closed field extending K we can find an F-isomorphic copy of the field L, say L', such that K and L' are algebraically disjoint over F. As F is algebraically closed, K and L' are linearly disjoint over F. Let γ' be a $\mathcal{D}$-structure on L' so that (L, γ) and (L', γ') are isomorphic over F. By Lemma 5.1 we can extend (K, ∂) and (L', γ') simultaneously to a $\mathcal{D}$-field structure on KL', which we can then extend further to a model, say $(K', \partial) \models \mathcal{D}\text{-CF}_0$. By model completeness, $(K, \partial) \preceq (K', \partial)$ and $(L', \gamma') \preceq (K', \partial)$. It follows that $(K, \partial) \equiv_F (L, \gamma)$. □

Lemma 5.3. *Suppose $(F, \partial) \subseteq (K, \partial)$ is a $\mathcal{D}$-field extension such that K is algebraically closed and F is inversive. Then $F^{\mathrm{alg}} \subseteq K$ is an inversive $\mathcal{D}$-subfield.*

Proof. Inversiveness comes for free once we see that F^{alg} is a $\mathcal{D}$-subfield. Let $a \in F^{\mathrm{alg}}$ and $P(x) \in F[x]$ be the minimal poynomial of a. Let $e : K \to \mathcal{D}(K)$ be the A-algebra homomorphism corresponding to ∂. We need to show that $e(a) \in \mathcal{D}(F^{\mathrm{alg}})$. Under the identification $\mathcal{D}(K) = \prod_{i=0}^{t} \mathcal{D}_i(K)$, we have $e(a) = \big(e_0(a), \dots, e_t(a)\big)$, where $e_i := \theta_i^K \circ e$, and it suffices to show that each $e_i(a) \in \mathcal{D}_i(F^{\mathrm{alg}})$. Now $\sigma_i(a) \in F^{\mathrm{alg}}$ and by the inversiveness assumption $P^{\sigma_i}(x)$ is the minimal polynomial of $\sigma_i(a)$ over F. So by Hensel's Lemma $\sigma_i(a)$ has a lifting to a root of $P^{e_i}(x)$ in $\mathcal{D}_i(F^{\mathrm{alg}})$. On the other hand, $e_i(a)$ also lifts $\sigma_i(a)$ to a root of $P^{e_i}(x)$ in $\mathcal{D}_i(K)$. As the extension is étale these liftings agree, and so $e_i(a) \in \mathcal{D}_i(F^{\mathrm{alg}})$, as desired. □

Corollary 5.4 (Completions of $\mathcal{D}\text{-CF}_0$)**.** *The completions of $\mathcal{D}\text{-CF}_0$ are determined by the difference-field structure on the algebraic closure of the prime $\mathcal{D}$-field. That is, two models (K, ∂) and (L, γ) of $\mathcal{D}\text{-CF}_0$ are elementarily equivalent if and only if $(A^{\mathrm{alg}}, \sigma \restriction_{A^{\mathrm{alg}}}) \approx_A (A^{\mathrm{alg}}, \tau \restriction_{A^{\mathrm{alg}}})$, where σ and τ are the sequences of automorphisms of K and L associated to ∂ and γ, respectively.*

Proof. First of all, both (K, ∂) and (L, γ) extend the prime $\mathcal{D}$-field A, which is itself inversive (the difference-field structure on A is trivial). Hence, by Lemma 5.3, $(A^{\mathrm{alg}}, \partial \restriction_{A^{\mathrm{alg}}})$ and $(A^{\mathrm{alg}}, \gamma \restriction_{A^{\mathrm{alg}}})$ are inversive $\mathcal{D}$-field extensions of A. By Lemma 4.10 their $\mathcal{D}$-field structures are determined by the action of the corresponding automorphisms on A^{alg}. Hence, if $(A^{\mathrm{alg}}, \sigma \restriction_{A^{\mathrm{alg}}}))$ and $(A^{\mathrm{alg}}, \tau \restriction_{A^{\mathrm{alg}}}))$ are isomorphic then $(A^{\mathrm{alg}}, \partial \restriction_{A^{\mathrm{alg}}})$ and $(A^{\mathrm{alg}}, \gamma \restriction_{A^{\mathrm{alg}}})$ are isomorphic, and so by Proposition 5.2, (K, ∂) and (L, γ) are elementarily equivalent. For the converse, if $(K, \partial) \equiv (L, \gamma)$ then there is an elementary embedding of (K, ∂) into an elementary extension (L', γ) of (L, γ). This elementary embedding will restrict to an isomorphism from $(A^{\mathrm{alg}}, \partial \restriction_{A^{\mathrm{alg}}})$ to its image in (L', γ), which is $(A^{\mathrm{alg}}, \gamma \restriction_{A^{\mathrm{alg}}})$. In particular, $(A^{\mathrm{alg}}, \sigma \restriction_{A^{\mathrm{alg}}}) \approx_A (A^{\mathrm{alg}}, \tau \restriction_{A^{\mathrm{alg}}})$. □

5.2. **Algebraic closure.** We characterise model-theoretic algebraic closure.

Proposition 5.5. *Suppose $(K, \partial) \models \mathcal{D}\text{-CF}_0$. For all $B \subseteq K$, $\operatorname{acl}(B) = \langle B \rangle^{\mathrm{alg}}$.*

Proof. Recall that $\langle B \rangle$ is the inversive closure of B, the smallest inversive $\mathcal{D}$-subfield of K containing B. As $\sigma_1, \dots, \sigma_t$ are $\mathcal{L}_{\mathcal{D}}$-definable, $\langle B \rangle \subseteq \operatorname{dcl}(B)$. Hence $F := \langle B \rangle^{\mathrm{alg}} \subseteq \operatorname{acl}(B)$. It remains to show that if $a \in K \setminus F$ then $\operatorname{tp}(a/F)$ is nonalgebraic.

Note that, by Lemma 5.3, F is an inversive $\mathcal{D}$-subfield of K. Since F is algebraically closed we can find, in some common field extension, an isomorphic copy of K over F, witnessed say by an F-isomorphism $\alpha : K \to K'$, and such that K is linearly disjoint from K' over F. Via α we can put a $\mathcal{D}$-field structure ∂' on K' that extends (F, ∂) and so that α is an isomorphism of $\mathcal{D}$-fields. Now we extend (K, ∂) and (K', ∂') to a $\mathcal{D}$-field structure on KK' using Lemma 5.1, and then further to a model of $\mathcal{D}\text{-CF}_0$. We have thus found a common elementary extension of (K, ∂) and (K', ∂'). In this elementary extension, $\alpha(a)$ will be a realisation of $\operatorname{tp}(a/F)$

that is distinct from a. Iterating, we find infinitely many realisations of $\mathrm{tp}(a/F)$ in some elementary extension, proving that this type is nonalgebraic. □

5.3. **Types.** We characterise types and deduce a quantifier reduction theorem. Recall that Θa is the (infinite) tuple whose co-ordinates are of the form θa_i as θ ranges over all finite words on the set $\{\partial_1, \dots, \partial_{\ell-1}, \sigma_1^{-1}, \dots, \sigma_t^{-1}\}$.

Proposition 5.6. *Suppose $(K, \partial) \models \mathcal{D}$-$\mathrm{CF}_0$, $k \subseteq K$ is an inversive $\mathcal{D}$-subfield, and $a, b \in K^n$. Then the following are equivalent:*

(i) $\mathrm{tp}(a/k) = \mathrm{tp}(b/k)$,

(ii) $\mathrm{tp}_\sigma\big(\Theta a/k\big) = \mathrm{tp}_\sigma\big(\Theta b/k\big)$ *(where $\mathrm{tp}_\sigma(c/k)$ denotes the type of c over k in the reduct to the language of difference fields),*

(iii) *there is an isomorphism from $\big(k\langle a\rangle, \partial\big)$ to $\big(k\langle b\rangle, \partial\big)$ sending a to b and fixing k that extends to an isomorphism from $\big(k\langle a\rangle^{\mathrm{alg}}, \sigma\big)$ to $\big(k\langle b\rangle^{\mathrm{alg}}, \sigma\big)$.*

Proof. (i) $\Longrightarrow$ (ii) is clear.

(ii) $\Longrightarrow$ (iii). Work in a sufficiently saturated elementary extension (L, ∂) of (K, ∂). Then (L, σ) is also saturated as a difference-field, and so $\mathrm{tp}_\sigma\big(\Theta a/k\big) = \mathrm{tp}_\sigma\big(\Theta b/k\big)$ is witnessed by a difference-field automorphism α of L over k, taking Θa to Θb. Then $\beta := \alpha \restriction_{k(\Theta a)}$ is the desired $\mathcal{D}$-field isomorphism from $k\langle a\rangle = k(\Theta a)$ to $k\langle b\rangle = k(\Theta b)$, and $\alpha \restriction_{k\langle a\rangle^{\mathrm{alg}}}$ is the desired extension.

(iii) $\Longrightarrow$ (i). Let $\alpha : k\langle a\rangle^{\mathrm{alg}} \to k\langle b\rangle^{\mathrm{alg}}$ be a difference-field isomorphism that takes a to b, fixes k, and restricts to a $\mathcal{D}$-field isomorphism between $k\langle a\rangle$ and $k\langle b\rangle$. Then α will take $\partial \restriction_{k\langle a\rangle^{\mathrm{alg}}}$ to a $\mathcal{D}$-field structure on $k\langle b\rangle^{\mathrm{alg}}$ whose associated endomorphism is $\sigma \restriction_{k\langle a\rangle^{\mathrm{alg}}}$. But by the uniqueness part of Lemma 4.10, this new $\mathcal{D}$-structure must co-incide with $\partial \restriction_{k\langle b\rangle^{\mathrm{alg}}}$. So $\alpha : k\langle a\rangle^{\mathrm{alg}} \to k\langle b\rangle^{\mathrm{alg}}$ is also an isomorphism of $\mathcal{D}$-fields. The equality of types is now an immediate consequence of 5.2 and 5.3. □

Corollary 5.7 (Quantifier Reduction)**.** *Every L-formula $\phi(x_1, \dots, x_n)$ is equivalent modulo $\mathcal{D}$-CF_0 to an L-formula of the form $\exists y\ \psi(x_1, \dots, x_n, y)$ where*

- *$\psi(x_1, \dots, x_n, y) = \xi(\bar{x}, \bar{y})$ where ξ is a quantifier-free ring formula, the co-ordinates of $\bar{x}$ are of the form θx_i where $\theta \in \Theta$ and $\bar{y} = (y, \sigma_1(y), \dots, \sigma_t(y))$,*
- *each disjunct of ξ written in disjunctive normal form includes a conjunct of the form $t_N(\bar{x}) \neq 0$ & $\sum_{j=0}^{N} t_j(\bar{x}) y^j = 0$ where each t_i is a polynomial.*

In particular, when the associated endomorphisms are all trivial the existential quantifier may be omitted and we have quantifier elimination.

Proof. This follows in the standard way from the determination of types given by the equivalence of (i) and (iii) in Proposition 5.6. Note that in condition (iii) of 5.6 the $\mathcal{D}$-field isomorphism between $k\langle a\rangle$ and $k\langle b\rangle$ is only required to lift to a *difference*-field isomorphism of the algebraic closures. It is this feature that allows us, in the quantifier reduction of the Corollary, to not have to consider all terms of the form θy with $\theta \in \Theta$, but rather only the $\sigma_i y$. □

5.4. **Independence and simplicity.** In this section we observe that $\mathcal{D}$-CF_0 is simple, and we give an algebraic characterisation of nonforking independence. The results here follow more or less axiomatically from the results of the previous sections, as established by Chatzidakis and Hrushovski in [3].

Let $(\mathbb{U}, \partial)$ be a sufficiently saturated model of $\mathcal{D}$-CF_0.

Definition 5.8. Suppose A, B, C are (small) subsets of $\mathbb{U}$. Then A *is independent from* B *over* C, denoted by $A \downfree_C B$, if $\mathrm{acl}(A \cup C)$ is algebraically independent (equivalently linearly disjoint) from $\mathrm{acl}(B \cup C)$ over $\mathrm{acl}(C)$.

Theorem 5.9. *Independence in* $(\mathbb{U}, \partial)$ *satisfies the following properties:*

(a) *Symmetry.* $A \downfree_C B$ *implies* $B \downfree_C A$.
(b) *Transitivity. Given* $A \subseteq B \subseteq C$ *and tuple* a,
$$a \downfree_A C \text{ if and only if } a \downfree_B C \text{ and } a \downfree_A B.$$
(c) *Invariance. If* $\alpha \in \mathrm{Aut}(\mathbb{U}, \partial)$ *then* $A \downfree_C B$ *implies* $\alpha(A) \downfree_{\alpha(C)} \alpha(B)$.
(d) *Finite character.* $A \downfree_C B$ *if and only if* $A \downfree_C B_0$ *for all finite* $B_0 \subset B$.
(e) *Local character. Given a set* B *and a tuple* a, *there exists a countable* $B_0 \subset B$ *such that* $a \downfree_{B_0} B$.
(f) *Extension. Given* $A \subseteq B$ *and tuple* a, *there exists a tuple* a' *such that* $\mathrm{tp}(a/A) = \mathrm{tp}(a'/A)$ *and* $a' \downfree_A B$.
(g) *Independence theorem. Suppose*
   - F *is an algebraically closed inversive* $\mathcal{D}$*-field,*
   - A *and* B *are supersets of* F *with* $A \downfree_F B$,
   - $a \downfree_F A$ *and* $b \downfree_F B$
   - $\mathrm{tp}(a/F) = \mathrm{tp}(b/F)$.

   Then there is $d \downfree_F AB$ *with* $\mathrm{tp}(d/A) = \mathrm{tp}(a/A)$ *and* $\mathrm{tp}(d/B) = \mathrm{tp}(b/B)$.

In particular, $\mathrm{Th}(\mathbb{U}, \partial)$ *is simple and* $\downfree$ *is nonforking independence.*

Proof. (a) through (e) follow easily from the corresponding properties for algebraic independence; part (e) using also the fact that if K is an inversive $\mathcal{D}$-field then $K\langle a\rangle$ is countably generated as a field over K.

(f). Let $F = \mathrm{acl}(A)$, $K = \mathrm{acl}(B)$, and $K_1 := F\langle a\rangle^{\mathrm{alg}}$. Let K_1' be a field-isomorphic copy of K_1 over F – say with $\alpha : K_1 \to K_1'$ witnessing this – such that K_1' is linearly disjoint from K over F. We can put a $\mathcal{D}$-field structure ∂' on K_1' extending (F, ∂) such that α is a $\mathcal{D}$-field isomorphism. Now by Lemma 5.1 we can find a model of $\mathcal{D}$-CF_0 extending both (K_1', ∂') and (K, ∂). By Proposition 5.2 and saturation we may assume this model is an elementary substructure of $(\mathbb{U}, \partial)$. Hence $\mathrm{tp}(\alpha(a)/F) = \mathrm{tp}(a/F)$ by the equivalence of (i) and (iii) in Proposition 5.6, and $\alpha(a) \downfree_A B$ by linear disjointedness.

(g). We follow the spirit of the argument used for $ACFA$ in [3]. Fix $c \models p(x) := \mathrm{tp}(a/F) = \mathrm{tp}(b/F)$. It suffices to find A', B' such that

(i) $\{A', B', c\}$ is independent over F,
(ii) $A'c \models \mathrm{tp}(Aa/F)$,
(iii) $B'c \models \mathrm{tp}(Bb/F)$, and
(iv) $A'B' \models \mathrm{tp}(AB/F)$.

Indeed, if $\alpha \in \mathrm{Aut}_F(\mathbb{U}, \partial)$ with $\alpha(A'B') = AB$, then $d := \alpha(c)$ will witness the conclusion.

Since $\mathrm{tp}(c/F) = \mathrm{tp}(a/F) = \mathrm{tp}(b/F)$, there exists $A'B'$ satisfying (ii) and (iii). Moreover, by extension, we may also assume that $A' \mathop{\smile\!\!\!\!\!\!|}_{Fc} B'$. Hence, by transitivity, we have (i) as well. The only thing missing is (iv).

Let $K_0 := \mathrm{acl}(A') \cdot \mathrm{acl}(B')$, and $K_1 := \mathrm{acl}(A'c) \cdot \mathrm{acl}(B'c)$, and $K_2 := \mathrm{acl}(A'B')$. So K_1 and K_2 are field extensions of K_0. We wish to give K_2 a $\mathcal{D}$-field structure γ such that

$$(K_2, \gamma) \approx_F (\mathrm{acl}(AB), \partial\restriction_{\mathrm{acl}(AB)}). \tag{5}$$

To do so, denote by α and β the F-automorphisms of the universe taking A to A' and B to B', respectively. Then since $A \mathop{\smile\!\!\!\!\!\!|}_F B$ and $A' \mathop{\smile\!\!\!\!\!\!|}_F B'$, $\alpha\restriction_{\mathrm{acl}(A)} \otimes \beta\restriction_{\mathrm{acl}(B)}$ induces an isomorphism over F between the fields $\mathrm{acl}(A)\cdot\mathrm{acl}(B)$ and $\mathrm{acl}(A')\cdot\mathrm{acl}(B')$, and hence between their field-theoretic algebraic closures $\big(\mathrm{acl}(A) \cdot \mathrm{acl}(B)\big)^{\mathrm{alg}} = \mathrm{acl}(AB)$ and $\big(\mathrm{acl}(A')\cdot\mathrm{acl}(B')\big)^{\mathrm{alg}} = \mathrm{acl}(A'B') = K_2$. We use this field isomorphism to define the desired γ on K_2 such that (5) holds.

Since $\alpha\restriction_{\mathrm{acl}(A)}$ and $\beta\restriction_{\mathrm{acl}(B)}$ are $\mathcal{D}$-field ismorphisms, we have that γ agrees with ∂ on each of $\mathrm{acl}(A')$ and $\mathrm{acl}(B')$. Hence γ must agree with ∂ on the composite K_0. That is, $(K_1, \partial\restriction_{K_1})$ and (K_2, γ) are $\mathcal{D}$-field extensions of $(K_0, \partial\restriction_{K_0})$. If we can find a common extension τ of $\partial\restriction_{K_1}$ and γ to the composite $K_1 \cdot K_2$, then we could extend $(K_1 \cdot K_2, \tau)$ to a model of $\mathcal{D}$-CF_0 which will be elementarily embeddable in $(\mathbb{U}, \partial)$ over F by Proposition 5.2. We will thus have achieved (iv) because of (5), without ruining (i) through (iii), thereby proving the independence theorem.

To find such an extension, by Lemma 5.1, it suffices to show that K_1 and K_2 are linearly disjoint over K_0. This follows from the following field-theoretic fact proved by Chatzidakis and Hrushovski in [3]: *If A, B, C are algebraically closed fields extending an algebraically closed field F, with C algebraically independent from AB over F, then $(AC)^{\mathrm{alg}}(BC)^{\mathrm{alg}}$ is linearly disjoint from $(AB)^{\mathrm{alg}}$ over AB.*[3] □

Definition 5.10 (Dimension)**.** Suppose a is a tuple and k is an algebraically closed inversive $\mathcal{D}$-subfield. We let $\dim_{\mathcal{D}}(a/k) := (\mathrm{trdeg}(\Theta_r(a)/k) : r < \omega)$ where $\Theta_r(a) := \big(\theta a : \theta \text{ a word of length} \leq r \text{ on } \{\partial_1, \ldots, \partial_\ell, \sigma_1^{-1}, \ldots, \sigma_t^{-1}\}\big)$. We view $\dim_{\mathcal{D}}(a/k)$ as an element of ω^ω equipped with the lexicographic ordering.

Note that this dimension is not preserved under interdefinability, and that a more robust notion would depend only on the the eventual growth of the sequence of transcendence degrees. This dimension should be regarded as an analogue of the Kolchin function in differential algebra. In some sense, it is too fine, but it will measure nonforking.

Lemma 5.11. *Suppose a is a tuple and $k \subseteq L$ are algebraically closed inversive $\mathcal{D}$-subfields. Then $a \mathop{\smile\!\!\!\!\!\!|}_k L$ if and only if $\dim_{\mathcal{D}}(a/L) = \dim_{\mathcal{D}}(a/k)$.*

---

[3]See Remark 2 following the proof of the Generalised Independence Theorem in [3].

Proof.

$$
\begin{aligned}
a \mathop{\smile\!\!\!\!\!\!\;|}_{k} L \iff &\ \operatorname{acl}(ka) \text{ is algebraically independent of } L \text{ over } k \quad \text{(by definition)}\\
\iff &\ k(\Theta a)^{\mathrm{alg}} \text{ is algebraically independent of } L \text{ over } k \quad \text{(by 5.5)}\\
\iff &\ k\big(\Theta_r(a)\big)^{\mathrm{alg}} \text{ is algebraically independent of } L \text{ over } k \text{ for all } r < \omega\\
\iff &\ \operatorname{trdeg}(\Theta_r(a)/L) = \operatorname{trdeg}(\Theta_r(a)/k) \text{ for all } r < \omega\\
\iff &\ \dim_{\mathcal{D}}(a/L) = \dim_{\mathcal{D}}(a/k)
\end{aligned}
$$

□

5.5. **Elimination of imaginaries.** We follow the same basic strategy for proving elimination of imaginaries as that of Chatzidakis and Hrushovski in [3].

Theorem 5.12. $\operatorname{Th}(\mathbb{U}, \partial)$ *eliminates imaginaries.*

Proof. The proof of elimination of imaginaries for ACFA given in §1.10 of [3] actually proves that a simple theory which satisfies the independence theorem over algebraically closed sets will admit weak elimination of imaginaries if the following holds: *given any imaginary element $e = f(a)$, where a is a tuple from the home sort and f is a definable function, there exists $c \models \operatorname{tp}\big(a/\operatorname{acl}^{\mathrm{eq}}(e) \cap \mathbb{U}\big)$ with $f(c) = e$ and $c \mathop{\smile\!\!\!\!\!\!\;|}_{\operatorname{acl}^{\mathrm{eq}}(e)\cap\mathbb{U}} a$.* Since in any theory of fields weak elimination of imaginaries implies full elimination of imaginaries, it suffices to prove the existence of such a c.

Let $E := \operatorname{acl}^{\mathrm{eq}}(e) \cap \mathbb{U}$. As pointed out in §1.10 of [3], Neumann's Lemma implies that there exists $b \models \operatorname{tp}(a/Ee)$ with $\operatorname{acl}^{\mathrm{eq}}(Ea) \cap \operatorname{acl}^{\mathrm{eq}}(Eb) \cap \mathbb{U} = E$. Let S denote the set of all such b. We first claim that S contains an element of maximal $\dim_{\mathcal{D}}$ over $\operatorname{acl}(Ea)$ in the lexicographic ordering. First, for each r, choose $b_r \in S$ so that $\big(\operatorname{trdeg}(\Theta_i(b_r)/\operatorname{acl}(Ea)) : i \leq r\big)$ is maximal possible. Let $n_r := \operatorname{trdeg}(\Theta_r(b_r)/\operatorname{acl}(Ea))$. Note that for all $i \leq r$, $\operatorname{trdeg}(\Theta_i(b_r)/\operatorname{acl}(Ea)) = n_i$. Now let $\Phi(x)$ be the partial type over $\operatorname{acl}(Ea)$ saying that

$x \models \operatorname{tp}(a/Ee)$,
$\operatorname{acl}^{\mathrm{eq}}(Ea) \cap \operatorname{acl}^{\mathrm{eq}}(Ex) \cap \mathbb{U} = E$, and,
for each $r < \omega$, $\operatorname{trdeg}(\Theta_r(x)/\operatorname{acl}(Ea)) \geq n_r$.

The b_r's witness that $\Phi(x)$ is finitely satisfiable, and hence by compactness it is satisfiable. Letting b realise $\Phi(x)$ we have that $b \models \operatorname{tp}(a/Ee)$, $\operatorname{acl}^{\mathrm{eq}}(Ea) \cap \operatorname{acl}^{\mathrm{eq}}(Eb) \cap \mathbb{U} = E$, and $\dim_{\mathcal{D}}(b/\operatorname{acl}(Ea))$ is maximal such.

Now we proceed as in §1.10 of [3]. Let $c \models \operatorname{tp}(b/\operatorname{acl}(Ea))$ with $c \mathop{\smile\!\!\!\!\!\!\;|}_{Ea} b$. Then $c \models \operatorname{tp}(a/Ee)$ and so $f(c) = e$. So it remains to show that $c \mathop{\smile\!\!\!\!\!\!\;|}_{E} a$.

We have that $\operatorname{acl}^{\mathrm{eq}}(Ec) \cap \operatorname{acl}^{\mathrm{eq}}(Eb) \subseteq \operatorname{acl}^{\mathrm{eq}}(Ea)$ by independence, and hence $\operatorname{acl}^{\mathrm{eq}}(Ec) \cap \operatorname{acl}^{\mathrm{eq}}(Eb) \cap \mathbb{U} \subseteq \operatorname{acl}^{\mathrm{eq}}(Ea) \cap \operatorname{acl}^{\mathrm{eq}}(Eb) \cap \mathbb{U} = E$. Letting c' be such that $\operatorname{tp}(bc/Ee) = \operatorname{tp}(ac'/Ee)$ we have that $c' \models \operatorname{tp}(a/Ee)$ and $\operatorname{acl}^{\mathrm{eq}}(Ec') \cap \operatorname{acl}^{\mathrm{eq}}(Ea) \cap \mathbb{U} = E$. So by maximality, $\dim_{\mathcal{D}}(c'/\operatorname{acl}(Ea)) \leq \dim_{\mathcal{D}}(b/\operatorname{acl}(Ea))$. Hence, as $\dim_{\mathcal{D}}$ is automorphism invariant, $\dim_{\mathcal{D}}(c/\operatorname{acl}(Eb)) \leq \dim_{\mathcal{D}}(b/\operatorname{acl}(Ea))$. But, on the other hand,

$$\dim_{\mathcal{D}}(c/\operatorname{acl}(Eb)) \geq \dim_{\mathcal{D}}(c/\operatorname{acl}(Eab)) = \dim_{\mathcal{D}}(c/\operatorname{acl}(Ea)) = \dim_{\mathcal{D}}(b/\operatorname{acl}(Ea))$$

where the first equality is by Lemma 5.11. Hence we have equality throughout, and $\dim_{\mathcal{D}}(c/\operatorname{acl}(Eb)) = \dim_{\mathcal{D}}(c/\operatorname{acl}(Eab))$ which, by Lemma 5.11 again, implies that

$c \mathop{\smile\!\!\!\!\!\!\;|}_{Eb} a$. Since we also have $c \mathop{\smile\!\!\!\!\!\!\;|}_{Ea} b$, and $\mathrm{acl}^{\mathrm{eq}}(Ea) \cap \mathrm{acl}^{\mathrm{eq}}(Eb) \cap \mathbb{U} = E$, we get $c \mathop{\smile\!\!\!\!\!\!\;|}_{E} ab$. In particular $c \mathop{\smile\!\!\!\!\!\!\;|}_{E} a$, as desired. □

6. The Zilber dichotomy for finite-dimensional minimal types

In this final chapter we begin to study the fine structure of definable sets in $\mathcal{D}$-CF_0. As is by now a standard approach, the first step is to prove a Zilber dichotomy type theorem for the types of SU-rank one as these form the building blocks of the finite rank definable sets. A second step, which we do not carry out here, would be to consider regular types more generally.

Our current methods only allow us to prove the Zilber dichotomy when the types in question are "finite-dimensional" in a sense that we now make precise. We continue to work in a sufficiently saturated model $(\mathbb{U}, \partial) \models \mathcal{D}$-$\mathrm{CF}_0$, and over a (small) inversive $\mathcal{D}$-subfield k.

Definition 6.1. A type $p = \mathrm{tp}(a/k)$ is called *finite-dimensional* if the $\mathcal{D}$-field generated by a over k is of finite transcendence degree over k.

From Proposition 5.5 we know that in general $\mathrm{acl}(ka) = k\langle a\rangle^{\mathrm{alg}}$, the field-theoretic algebraic closure of the *inversive* $\mathcal{D}$-field generated by a over k. The anonymous referee of an earlier version of this paper pointed out to us that in the finite-dimensional case the inversiveness comes for free:

Lemma 6.2. *Suppose* $\mathrm{tp}(a/k)$ *is finite-dimensional. Then* $\mathrm{acl}(ka)$ *is the field-theoretic algebraic closure of the* $\mathcal{D}$*-field generated by* a *over* k*.*

Proof. Let L be the $\mathcal{D}$-field generated by a over k. So if Ξ is the set of all finite words on $\{\partial_1, \dots, \partial_{\ell-1}\}$, then $L = k(\xi a : \xi \in \Xi)$. We need to show that L^{alg} is already inversive. Let $b \in L^{\mathrm{alg}}$, and fix one of the associated endomorphisms σ_i. By finite-dimensionality, that is by the finite transcendence of L^{alg} over k, for some $r \geq 0$, $\{b, \sigma_i b, \dots, \sigma_i^r b\}$ is algebraically dependent over k. Applying σ_i^{-1} sufficiently many times to the algebraic relation witnessing this, we get that $b \in k(\sigma_i b, \dots, \sigma_i^r b)^{\mathrm{alg}}$. Applying σ_i^{-1} one more time, we get $\sigma_i^{-1} b \in k(b, \sigma_i b, \dots, \sigma_i^{r-1} b)^{\mathrm{alg}} \subseteq L^{\mathrm{alg}}$. So L^{alg} is inversive, as desired. □

Corollary 6.3. *Suppose* $\mathrm{tp}(a/k)$ *is finite-dimensional and let* Ξ *be the set of all finite words on* $\{\partial_1, \dots, \partial_{\ell-1}\}$*. For* L *any inversive* $\mathcal{D}$*-field extending* k*,*

$$a \mathop{\smile\!\!\!\!\!\!\;|}_{k} L \iff \mathrm{trdeg}\big(k(\xi a : \xi \in \Xi)/k\big) = \mathrm{trdeg}\big(L(\xi a : \xi \in \Xi)/L\big)$$

Proof. By Lemma 6.2, $\mathrm{acl}(ka) = k(\xi a : \xi \in \Xi)^{\mathrm{alg}}$. So by definition, $a \mathop{\smile\!\!\!\!\!\!\;|}_{k} L$ if and only if and only if $k(\xi a : \xi \in \Xi)^{\mathrm{alg}}$ is algebraically independent of L^{alg} over k^{alg}. Since $k(\xi a : \xi \in \Xi)$ is of finite transcendence degree over k, this last condition is equivalent to the one stated in the corollary. □

Definition 6.4. The *field of constants* is $C := \{x \in \mathbb{U} : e(x) = s^{\mathbb{U}}(x)\}$.

The goal of this chapter is to prove that if p is a finite-dimensional type of SU-rank one, then either p is one-based or it is almost internal to the field of constants. We follow here the strategy of Pillay and Ziegler [16] by proving first a canonical base property (see 6.18 for a precise statement in our context) using an appropriate notion of *jet space*. For finite-dimensional differential and difference varieties, Pillay and Ziegler gave a rather direct construction using δ-modules and σ-modules respectively. Significant complications arise when one tries to generalise their construction to finite-dimensional $\mathcal{D}$-varieties, and while we expect that a $\mathcal{D}$-modules approach to jet spaces can be made to work, such a theory is not yet ready. Instead we will use the theory of jet spaces that we developed in [14], a theory that does not assume finite-dimensionality and is well-suited to the present context. However, some preliminaries are necessary to relate the formalisms of that paper and the current one.

Assumptions 4.1 and the notation of Chapter 4 remain in place.

6.1. **Iterativity.** We begin by describing how $\mathcal{D}$ gives rise to a generalised iterative Hasse-Schmidt system in the sense of [14]. The construction here is essentially the same as (though dual to) that of Kamensky (Proposition 2.3.2 of [10]).

First of all, one can always form a completely free iterative Hasse-Schmidt system by simply iterating $\mathcal{D}$ with itself. That is, one defines the projective system of finite free algebra schemes $(\mathcal{D}^{(n)}, s_n, \psi_n,)$ by

$$\begin{aligned}
&\mathcal{D}^{(n+1)}(R) = \mathcal{D}\big(\mathcal{D}^{(n)}(R)\big)\\
&s^R_{n+1} := s^{\mathcal{D}^n(R)} \circ s^R_n : R \to \mathcal{D}^{(n+1)}(R)\\
&\psi^R_{n+1} := (\psi^R_n)^\ell \circ \psi^{\mathcal{D}^{(n)}(R)} : \mathcal{D}^{(n+1)}(R) \to (R^\ell)^{n+1}\\
&f^R_{n+1} := \pi^{\mathcal{D}^{(n)}(R)} : \mathcal{D}^{(n+1)}(R) \to \mathcal{D}^{(n)}(R)
\end{aligned}$$

for any A-algebra R. For details on composing finite free $\mathbb{S}$-algebras see §4.2 of [13]. Equipped with the trivial iterativity maps (since $\mathcal{D}^{(n+m)} = \mathcal{D}^{(n)} \circ \mathcal{D}^{(m)}$), this becomes a generalised iterative Hasse-Schmidt system (see 2.2 and 2.17 of [14]).

However, the above construction does not take into account the fact that in our $\mathcal{D}$-rings (R, e), the coefficient of ϵ_0 in $e(a)$ is always a. In other words, the fact that e is a section to $\pi = \mathrm{pr}_1 \circ\psi$. We will thus need to define a sequence of subalgebra schemes $\mathcal{D}_n \subseteq \mathcal{D}^{(n)}$ by identifying the appropriate co-ordinates. This is done as follows. Given an A-algebra R, fix the R-basis $\big\{\epsilon_{i_1} \otimes \cdots \otimes \epsilon_{i_n} : 0 \le i_j \le \ell - 1\big\}$ for $\mathcal{D}^{(n)}(R)$. Define $(i_1, \dots, i_n) \sim (j_1, \dots, j_n)$ if $(i_1, \dots, i_n)$ and $(j_1, \dots, j_n)$ yield the same ordered tuple when all the zeros are dropped. Then $\mathcal{D}_n(R)$ is the subalgebra of elements

$$\Big\{\sum r_{i_1,\dots,i_n}(\epsilon_{i_1} \otimes \cdots \otimes \epsilon_{i_n}) \mid r_{i_1,\dots,i_n} = r_{j_1,\dots,j_n} \text{ whenever } (i_1, \dots, i_n) \sim (j_1, \dots, j_n)\Big\}$$

It follows from this that $\psi_n : \mathcal{D}^{(n)} \to (\mathbb{S}^\ell)^n$ maps $\mathcal{D}_n$ onto the diagonal defined by equating the $(i_1, \dots, i_n)$th and $(j_1, \dots, j_n)$th co-ordinates whenever $(i_1, \dots, i_n) \sim (j_1, \dots, j_n)$. This diagonal is canonically identified with the free $\mathbb{S}$-module scheme $\mathbb{S}^{L_n}$ where $L_n := \{(i_1, \dots, i_m) : 0 \le m \le n, 0 < i_j \le \ell - 1\}$.

Remark 6.5. We can define $\mathcal{D}_n$ in a co-ordinate free manner as follows. Given $n > 0$ and $1 \le i \le n$, consider the morphism of algebra schemes

$$\lambda_{i,n} := \mathcal{D}^{i-1}(f_{n-i+1}) : \mathcal{D}^{(n)} \to \mathcal{D}^{(n-1)}$$

Then $\mathcal{D}_n$ is the equaliser of $\lambda_{1,n}, \dots, \lambda_{n,n}$.

The following properties follow:

- $s_n : \mathbb{S} \to \mathcal{D}^{(n)}$ maps $\mathbb{S}$ to $\mathcal{D}_n$, so the latter becomes an $\mathbb{S}$-algebra scheme.
- $\psi_n : \mathcal{D}^{(n)} \to (\mathbb{S}^\ell)^n$ maps $\mathcal{D}_n$ to $\mathbb{S}^{L_n}$ isomorphically as an $\mathbb{S}$-module
- $f_n : \mathcal{D}^{(n)} \to \mathcal{D}^{(n-1)}$ restricts to a surjective morphism of $\mathbb{S}$-algebra schemes from $\mathcal{D}_n$ to $\mathcal{D}_{n-1}$.

- As subalgebra schemes of $\mathcal{D}^{(m+n)}$, $\mathcal{D}_{m+n} \subseteq \mathcal{D}_m \circ \mathcal{D}_n$.

Hence, $\underline{\mathcal{D}} := (\mathcal{D}_n)$ is a generalised iterative Hasse-Schmidt system, where the iterativity maps $\mathcal{D}_{m+n} \to \mathcal{D}_m \circ \mathcal{D}_n$ are just the inclusions.

Proposition 6.6. *Suppose (R, e) is a $\mathcal{D}$-ring. Then there is a unique iterative $\underline{\mathcal{D}}$-ring structure $E = \big(E_n : R \to \mathcal{D}_n(R) : n < \omega\big)$ on R with $E_1 = e$. In terms of co-ordinates, this $\underline{\mathcal{D}}$-ring structure is given by*

$$E_n(a) \quad = \sum_{(i_1,\ldots,i_m)\in L_n} \partial_{i_1}\cdots\partial_{i_m}(a)\epsilon_{i_1,\ldots,i_m} \tag{6}$$

where $L_n := \{(i_1, \ldots, i_m) : 0 \le m \le n, 0 < i_j \le \ell-1\}$ and $\{\epsilon_{i_1,\ldots,i_m}\}$ is the R-basis for $\mathcal{D}_n(R)$ obtained from the standard basis for R^{L_n} via ψ_n^R.

Proof. First of all, recall that $E = \big(E_n : R \to \mathcal{D}_n(R) : n < \omega\big)$ is an iterative $\underline{\mathcal{D}}$-ring structure on R, according to Definitions 2.2 and 2.17 of [14], if the maps are all ring homomorphisms and

(i) $E_0 = \mathrm{id}$
(ii) $f^R_{m,n} \circ E_m = E_n$ for all $m \ge n$, where $f_{m,n} : \mathcal{D}_m \to \mathcal{D}_n$ is $f_m \circ \cdots \circ f_{n+1}$,
(iii) $E_{m+n} = \mathcal{D}_m(E_n) \circ E_m$ for all m, n.

For existence, we define $E_n : R \to \mathcal{D}^{(n)}(R)$ by composing e with itself n-times. That is, recursively, $E_0 = \mathrm{id}$ and $E_{n+1} = \mathcal{D}(E_n) \circ e$. That E_n maps R to $\mathcal{D}_n(R)$, and that it has the form claimed in (6), is not difficult to check using the fact that $e(a) = a + \partial_1(a)\epsilon + \cdots + \partial_{\ell-1}(a)\epsilon_{\ell-1}$. Properties (i) through (iii) follow more or less immediately from (6).

For uniqueness, note that the assumption that $E_1 = e$ and property (iii) force $E_{n+1} = \mathcal{D}(E_n) \circ e$, so that the above construction is the only one possible. □

6.2. **$\underline{\mathcal{D}}$-varieties and generic points.** We return now to our saturated model $(\mathbb{U}, \partial)$ of $\mathcal{D}$-CF_0, and equip it with the definable iterative $\underline{\mathcal{D}}$-field structure E given by Proposition 6.6. We fix also an inversive $\mathcal{D}$-subfield k.

In Chapter 3 of [14] the rudiments of $\underline{\mathcal{D}}$-algebraic geometry are developed. Let us recap some of the notions introduced there. We work inside a fixed irreducible algebraic variety X over k. While the treatment in [14] is more general, for the sake of concreteness and also with an eye toward our model-theoretic intentions, we will assume that X is affine. By a *$\underline{\mathcal{D}}$-subvariety of X over k* is meant a sequence $\underline{Z} = (Z_n)_{n<\omega}$ of subvarieties $Z_n \subseteq \tau_n X$ over k such that for all $0 < n < \omega$,

(1) $\hat{f}_n : \tau_n X \to \tau_{n-1} X$ restricts to a morphism from $Z_n \to Z_{n-1}$, and
(2) $Z_n \subseteq \tau(Z_{n-1})$.

Here, $\tau_n X = \tau(X, \mathcal{D}_n, E_n)$ is the nth prolongation of X in the sense of §2.1 of [14]. In particular, $\tau_n X(\mathbb{U})$ is canonically identified with $X\big(\mathcal{D}_n(\mathbb{U})\big)$. For general properties of these prolongation functors see §4 of [13].

Remark 6.7. The iterativity condition of Definition 3.1 of [14] here simplifies to (2) above because the iterativity map here is just the containment $\mathcal{D}_n \subseteq \mathcal{D}_{n-1} \circ \mathcal{D}$ as subschemes of $\mathcal{D}^{(n)}$.

The map E_n induces a definable map $\nabla_n : X(\mathbb{U}) \to \tau_n X(\mathbb{U})$ which with respect to the standard bases is given by

$$\nabla_n(p) := \big(\partial_{i_1}\cdots\partial_{i_m}(p) : 0 \le m \le n, 0 < i_j \le \ell - 1\big)$$

We have seen this map appear already in the proof of Theorem 4.6.

We say $\underline{Z}$ is (*absolutely*) *irreducible* if each Z_n is (absolutely) irreducible.

By the $\mathbb{U}$-*rational points* (or simply the *rational points*) of $\underline{Z}$ is meant the type-definable set

$$\underline{Z}(\mathbb{U}) := \{p \in X(\mathbb{U}) : \nabla_n(p) \in Z_n(\mathbb{U}), \text{ for all } n < \omega\}$$

A rational point $p \in \underline{Z}(\mathbb{U})$ is k-*generic* if $\nabla_n(p)$ is generic in Z_n over k in Weil's sense that there is no proper k-subvariety $Y \subsetneq Z_n$ with $\nabla_n(p) \in Y(\mathbb{U})$), for all n.

Of course nothing so far has guaranteed the existence of generic points, or even of rational points. In [14] this is dealt with by working in "rich" $\underline{\mathcal{D}}$-fields. Here we do not have richness of $(\mathbb{U}, \partial)$, however, we can characterise precisely which $\underline{\mathcal{D}}$-varieties do have generic points. Indeed, what is required is the following higher-order analogue of the condition appearing in axiom III of $\mathcal{D}$-CF_0 (cf. Theorem 4.6).

Definition 6.8 (σ-dominance)**.** Suppose $\underline{Z} = (Z_n)_{n<\omega}$ is a $\underline{\mathcal{D}}$-subvariety of an algebraic variety X. We will say that $\underline{Z}$ is σ-*dominant* if for each $n > 0$ and each $i = 0, \dots, t$, the morphism $\hat{\pi}_i : \tau(Z_{n-1}) \to Z_{n-1}^{\sigma_i}$ restricts to a dominant morphism from Z_n to $Z_{n-1}^{\sigma_i}$. (See §4.2 to recall what $\pi = \pi_0, \dots, \pi_t$ are.)

Remark 6.9. It follows from σ-dominance that $\hat{f}_n \restriction_{Z_n} : Z_n \to Z_{n-1}$ is dominant. Indeed, this is because $\pi_0 = \pi$ and f_n is just π applied to $\mathcal{D}^{(n)}$. Hence a σ-dominant $\underline{\mathcal{D}}$-variety is in particular *dominant* in the sense of [14].

Proposition 6.10. *Suppose $\underline{Z}$ is an absolutely irreducible $\underline{\mathcal{D}}$-subvariety of X over k. Then $\underline{Z}$ has a k-generic point if and only if $\underline{Z}$ is σ-dominant.*

Proof. First suppose that $\underline{Z}$ has a k-generic point $p \in \underline{Z}(\mathbb{U})$. Then $\nabla_n(p)$ is generic in $Z_n(\mathbb{U})$. To prove σ-dominance we will show that $\hat{\pi}_i\big(\nabla_n(p)\big)$ is generic in $Z_{n-1}^{\sigma_i}$ over k. Under the identification of $\tau Z_{n-1}(\mathbb{U})$ with $Z_{n-1}\big(\mathcal{D}(\mathbb{U})\big)$, $\nabla_n(p)$ corresponds $e\big(\nabla_{n-1}(p)\big)$, and we need to show that $\pi_i\big(e(\nabla_{n-1}(p))\big) = \sigma_i\big(\nabla_{n-1}(p)\big)$ is generic in $Z_{n-1}^{\sigma_i}$ over k. But this follows from the fact that $\nabla_{n-1}(p)$ is generic in Z_{n-1} and σ_i is an automorphism.

For the converse we assume that $\underline{Z}$ is σ-dominant and seek a generic rational point. Without loss of generality we may assume that k is algebraically closed. By saturation it suffices to fix $n < \omega$ and show that there exists $p \in X(\mathbb{U})$ such that $\nabla_n(p)$ is generic in Z_n over k. For this we follow the general strategy in the proof of Theorem 4.6. Let L be an algebraically closed field extension of k, $b \in Z_n(L)$ a generic point over k, and $a := \hat{f}_{n,0}(b) \in X(L)$. We will show how to extend ∂ from k to a $\mathcal{D}$-field structure on some extension of L, such that $\nabla_n(a) = b$. This $\mathcal{D}$-field structure could then be further extended to a model of $\mathcal{D}$-CF_0, which by Proposition 5.2 and saturation can then be embedded into $(\mathbb{U}, \partial)$ over k; thus establishing the existence of the desired $p \in X(\mathbb{U})$.

Under the identification of $\tau Z_{n-1}(L)$ with $Z_{n-1}\big(\mathcal{D}(L)\big)$, let b' be the tuple from $\mathcal{D}(L)$ corresponding to b. We have that $P^e(b') = 0$ for all $P(x) \in I(Z_{n-1}/k)$. On the other hand, as $Z_n \to Z_{n-1}$ is dominant (cf. Remark 6.9), $I(Z_{n-1}/k) = I(b_{n-1}/k)$. It follows that e on k extends to a ring homomorphism $\eta : k[b_{n-1}] \to \mathcal{D}(L)$, where b_{n-1} is the image of b under $Z_n \to Z_{n-1}$, by $\eta(b_{n-1}) = b'$. The assumption of σ-dominance implies that $\pi_i \circ \eta : k[b_{n-1}] \to L$ is injective, for each $i = 1, \dots, t$, so that by Lemmas 4.9 and 4.11 we can extend η to a $\mathcal{D}$-field structure on some field L' extending L. In this $\mathcal{D}$-field, the fact that $\eta(b_{n-1}) = b'$ means that $\nabla(b_{n-1}) = b$.

It then follows, that for any $r < n$,

$$\begin{aligned}\nabla\big(\hat{f}_{n,r}(b)\big) &= \nabla\big(\hat{f}_{n-1,r}(b_{n-1})\big)\\ &= \hat{f}_{n,r+1}\big(\nabla(b_{n-1})\big) \qquad \text{cf. Proposition 4.7(a) of [13]}\\ &= \hat{f}_{n,r+1}(b)\end{aligned}$$

Iterating, and recalling that $\hat{f}_{n,0}(b) = a$ and $\hat{f}_{n,n}(b) = b$, we get that $\nabla_n(a) = b$, as desired. □

6.3. **Jet spaces.** In [13] and [14], following the work of Pillay and Ziegler [16], we effected a linearisation of generalised Hasse-Schmidt varieties by introducing *jet spaces*. We now specialise this theory to our present context (Fact 6.11 below), and also prove a finiteness theorem (Proposition 6.15 below) that was not done in the earlier papers but is essential here. We continue to work in a fixed affine algebraic variety X over an inversive $\mathcal{D}$-subfield k.

To each point p of X, and for each $m > 0$, we can associate a linear algebraic variety called the *mth algebraic jet space of X at p*, denoted by $\operatorname{Jet}^m X_p$. It is a kind of higher-order tangent space; see §5 of [13] for a review of this notion. Now suppose that $\underline{Z} = (Z_n)_{n<\omega}$ is a $\underline{\mathcal{D}}$-subvariety of X over k and $p \in \underline{Z}(\mathbb{U})$. One would like to associate a $\underline{\mathcal{D}}$-jet space to $\underline{Z}$ at p. A natural thing would be to consider $\big(\operatorname{Jet}^m(Z_n)_{\nabla_n(p)}\big)_{n<\omega}$. However, this sequence does *not* determine a $\underline{\mathcal{D}}$-subvariety of $\operatorname{Jet}^m X_p$, simply because $\operatorname{Jet}^m(Z_n)_{\nabla_n(p)}$ lives in $\operatorname{Jet}^m(\tau_n X)_{\nabla_n(p)}$ rather than in $\tau_n(\operatorname{Jet}^m X_p)$ as would be required by the definition of $\underline{\mathcal{D}}$-subvariety. We addressed this issue in [13] and [14] by studying a certain canonical linear *interpolating map* $\phi : \operatorname{Jet}^m(\tau_n X)_{\nabla_n(p)} \to \tau_n(\operatorname{Jet}^m X_p)$. See §2.3 of [14] for a brief description of ϕ. In §4 of [14] we were then able to define the *mth $\underline{\mathcal{D}}$-jet space of $\underline{Z}$ at p*, denoted by $\operatorname{Jet}^m_{\underline{\mathcal{D}}}(\underline{Z})_p$, as the $\mathcal{D}$-subvariety defined by taking the Zariski closure of the images of the $\operatorname{Jet}^m(Z_n)_{\nabla_n(p)}$ under these interpolating maps. In particular we show (Lemma 4.4 of [14]) that for generic p,

$$(7) \quad \operatorname{Jet}^m_{\underline{\mathcal{D}}}(\underline{Z})_p(\mathbb{U}) = \big\{\lambda \in \operatorname{Jet}^m X_p(\mathbb{U}) : \nabla_n(\lambda) \in \phi\big(\operatorname{Jet}^m(Z_n)_{\nabla_n(p)}(\mathbb{U})\big), \forall n \geq 0\big\}$$

As $\operatorname{Jet}^m(Z_n)_{\nabla_n(p)}(\mathbb{U})$ is a $\mathbb{U}$-linear subspace of $\operatorname{Jet}^m(\tau_n X)_{\nabla_n(p)}(\mathbb{U})$, ϕ is $\mathbb{U}$-linear, and ∇_n is C-linear, we get that $\operatorname{Jet}^m_{\underline{\mathcal{D}}}(\underline{Z})_p(\mathbb{U})$ is a C-linear subspace of $\operatorname{Jet}^m X_p(\mathbb{U})$. Moreover, $\operatorname{Jet}^m_{\underline{\mathcal{D}}}(\underline{Z})_p$ is the fibre above p of a bundle $\operatorname{Jet}^m_{\underline{\mathcal{D}}}(\underline{Z}) \to \underline{Z}$ which is a $\underline{\mathcal{D}}$-subvariety of the algebraic jet bundle $\operatorname{Jet}^m X \to X$. We refer the reader to [14] for more details on these spaces.

One thing that will be important for us is that if $\underline{Z}$ is σ-dominant then so is $\operatorname{Jet}^m_{\underline{\mathcal{D}}}(\underline{Z})$. Indeed, in Proposition 4.7 of [14] it is proved that the dominance of the $\hat{\pi}_0$ maps are preserved when one takes jets, and the very same proof works for the other $\hat{\pi}_i$ maps as well – the key lemma behind all these cases being the "compatibility of the interpolating map with comparing of prolongations" which is 6.4(c) of [13]. Similarly, the proof of 4.7 of [14] also shows that, if p is a k-generic rational point of $\underline{Z}$, and $\underline{Z}$ is σ-dominant, then so is $\operatorname{Jet}^m_{\underline{\mathcal{D}}}(\underline{Z})_p$.

We need one more piece of notation before stating the main result of [14] specialised to the present context. Given $\underline{Z} = (Z_n)_{n<\omega}$ and $r < \omega$, we let $\nabla_r\underline{Z} := (Z_{r+n})_{n<\omega}$. It is clear that $\nabla_r\underline{Z}$ is a $\underline{\mathcal{D}}$-subvariety of Z_r. If $\underline{Z}$ is σ-dominant, then this is also the case for $\nabla_r \underline{Z}$. Finally, assuming σ-dominance, one can show that the set of rational points of $\nabla_r \underline{Z}$ is exactly $\nabla_r\big(\underline{Z}(\mathbb{U})\big)$, see the proof[4] of 3.16 of [14].

Fact 6.11. *Suppose L and L' are inversive $\mathcal{D}$-subfields extending k, $\underline{Z}$ and $\underline{Z}'$ are absolutely irreducible $\underline{\mathcal{D}}$-subvarieties of X over L and L' respectively, and $p \in \underline{Z}(\mathbb{U}) \cap \underline{Z'}(\mathbb{U})$ is L-generic in $\underline{Z}$ and L'-generic in $\underline{Z}'$. If $\mathrm{Jet}^m_{\underline{\mathcal{D}}}(\nabla_r \underline{Z})_{\nabla_r(p)}(\mathbb{U}) = \mathrm{Jet}^m_{\underline{\mathcal{D}}}(\nabla_r \underline{Z}')_{\nabla_r(p)}(\mathbb{U})$ for all $m \geq 1$ and $r \geq 0$, then $\underline{Z} = \underline{Z}'$.*

Proof. This follows from the proof of Theorem 4.8 of [14] specialised to our context. The only reason we cannot apply that theorem directly is because the "richness" assumptions of that theorem are not necessarily satisfied here. However, richness is only used in the proof to ensure that all the relevant $\underline{\mathcal{D}}$-varieties have (densely) many rational points. Hence, because of Proposition 6.10 of this paper, all one needs for that proof to go through is that the relevant $\underline{\mathcal{D}}$-varieties here be σ-dominant. This is the case for $\underline{Z}$ and $\underline{Z}'$ because by assumption they have a generic rational point, and as discussed in the preceding paragraphs taking jets and ∇_r preserves σ-dominance.

Let us also remark that Theorem 4.8 of [14] asks for p to be in the "good locus" of $\underline{Z}$ (and also of $\underline{Z}'$). That is, p should be smooth on X, $\hat{f}_n$ restricted to Z_n should be smooth at $\nabla_n(p)$, and also $\nabla_n(p)$ should land inside a certain L-definable nonempty Zariski open subset of Z_n mentioned in Lemma 4.4 of [14], for all $n > 0$. Our assumption here that p is L-generic (recalling also that we are in characteristic zero) ensures that p is in the good locus. □

As one might expect, the linearisation that Fact 6.11 gives us is particularly useful when the $\underline{\mathcal{D}}$-jet spaces are finite dimensional as vector spaces over the constants. In the rest of this section we aim to prove that if $\underline{Z} = (Z_n)_{n<\omega}$ is "finite-dimensional" in the sense that $\dim Z_n$ is bounded independently of n, then the jet space $\mathrm{Jet}^m_{\underline{\mathcal{D}}}(\underline{Z})_p(\mathbb{U})$ is a finite dimensional C-vector space. Towards this end we begin with a study of prolongations of schemes defined over the constants. With our first technical lemma we observe that the constants are constants for all of the higher exponentials as well.

Lemma 6.12. *The maps E_n and s_n agree on C for every natural number n.*

Proof. This lemma follows by induction and the iterative construction, once one observes that the following diagram commutes for any A-algebra R

$$\begin{array}{ccc} R & \xrightarrow{\ s\ } & \mathcal{D}(R) \\ {\scriptstyle s_n}\downarrow & & \downarrow{\scriptstyle \mathcal{D}(s_n)} \\ \mathcal{D}_n(R) & \xrightarrow[s^{\mathcal{D}_n(R)}]{} & \mathcal{D}\big(\mathcal{D}_n(R)\big) \end{array}$$

To see that the above diagram does indeed commute one writes it in terms of tensor products as

$$\begin{array}{ccc} R & \xrightarrow{\ s\ } & R \otimes_A \mathcal{D}(A) \\ {\scriptstyle s_n}\downarrow & & \downarrow{\scriptstyle s_n \otimes_A \mathcal{D}(A)} \\ R \otimes_A \mathcal{D}_n(A) & \xrightarrow[s^{\mathcal{D}_n(R)}]{} & R \otimes_A \mathcal{D}_n(A) \otimes_A \mathcal{D}(A) \end{array}$$

[4] Actually the proof in this case is much easier as the iterativity maps are trivial.

which commutes because all the maps in this diagram are the natural algebra structure maps. □

In Section 4.1 of [13] maps between prolongation spaces associated to maps of $\mathbb{S}$-algebras are constructed. More precisely, given a map $\alpha : \mathcal{E} \to \mathcal{F}$ of finite free $\mathbb{S}$-algebras, an A-algebra k, and $\mathcal{E}$-ring and $\mathcal{F}$-ring structures $e : k \to \mathcal{E}(k)$ and $f : k \to \mathcal{F}(k)$ on k for which the diagram

$$\begin{array}{ccc} \mathcal{E}(k) & \xrightarrow{\alpha^k} & \mathcal{F}(k) \\ {\scriptstyle e}\nwarrow & & \nearrow{\scriptstyle f} \\ & k & \end{array}$$

commutes, for any k-scheme X, there is a map $\widehat{\alpha} : \tau(X, \mathcal{E}, e) \to \tau(X, \mathcal{F}, f)$ so that for any k-algebra R the diagram

$$\begin{array}{ccc} \tau(X,\mathcal{E},e)(R) & \xrightarrow{\widehat{\alpha}} & \tau(X,\mathcal{F},f)(R) \\ \| & & \| \\ X(\mathcal{E}^e(R)) & \xrightarrow{\alpha} & X(\mathcal{F}^f(R)) \end{array}$$

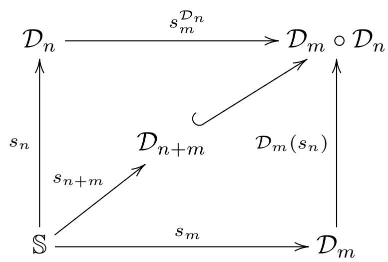
commutes where the vertical arrows come from the natural identifications. Here $\mathcal{E}^e(R)$ denotes the ring $\mathcal{E}(R)$ with the k-algebra structure induced by the functor $\mathcal{E}$ together with $e : k \to \mathcal{E}(k)$. It is clear from this interpretation that if $\beta : \mathcal{F} \to \mathcal{G}$ is another map of finite free-$\mathbb{S}$-algebras and $g : k \to \mathcal{G}(k)$ makes k into a $\mathcal{G}$-ring with $\beta^k \circ f = g$, then $\widehat{\beta \circ \alpha} = \widehat{\beta} \circ \widehat{\alpha}$.

Specialising to $k = C$ and $\alpha = s_n : \mathbb{S} \to \mathcal{D}_n$, we obtain a *constant* section map $z_n := \widehat{s_n} : X = \tau(X, \mathbb{S}, \mathrm{id}_k) \to \tau(X, \mathcal{D}_n, E_n) = \tau_n X$. By Proposition 4.18(c) of [13], the map $z_n : X \to \tau_n X$ is a closed immersion. That $\underline{zX} := (z_n X)_{n<\omega}$ is a σ-dominant $\underline{\mathcal{D}}$-subscheme of X follows from the functoriality in the $\mathbb{S}$-algebra of the prolongation space construction. For example, from the commutativity of the diagram

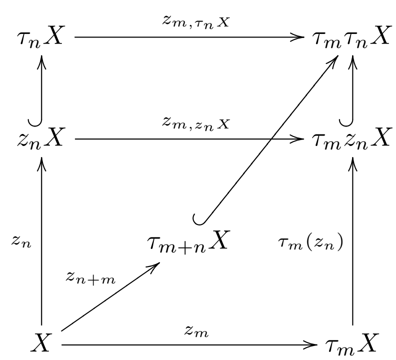
$$\begin{array}{ccc} & & \mathcal{D}_{n+1} \\ & {\scriptstyle s_{n+1}}\nearrow & \\ \mathbb{S} & & \downarrow{\scriptstyle f_n} \\ & {\scriptstyle s_n}\searrow & \\ & & \mathcal{D}_n \end{array}$$

we deduce that $\widehat{f_n}$ takes $z_{n+1}X$ to $z_n X$.

Likewise, from the commutativity of



and the fact that z_m is a natural transformation (see Proposition 4.18 of [13]), we obtain the following commutative diagram



from which we deduce that via the inclusion of $\tau_{n+m}X$ in $\tau_m\tau_n X$, $z_{n+m}X$ is contained in $\tau_m z_n X$. Thus, $\underline{zX}$ is a $\underline{\mathcal{D}}$-subscheme of X.

Finally, to check that this $\underline{\mathcal{D}}$-scheme is σ-dominant, observe that if $\pi_i : \mathcal{D} \to \mathbb{S}$ is a projection map corresponding to one of the distinguished endormorphisms σ_i, the commutative diagram of $\mathbb{S}$-algebras $\mathbb{S} \underset{\pi_i}{\overset{s}{\rightleftarrows}} \mathcal{D}$ shows that $\hat{\pi}_i \circ z = \mathrm{id}$. It follows that for every $n > 0$, the restriction of $\hat{\pi}_i$ to $z_n X = z(z_{n-1}X)$ is dominant onto $z_{n-1}X = (z_{n-1}X)^{\sigma_i}$, where the final equality is because σ_i is the identity on C.

With the next lemma we show that the $\mathbb{U}$-rational points of $\underline{zX}$ are exactly the C-rational points of X.

Lemma 6.13. $\underline{zX}(\mathbb{U}) = X(C)$

Proof. Suppose $p \in \underline{zX}(\mathbb{U})$. In particular, $\nabla(p) = z_1(p)$. Now as ∇ is induced by e and z_1 is induced by $s_1 = s$ and $\nabla(p) = z_1(p)$ (as both $\nabla(p)$ and $z_1(p)$ lie on $z_1 X(\mathbb{U})$ over p but there is only one point in this fibre as z_1 is a morphism), we conclude that $e(p) = s(p)$. That is, $p \in X(C)$.

For the converse, by Lemma 6.12 the maps s_n and E_n agree on C for all n. Thus, if $p \in X(C)$, then $\nabla_n(p) = z_n(p)$ for all natural numbers n. In particular, $\nabla_n(p) \in z_n X(\mathbb{U})$ for all n so that $p \in \underline{zX}(\mathbb{U})$. □

We show now that injectivity of a map of algebraic groups on the C-points is reflected by the injectivity of higher prolongations of the map when restricted to the constant section.

Lemma 6.14. *Let G be an algebraic group over C and let $\Lambda : G_{\mathbb{U}} \to H$ be a morphism of algebraic groups over $\mathbb{U}$ where $G_{\mathbb{U}}$ is the base change of G to $\mathbb{U}$. If the* *kernel of the restriction of* Λ *to* $G(C)$ *is trivial, then for* $n \gg 0$*, the restriction of* $\tau_n(\Lambda) : \tau_n(G_{\mathbb{U}}) \to \tau_n(H)$ *to* $z_n G(\mathbb{U})$ *has trivial kernel.*

Proof. For each natural number n, define $K_n := \ker(\tau_n(\Lambda) \restriction z_n G)$ as an algebraic subgroup of $z_n G$. We aim to show that the K_n's are eventually trivial. Define $V_n := \bigcap_{m \geq n} \widehat{f}_{m,n}(K_m)$ where $\widehat{f}_{m,n} : \tau_m G \to \tau_n G$ is the map in the projective system defining the $\underline{\mathcal{D}}$-variety $\underline{G} = (\tau_n G)$. Note that $\widehat{f}_{m,n}$ is a morphism of algebraic groups because on points it corresponds to $G\big(\mathcal{D}_m(\mathbb{U})\big) \to G\big(\mathcal{D}_n(\mathbb{U})\big)$ induced by the ring homomorphism $f^{\mathbb{U}}_{m,n} : \mathcal{D}_m(\mathbb{U}) \to \mathcal{D}_n(\mathbb{U})$. Since each K_m is an algebraic group, V_n is an algebraic subgroup of $z_n G$. If the K_n were non-trivial for arbitrarily large n, then because $\widehat{f}_{m,n}$ maps K_m to K_n, the V_n would also be non-trivial. So it suffices to show that $V_n = \{1\}$ for all n.

From the descending intersection defining V_n, note that $\widehat{f}_n : V_n \to V_0$ is surjective. Thus, $V_n = z_n V_0$. We now prove that $\underline{V} = (V_n)$ is a σ-dominant $\underline{\mathcal{D}}$-variety. Note that if we knew V_0 were defined over the constants, this would have followed automatically from our discussion of constant sections.

By definition, $K_{n+1} = \ker \tau_{n+1}(\Lambda) \restriction z_{n+1} G$. On the other hand, note that whenever one has a morphism of algebraic groups $\rho : A \to B$, then $\tau(\ker \rho) = \ker(\tau\rho)$. Indeed, this follows from the fact that, for any algebra R, the identification of $\tau A(R)$ with $A\big(\mathcal{D}^e(R)\big)$ identifies $\tau\rho$ with ρ evaluated on $A\big(\mathcal{D}^e(R)\big)$. Hence $\tau K_n = \ker(\tau\tau_n(\Lambda) \restriction \tau z_n G)$. Thus, from the inclusion $z_{n+1} G \hookrightarrow \tau z_n G$, we obtain an inclusion $K_{n+1} \hookrightarrow \tau K_n$. Taking intersections, we see that V_{n+1} is included in τV_n. Thus, $\underline{V} = (V_n)$ is a $\underline{\mathcal{D}}$-subvariety of $\underline{zG} = (z_n G)$. For σ-dominance, fixing $i = 0, \ldots, t$ and $n > 0$, we have

$$\begin{array}{rcll} V_{n-1} & = & \hat{\pi}_i\big(zV_{n-1}\big) & \text{as } \hat{\pi}_i \circ z = \mathrm{id} \\ & = & \hat{\pi}_i(V_n) & \\ & \subseteq & V_{n-1}^{\sigma_i} & \text{as } \hat{\pi}_i : \tau(V_{n-1}) \to V_{n-1}^{\sigma_i} \end{array}$$

And so we have equality throughout, and $\underline{V}$ is a σ-dominant $\underline{\mathcal{D}}$-variety.

However, $\underline{V}(\mathbb{U}) \subseteq \underline{zG}(\mathbb{U}) \cap \ker(\Lambda)(\mathbb{U}) = G(C) \cap \ker(\Lambda)(\mathbb{U}) = \{1\}$, the penultimate equality being Lemma 6.13. Thus, by Proposition 6.10, we must have $V_n(\mathbb{U}) = \{1\}$ for all n, as desired. □

We are now in a position to show that the jet spaces of a finite dimensional $\underline{\mathcal{D}}$-variety are always finite dimensional C-vector spaces.

Proposition 6.15. *Suppose* $\underline{Z} = (Z_n)$ *is a* $\underline{\mathcal{D}}$*-subvariety of* X *over* k *such that* $\dim Z_n$ *is bounded independently of* n*. Suppose* $p \in \underline{Z}(\mathbb{U})$ *is* k*-generic. Then for each* $m \geq 1$*,* $\mathrm{Jet}^m_{\underline{\mathcal{D}}}(\underline{Z})_p(\mathbb{U})$ *is a finite dimensional* C*-vector space.*

Proof. Fix $m \geq 1$. As explained at the beginning of this section, see (7) in particular, we have that $\mathrm{Jet}^m_{\underline{\mathcal{D}}}(\underline{Z})_p = (T_n)_{n<\omega}$ where $T_n := \phi\big(\,\mathrm{Jet}^m(Z_n)_{\nabla_n(p)}\big)$. As $\dim Z_n$ is bounded, so is $\dim(\mathrm{Jet}^m(Z_n)_{\nabla_n(p)})$, and hence also $\dim T_n$. Let N be a bound on $\dim T_n$. We will show that the C-dimension of $\mathrm{Jet}^m_{\underline{\mathcal{D}}}(\underline{Z})_p(\mathbb{U})$ is bounded by N.

Toward a contradiction, set $\mu := N + 1$ and suppose $\lambda_1, \ldots, \lambda_\mu \in \mathrm{Jet}^m_{\underline{\mathcal{D}}}(\underline{Z})_p(\mathbb{U})$ are C-linearly independent. Consider the map $g : \mathbb{G}_{\mathrm{a}}^\mu \to \mathrm{Jet}^m(X)_p$ given by $(x_1, \ldots, x_\mu) \mapsto \sum_{i=1}^{\mu} x_i \lambda_i$. Note that g restricted to the C-points is injective, and hence Lemma 6.14 will apply. Recall that by Lemma 6.13, $\mathbb{G}_{\mathrm{a}}^\mu(C) = \underline{z\mathbb{G}_{\mathrm{a}}^\mu}(\mathbb{U})$. We claim that for each n, $\tau_n g$ restricts to a morphism from $z_n \mathbb{G}_{\mathrm{a}}^\mu$ to T_n. It suffices to check that $\tau_n g$ takes a k-generic point of $z_n \mathbb{G}_{\mathrm{a}}^\mu$ to T_n. But if we take a k-generic point of $\underline{z\mathbb{G}_{\mathrm{a}}^{\mu}}$, say $q \in \mathbb{G}_{\mathrm{a}}^{\mu}(C)$, then $\nabla_n(q)$ is k-generic in $z_n\mathbb{G}_{\mathrm{a}}^{\mu}$, and $\tau_n g\big(\nabla_n(q)\big) = \nabla_n\big(g(q)\big)$ by Proposition 4.7(a) of [13]. This latter is in T_n since $g(q) \in \operatorname{Jet}^m_{\underline{\mathcal{D}}}(\underline{Z})_p(\mathbb{U})$. So $\tau_n g$ does restrict to a morphism from $z_n\mathbb{G}_{\mathrm{a}}^{\mu}$ to T_n. Lemma 6.14 tells us that this morphism is an embedding, for sufficiently large n. Hence eventually T_n has dimension at least μ, which is a contradiction. □

6.4. **A canonical base property.** Using jet spaces we obtain a description of canonical bases and deduce therefrom the Zilber dichotomy for finite-dimensional rank one types. Canonical bases for simple theories were introduced as hyperimaginary elements in [7], to which we refer the reader for further details. Here we show that the canonical bases in $\mathcal{D}$-CF_0 are interalgebraic with infinite sequences of real elements. As might be expected from our description of types in §5.3, the canonical base of a type $\operatorname{tp}(a/L)$ will need to take into account not just the L-loci of the $\nabla_n(a)$, but indeed of the $\Theta_r(a)$, where recall that

$$\Theta_r(a) := \big(\theta a : \theta \text{ a word of length} \leq r \text{ on } \{\partial_1, \dots, \partial_\ell, \sigma_1^{-1}, \dots, \sigma_t^{-1}\}\big).$$

Theorem 6.16. *Suppose L is an algebraically closed inversive $\mathcal{D}$-subfield and a is a finite tuple from $\mathbb{U}$. Let $\underline{Z} := \underline{\mathcal{D}}$-locus$(a/L)$ in the sense that $Z_n = \operatorname{locus}(\nabla_n a/L)$ for all $n < \omega$. For each $r < \omega$, let $\Theta_r\underline{Z} := \big(\operatorname{locus}(\nabla_n\Theta_r a)/L)\big)_{n<\omega}$. Then*

$$\operatorname{Cb}(a/L) \subseteq \operatorname{acl}\left(\{a\} \cup \bigcup_{m\geq 1, r\geq 0} \operatorname{Jet}^m_{\underline{\mathcal{D}}}(\Theta_r\underline{Z})_{\Theta_r a}(\mathbb{U})\right)$$

If $\operatorname{tp}(a/L)$ *is finite-dimensional then*

$$\operatorname{Cb}(a/L) \subseteq \operatorname{acl}\left(\{a\} \cup \bigcup_{m\geq 1, r\geq 0} \operatorname{Jet}^m_{\underline{\mathcal{D}}}(\nabla_r\underline{Z})_{\nabla_r(a)}(\mathbb{U})\right)$$

Proof. Note that the $\underline{\mathcal{D}}$-locus of a tuple over an algebraically closed inversive $\mathcal{D}$-field will always be a σ-dominant absolutely irreducible $\underline{\mathcal{D}}$-subvariety. Hence this is the case for $\underline{Z}$, $\Theta_r\underline{Z}$, and $\nabla_r\underline{Z}$.

Let $K \subseteq L$ be the inversive $\mathcal{D}$-subfield generated by the minimal fields of definition of all the $\operatorname{locus}(\Theta_r a/L)$, as $r < \omega$ varies.

Claim 6.17. $\operatorname{Cb}(a/L) \subseteq \operatorname{acl}(K)$ *and* $K \subseteq \operatorname{dcl}\big(\operatorname{Cb}(a/L)\big)$

Proof of Claim 6.17. By Lemma 5.11 we have that $a \mathop{\smile\!\!\!\!\!\!\downarrow}_K L$. We also know that $\operatorname{tp}(a/K^{\mathrm{alg}})$ is an amalgamation base because the independence theorem holds over algebraically closed sets. It follows that $\operatorname{Cb}(a/L)$ is in the definable closure of K^{alg} and hence in the algebraic closure of K.

For the other containment, let $L_0 = \operatorname{dcl}\big(\operatorname{Cb}(a/L)\big)$. Then, as $a \mathop{\smile\!\!\!\!\!\!\downarrow}_{L_0} L$, we have that $\operatorname{locus}(\Theta_r a/L) = \operatorname{locus}(\Theta_r a/L_0)$, for each $r < \omega$. Hence the minimal field of definition of each $\operatorname{locus}(\Theta_r a/L)$ is a subfield of L_0. It follows that K, which is in the definable closure of these minimal fields of definition, must also be contained in L_0, as desired. , so $K \subseteq L_0$. □

Given the above claim it suffices to show that K is in the definable closure of $\{a\} \cup \bigcup_{m\geq 1, r\geq 0} \operatorname{Jet}^m_{\underline{\mathcal{D}}}(\Theta_r\underline{Z})_{\Theta_r a}(\mathbb{U})$. Suppose that α is an automorphism that fixes a and all the $\operatorname{Jet}^m_{\underline{\mathcal{D}}}(\Theta_r\underline{Z})_{\Theta_r a}(\mathbb{U})$ pointwise. Then α takes L to $L' := \alpha(L)$ and it takes

$\Theta_r\underline{Z}$ to the $\underline{\mathcal{D}}$-suvariety $\underline{Y}^{(r)} := \big(\mathrm{locus}(\nabla_n\Theta_r a)/L')\big)_{n<\omega}$. Note that $\Theta_r a$ is generic in $\Theta_r\underline{Z}$ over L and in $\underline{Y}^{(r)}$ over L'. Note also that

$$\mathrm{Jet}^m_{\underline{\mathcal{D}}}(\nabla_s\Theta_r\underline{Z})_{\nabla_s(\Theta_r a)}(\mathbb{U}) = \mathrm{Jet}^m_{\underline{\mathcal{D}}}(\nabla_s\underline{Y}^{(r)})_{\nabla_s(\Theta_r a)}(\mathbb{U})$$

for all $m \geq 1$ and $s \geq 0$. Indeed, there is a co-ordinate projection taking $\Theta_{r+s}(a)$ to $\nabla_s(\Theta_r a)$ that will induce a definable surjection from $\mathrm{Jet}^m_{\underline{\mathcal{D}}}(\Theta_{r+s}\underline{Z})_{\Theta_{r+s}(a)}(\mathbb{U})$ to $\mathrm{Jet}^m_{\underline{\mathcal{D}}}(\nabla_s\Theta_r\underline{Z})_{\nabla_s(\Theta_r a)}(\mathbb{U})$, and since the former is fixed pointwise by α, so is the latter. It follows by Fact 6.11, then, that $\Theta_r\underline{Z} = \underline{Y}^{(r)}$. In particular, α fixes $\mathrm{locus}(\Theta_r a/L)$ for all $r \geq 0$. Hence $\alpha\restriction_K = \mathrm{id}$, as desired.

In the finite-dimensional case we can carry out the same argument but with ∇_r instead of Θ_r. That is, we set K to be the inversive $\mathcal{D}$-field generated by the minimal fields of definition of $\mathrm{locus}(\nabla_r a/L)$, as r varies. Using Corollary 6.3 instead of Lemma 5.11, the above argument goes through with this K, and we get the desired description of the canonical base in the finite-dimensional case. □

The following is a generalisation of Theorems 1.1 and 1.2 of [16].

Corollary 6.18 (The canonical base property for finite-dimensional types)**.** *Suppose* $\mathrm{tp}(a/k)$ *is finite-dimensional and* L *is an algebraically closed inversive* $\mathcal{D}$*-field extending* k *such that* $\mathrm{Cb}(a/L)$ *is interalgebraic with* L *over* k*. Then* $\mathrm{tp}(L/k\langle a\rangle)$ *is almost internal to the constants.*

Proof. Let $\underline{X} := \underline{\mathcal{D}}\text{-locus}(a/k)$ and $\underline{Z} := \underline{\mathcal{D}}\text{-locus}(a/L)$. The finite-dimensionality of $\mathrm{tp}(a/k)$ implies in particular that for each $r < \omega$, $\dim\mathrm{locus}(\nabla_n\nabla_r a/k)$ is bounded independently of n, and so by Proposition 6.15, for each $m \geq 1$, $\mathrm{Jet}^m_{\underline{\mathcal{D}}}(\nabla_r\underline{X})_{\nabla_r(a)}(\mathbb{U})$ is a finite dimensional C-vector space. Let B be a countable set that contains a C-basis for $\mathrm{Jet}^m_{\underline{\mathcal{D}}}(\nabla_r\underline{X})_{\nabla_r(a)}(\mathbb{U})$, for all m and r. As these jet spaces are type-definable over $k\langle a\rangle$, we can choose B so that $B \mathop{\smile\!\!\!\!\!\mid}_{k\langle a\rangle} L$. Then

$$\begin{aligned} L &\subseteq \mathrm{acl}\left(k\langle a\rangle \cup \bigcup_{m\geq 1, r\geq 0}\mathrm{Jet}^m_{\underline{\mathcal{D}}}(\nabla_r\underline{Z})_{\nabla_r a}(\mathbb{U})\right) && \text{by Theorem 6.16} \\ &\subseteq \mathrm{acl}\left(k\langle a\rangle \cup B \cup C\right) && \text{as the } \mathrm{Jet}^m_{\underline{\mathcal{D}}}(\nabla_r\underline{Z})_{\nabla_r(a)}(\mathbb{U}) \ \leq\ \mathrm{Jet}^m_{\underline{\mathcal{D}}}(\nabla_r\underline{X})_{\nabla_r(a)}(\mathbb{U}) \end{aligned}$$

So $\mathrm{tp}(L/k\langle a\rangle)$ is almost internal to the constant field C, as desired. □

Corollary 6.19 (The Zilber dichotomy for finite-dimensional types)**.** *Suppose* p *is a finite-dimensional* SU*-rank one type over a substructure of some model of* $\mathcal{D}\text{-}\mathrm{CF}_0$*. Then* p *is either one-based or almost internal to the constants.*

Proof. Work in a saturated model $(\mathbb{U}, \partial)$ of $\mathcal{D}\text{-}\mathrm{CF}_0$, and suppose $p = \mathrm{tp}(a/k)$ for some inversive $\mathcal{D}$-field k. If p is not one-based then there exists $d \in \mathrm{dcl}(k \cup p^{\mathbb{U}})$ and an algebraically closed inversive $\mathcal{D}$-field L extending k such that $\mathrm{Cb}(d/L)$ is interalgebraic with L over k, but $\mathrm{tp}(L/kd)$ is nonalgebraic. Since $\mathrm{tp}(d/L)$ is finite-dimensional the CBP (Corollary 6.18) applies and we have that $\mathrm{tp}(L/kd)$ is almost C-internal. So $L \subseteq \mathrm{acl}(kBdc)$ where $L \mathop{\smile\!\!\!\!\!\mid}_{kd} B$ and c is a tuple from C. Hence $L \not\mathop{\smile\!\!\!\!\!\mid}_{kBd} c$. On the other hand, being (interalgebraic over k with) the canonical base, L is in the definable closure of a Morley sequence in $\mathrm{tp}(d/L)$. But as $d \in \mathrm{dcl}(k \cup p^{\mathbb{U}})$, this implies that $L \subseteq \mathrm{acl}(ka_1 \dots a_n)$ for some independent realisations $a_1, \dots, a_n$ of p, which we may assume to be independent of Bd over k. So $(a_1, \dots, a_n) \not\mathop{\smile\!\!\!\!\!\mid}_{kBd} c$, and hence for some $i < n$, $a_{i+1} \in \operatorname{acl}(kBda_1 \dots a_i c)$. But as $a_{i+1} \mathop{\smile\!\!\!\!\!\!\mid}_k Bda_1 \dots a_i$ by choice, this witnesses that p is almost internal to C. $\square$

Remark 6.20. We do not know if the restriction of the above corollary to finite-dimensional types is necessary. In the case of partial differential fields, by which we mean differentially closed field of characteristic zero for finitely many commuting derivations, it follows from the analysis of regular non one based types in [12] that those of rank one are in fact finite-dimensional. (This was observed first in the difference-differential case by Bustamante [1].) We expect the same to hold here.

7. Appendix: On the assumptions

In proving the existence of the model companion we restricted ourselves to characteristic zero, and we also imposed on A the properties described in Assumptions 4.1. In this appendix, we discuss the extent to which these restrictions are necessary.

7.1. **No model companion in positive characteristic.** To begin with, in most cases the restriction to characteristic zero is necessary. While model companions are known to exist in positive characteristic for the differential and difference cases, at the level of generality considered in this paper model companions do not necessarily exist in characteristic $p > 0$. We will prove this by showing that the condition of having a pth root in some $\mathcal{D}$-field extension is not in general first-order.

For the time being Assumptions 4.1 remain in place.

Proposition 7.1. *Suppose (K, ∂) is a $\mathcal{D}$-field of characteristic $p > 0$ and $a \in K$. Then the following are equivalent:*

(i) *There is a $\mathcal{D}$-field extension of K in which a has a pth root.*
(ii) *For each $n < \omega$, $E_n(a) \in \mathcal{D}_n^p(K)$. Here $\mathcal{D}_n^p$ is the $\mathbb{S}$-algebra scheme that is the image of $\mathcal{D}_n$ under the $\mathbb{S}$-algebra morphism of raising to the power p.*

Proof. First of all, note that as raising to the pth power is an algebraic group homomorphism it is a closed morphism. Hence $E_n(a) \in \mathcal{D}_n^p(K)$ if and only if $E_n(a)$ is a pth power in $\mathcal{D}_n(K^{\text{alg}})$, if and only if $E_n(a)$ is a pth power in $\mathcal{D}_n(L)$ for some field extension L of K.

(i) $\Longrightarrow$ (ii). Let (L, ∂) extend (K, ∂) with $b \in L$ such that $b^p = a$. Then

$$E_n(b)^p = E_n(b^p) = E_n(a)$$

in $\mathcal{D}(L)$ showing that $E_n(a)$ is a pth power for each $n < \omega$, as desired.

(ii) $\Longrightarrow$ (i). Passing to an elementary extension if necessary, we may assume that K is $\aleph_0$-saturated as a field. Now for each n there exists $b_n \in \mathcal{D}_n(K^{\text{alg}})$ such that $b_n^p = E_n(a)$. Since the minimal polynomial of b_0 over K is $x^p - a$, we can extend $E_n : K \to \mathcal{D}_n(K)$ to an A-algebra homomorphism $E_n' : K(b_0) \to \mathcal{D}_n(K^{\text{alg}})$ by setting $E_n'(b_0) := b_n$.

Now, note that $f_n(b_n)^p = E_{n-1}(a)$, and so by $\aleph_0$-saturation, we may assume that $f_n(b_n) = b_{n-1}$ for each $n > 0$. In other words, there exists $\{b^{(\alpha)} : \alpha \in L_{<\omega}\} \subseteq K^{\text{alg}}$ such that for every n

$$b_n = \sum_{\alpha \in L_n} b^{(\alpha)} \epsilon_\alpha$$

where $(\epsilon_\alpha : \alpha \in L_n)$ is the basis for $\mathcal{D}_n(A)$ over A induced by ψ_n from the standard basis for A^{L_n}. Let $L = K\big(b^{(\alpha)}\big)_{\alpha \in L_{<\omega}}$ and extend $e : K \to \mathcal{D}(K)$ to L by

$$e'(b^{(\alpha)}) := \sum_{j=0}^{\ell-1} b^{(j^\frown \alpha)} \epsilon_j$$

That this is an A-algebra homomorphism extending e follows from the fact that the E'_n defined above were A-algebra homomorphisms extending E_n. That $\pi \circ e = \mathrm{id}$ is clear from construction. We have thus given L a $\mathcal{D}$-field structure extending (K, ∂), and we have a pth root of a in L, namely b_0. $\square$

Using Proposition 7.1 we show that under a weak hypothesis on $\mathcal{D}(A)$, the theory of $\mathcal{D}$-fields does not have a model companion.

Proposition 7.2. *If p is a prime and there is some nilpotent $\epsilon \in \mathcal{D}(A)$ with $\epsilon^p \neq 0$, then the theory of $\mathcal{D}$-fields of characteristic p does not have a model companion.*

In particular, the class of fields of characteristic p equipped with a truncated higher derivation of length greater than p does not have a model companion– see Example 3.6(b).

Proof. Suppose the theory of $\mathcal{D}$-fields does have a model companion, T, and seek a contradiction. By Proposition 7.1, in an existentially closed $\mathcal{D}$-field the partial type $\Phi(x) := \{E_n(x) \in \mathcal{D}_n^p\}_{n=0}^\infty$ is equivalent to the formula $(\exists y) y^p = x$. As every model of T is existentially closed, this equivalence is entailed by T. We get by compactness that there is some $d \geq 0$ such that in every model of T, $\bigwedge_{n=0}^d (E_n(x) \in \mathcal{D}_n^p)$ implies $\Phi(x)$. But as these formulas are quantifier-free, and every $\mathcal{D}$-field embeds into a model of T, we have that in every $\mathcal{D}$-field (L, ∂)

$$\bigwedge_{n=0}^{d} E_n(x) \in \mathcal{D}_n^p(L) \implies \bigwedge_{n=0}^{\infty} E_n(x) \in \mathcal{D}_n^p(L) \tag{8}$$

We will construct a counterexample to this claim.

Since $\mathcal{D}$ is an $\mathbb{S}$-algebra, it is itself of characteristic p. Hence, the Frobenius defines a morphism of ring schemes $F : \mathcal{D} \to \mathcal{D}$ given on points as $F : \mathcal{D}(R) \to \mathcal{D}(R)$ via $a \mapsto a^p$ where the p^{th} power is taken with respect to the multiplication in $\mathcal{D}$. Let $\mathcal{N}$ be the nilradical of $\mathcal{D}$ considered as an algebraic subgroup scheme of $(\mathcal{D}, +)$. Visibly, $\mathcal{N}$ is mapped back to itself by F. Since the nilradical is non-trivial, the kernel of $F : \mathcal{N} \to \mathcal{N}$ has positive dimension. Hence, its image has dimension strictly less than $\dim \mathcal{N}$. So $\mathcal{N}(A^{\text{alg}}) \smallsetminus \mathcal{D}(A^{\text{alg}})^p$ is nonempty. As $\mathcal{N}(A^{\text{alg}})$ is generated by $\mathcal{N}(A)$ as a vector space over A^{alg}, it follows that $\mathcal{N}(A) \smallsetminus \mathcal{D}(A^{\text{alg}})^p$ is nonempty. Fix $\eta \in \mathcal{N}(A) \smallsetminus \mathcal{D}(A^{\text{alg}})^p$.

Let $m = d+1$ and let $L := A(x_1, \ldots, x_m)$ be the field of rational functions in m variables over A. Define $e : A[x_1, \ldots, x_m] \to \mathcal{D}(L)$ by $e(x_i) := x_i + x_{i+1}\epsilon^p$ for $i < m$ and $e(x_m) := x_m + \eta$. Note that as η and ϵ are nilpotent, for any $f \in A[x_1, \ldots, x_m]$, $e(f) = f + \nu$ where $\nu \in \mathcal{N}(L)$. It follows that $e(f)$ is invertible in $\mathcal{D}(L)$ if f is nonzero. We may therefore extend e to a $\mathcal{D}$-field structure on L. For each natural number n, let $E_n : L \to \mathcal{D}_n(L)$ be the map obtained from e by iteration.

We first show by induction on $n < m$ that $E_n(x_1) = \sum_{j=0}^n x_{j+1} a_{j,n}$ where each $a_{j,n} \in \mathcal{D}_n(A)^p$. For $n = 1$ this is clear: $e(x_1) = x_1 + x_2\epsilon^p$. Assume the result is proved for $n-1$ and compute

$$E_n(x_1) = \sum_{j=0}^{n-1} x_{j+1}a_{j,n-1} \otimes 1 + \sum_{j=0}^{n-1} x_{j+2}a_{j,n-1} \otimes \epsilon^p$$

which is of the required form.

It follows from the above that $E_n(x_1) \in \mathcal{D}_n(L^{\mathrm{alg}})^p = \mathcal{D}_n^p(L)$, for all $n < m$.

On the other hand,

$$E_m(x_1) = \sum_{j=0}^{m-1} x_{j+1}a_{j,m-1} \otimes 1 + \sum_{j=0}^{m-2} x_{j+2}a_{j,m-1} \otimes \epsilon^p + a_{m-1,m-1} \otimes \eta$$

Since the last summand in the above, namely $a_{m-1,m-1} \otimes \eta$, is not in $\mathcal{D}_n^p(L)$, but all the others are, it follows that $E_m(x_1) \notin \mathcal{D}_n^p(L)$. This contradicts (8). □

7.2. **Removing Assumptions 4.1.** On the other hand, if we restrict to characteristic zero, then model companions exist even in the absence of Assumptions 4.1. As we do not yet see a pressing reason to develop the theory in full generality, we restrict ourselves here to a sketch of a proof.

We explain first why Assumption 4.1(ii) is unnecessary. That is, we will describe the model companion still assuming that A is a field of characteristic zero, but *without* assuming in the decomposition $\mathcal{D}(A) = \prod_{i=0}^t B_i$ that the residue field of each B_i is A.

For each i fix an irreducible polynomial $P_i(x)$ of degree d_i such that the residue field of B_i is the finite extension $A[x]/(P_i)$. Denote by $\mathcal{E}_i$ the $\mathbb{S}$-algebra scheme such that $\mathcal{E}_i(R) = R[x]/(P_i)$ for any A-algebra R. In particular this fixes a basis for $\mathcal{E}_i$, namely the image of $\{1, x, \dots, x^{d_i-1}\}$. Note that we may assume $\mathcal{E}_0 = \mathbb{S}$; one of the B_i will still correspond to the kernel of π and hence will have residue field A.

We have as before $\mathcal{D} = \prod_{i=0}^t \mathcal{D}_i$ and $\theta_i : \mathcal{D} \to \mathcal{D}_i$, but now $\rho_i : \mathcal{D}_i \to \mathcal{E}_i$ are the $\mathbb{S}$-algebra homomorphisms which when evaluated at A are the residue maps on B_i. Note that when evaluated on another A-algebra R, even if R is a field extension, $\mathcal{D}_i(R)$ need no longer be a local ring and $\mathcal{E}_i(R)$ may no longer be a field. Nevertheless, $\rho_i^R : \mathcal{D}_i(R) \to \mathcal{E}_i(R)$ will be a surjective ring homomorphism, it is obtained from the residue map ρ_i^A by base change to R. As before we set $\pi_i := \rho_i \circ \theta_i : \mathcal{D} \to \mathcal{E}_i$.

Suppose we are given a $\mathcal{D}$-ring (R, ∂). For each $i = 0, \dots, t$, instead of an associated endomorphism we now only have the *associated A-algebra homomorphisms* $\sigma_i := \pi_i^R \circ e : R \to \mathcal{E}_i(R)$, which with respect to the basis for $\mathcal{E}_i$ fixed above can be written as

$$\sigma_i(a) = \sum_{j=0}^{d_i-1} \alpha_{ij}(a)x^j$$

The $\alpha_{ij} : R \to R$ will be A-linear maps that are 0-definable in (R, ∂); indeed they are fixed A-linear combinations of the original operators ∂. (We are working in the language $\mathcal{L}_{\mathcal{D}}$ of A-algebras equipped with the operators $\partial_1, \dots, \partial_{\ell-1}$, see page 6.) Note that $d_0 = 1$, $\pi_0 = \pi$, and $\sigma_0 = \alpha_{0,0} = \mathrm{id}$.

Extending our earlier notation we now let $\mathcal{K}$ be the class of $\mathcal{D}$-rings (R, ∂) such that R is an integral A-algebra and for each $i = 1, \dots, t$, $\sigma_i : R \to \mathcal{E}_i(R)$ is *injective and has no non-trivial zero divisors in its image.* Notice that as $\mathcal{E}_i(R)$ need not be an integral domain, this latter constraint on the σ_i is not vacuous. The class $\mathcal{K}$ is universally axiomatisable; this follows from the fact that the α_{ij} are quantifier-free definable, as is the ring structure on R^{d_i} induced by the ring structure on $\mathcal{E}_i(R)$ via the basis $\{1, x, \cdots, x^{d_i-1}\}$.

More generally, the class $\mathcal{M}$ is now the class of triples (R, S, ∂) where $R \subseteq S$ are integral A-algebras and $\partial = (\partial_1, \ldots, \partial_{\ell-1})$ is a sequence of maps from R to S such that $e : R \to \mathcal{D}(S)$ given by $e(r) := r\epsilon_0 + \partial_1(r)\epsilon_1 + \cdots + \partial_{\ell-1}(r)\epsilon_{\ell-1}$ has the following properties:

(i) e is an A-algebra homomorphism,
(ii) for each $i = 1, \ldots, t$, $\sigma_i := \pi_i^S \circ e : R \to \mathcal{E}_i(S)$ is injective and has no non-trivial zero divisors in its image.

Note that $\sigma_0 := \pi_0^S \circ e = \pi^S \circ e : R \to S$ is then the identity on R.

Lemma 7.3. (a) *If $(R, L, \partial) \in \mathcal{M}$ with L a field, then we can (uniquely) extend ∂ to the fraction field F of R so that $(F, L, \partial) \in \mathcal{M}$.*
(b) *Suppose $(F, L, \partial) \in \mathcal{M}$ where F and L are fields and L is algebraically closed. If $\sigma_i : F \to \mathcal{E}_i(L)$ are the embeddings associated to ∂, and σ_i' are extensions of σ_i to F^{alg}, then there is an extension ∂' of ∂ to F^{alg} with associated embeddings σ_i'.*

Proof. Part (a) is proved along the lines of Lemma 4.9. In order to extend e to the fraction field F we need to show that e takes nonzero elements of R to units in $\mathcal{D}(L)$. Equivalently, for each $i = 1, \ldots, t$, we need to show that $\theta_i \circ e$ takes nonzero elements of R to units in $\mathcal{D}_i(L)$. Note that as L is a field, $\mathcal{D}_i(L)$ is a product of local A-algebras and $\mathcal{E}_i(L)$ is the product of the residue fields of these local A-algebras. The units of $\mathcal{D}_i(L)$ are therefore precisely those elements whose images in $\mathcal{E}_i(L)$ under ρ_i are not zero divisors. What we therefore need to verify is that $\rho_i \circ \theta_i \circ e$ is injective and has no non-trivial zero divisors in its image. But $\rho_i \circ \theta_i \circ e = \sigma_i$ and the desired property is true since $(R, L, \partial) \in \mathcal{M}$. That the corresponding extension (F, L, ∂) lands back inside $\mathcal{M}$ is clear.

For part (b) it suffices to prove, as in the proof of Lemma 4.10, that for any $a \in F^{\text{alg}}$ we can extend e to $F(a)$ in such a way that $\pi_i^L \circ e(a) = \sigma_i'(a)$ for each $i = 0, \ldots, t$. Let $P(x) \in F[x]$ be the minimal polynomial of a over F and let $c_i := \sigma_i'(a) \in \mathcal{E}_i(L)$. Note that as $P(a) = 0$ but $\frac{d}{dx}P(a) \neq 0$ in F^{alg}, and since σ_i' is a ring homomorphism, we have that $P^{\sigma_i}(c_i) = 0$ while $\frac{d}{dx}P^{\sigma_i}(c_i)$ is a unit in $\mathcal{E}_i(L)$. We wish to lift this root to $\mathcal{D}_i(L)$. While it is not the case that $\rho_i^L : \mathcal{D}_i(L) \to \mathcal{E}_i(L)$ is the residue map of a local algebra, it is still surjective with nilpotent kernel. This is because the kernel of ρ_i^L is obtained from the kernel of ρ_i^A by tensoring with L over A, and the kernel of the latter is the maximal ideal of B_i which is nilpotent. Hence by a Hensel's Lemma type argument (see Theorem 7.3 of [5]) we can lift c_i to a root b_i of $P^{e_i}(x)$ in $\mathcal{D}_i(L)$. Then $b = (b_0, \ldots, b_t) \in \mathcal{D}(L)$ is a root of $P^e(x) \in \mathcal{D}(L)[x]$, and we can extend e to $F(a)$ by sending a to b. By construction $\pi_i^L e(a) = \sigma_i'(a)$. □

Suppose now that (K, ∂) is an algebraically closed $\mathcal{D}$-field. Recall that we wrote each $B_i = A[x]/(P_i)$. Let $b_{i1}, \ldots, b_{id_i}$ be the distinct roots of P_i in K. Then using these roots we can decompose $\mathcal{E}_i(K)$ into a power of K itself:

$$\mathcal{E}_i(K) = K[x]/(P_i) = \prod_{k=1}^{d_i} K[x]/(x - b_{ik}) = K^{d_i}$$

Composing the associated homomorphism σ_i with the co-ordinate projections we get a d_i-tuple of *associated endomorphisms* of K, $(\sigma_{i1}, \dots, \sigma_{id_i})$ where

$$\sigma_{ik} := \sum_{j=0}^{d_i-1} b_{ik}^j \alpha_{ij}$$

In fact, under the identification $\mathcal{E}_i(K) = K^{d_i}$ we have $\sigma_i = (\sigma_{i1}, \dots, \sigma_{id_i})$, and so by the *associated difference field* (K, σ) we mean the field K equipped with all of these endomorphisms. It then also makes sense to say that (K, ∂) is *inversive* if each σ_{ik} is an automorphism. It is important to note, though, that the σ_{ik}, while still definable in (K, ∂), are now not 0-definable but b_{ik}-definable. Note also that we have only defined the associated difference field of an *algebraically closed* $\mathcal{D}$-field.

Example 7.4. Consider Example 4.2 where

$$A = \mathbb{Q} \text{ and } \mathcal{D}(A) = \mathbb{Q} \times \mathbb{Q}[x]/(x^2 - 2)$$

Then $t = 1$ and the homomorphism associated to a $\mathcal{D}$-ring $(R, \partial_1, \partial_2)$ is $\sigma_1 : R \to R[x]/(x^2 - 2)$ where $\sigma_1(a) = \partial_1(a) + \partial_2(a)x$. So $\alpha_{10} = \partial_1$ and $\alpha_{11} = \partial_2$. If $(K, \partial_1, \partial_2)$ is an algebraically closed $\mathcal{D}$-field then the associated endomorphisms are $\sigma_{10} = \partial_1 + \sqrt{2}\partial_2$ and $\sigma_{11} = \partial_1 - \sqrt{2}\partial_2$.

Lemma 7.5. *Suppose $(F, L, \partial) \in \mathcal{M}$ where F and L are fields. Then there exists an algebraically closed extension K of L and an extension of ∂ to an inversive $\mathcal{D}$-field structure on K.*

Proof. Replacing L with L^{alg} we may assume that L is algebraically closed. We can thus write $\sigma_i = (\sigma_{i1}, \dots, \sigma_{id_i})$ where the embeddings $\sigma_{ik} : F \to L$ are obtained by composing $\sigma_i : F \to \mathcal{E}_i(L)$ with the kth projection in the decomposition $\mathcal{E}_i(L) = L^{d_i}$. We can extend these σ_{ik} to automorphisms σ'_{ik} of some algebraically closed $K \supseteq L$. So $\sigma'_i := (\sigma'_{i1}, \dots, \sigma'_{id_i}) : K \to \mathcal{E}_i(K)$ extends σ_i. Now, as in Lemma 4.11, we fix a transcendence basis B for K over F and easily extend ∂ to $F[B]$ so that $(F[B], K, \partial) \in \mathcal{M}$ and the associated homomorphisms are $\sigma'_i \restriction F[B]$. By Lemma 7.3(a) we can extend ∂ to $F(B)$ preserving this property. By Lemma 7.3(b) we can extend ∂ further to a $\mathcal{D}$-structure on $K = F(B)^{\text{alg}}$ in such a way that the associated homomorphisms are $\sigma'_1, \dots, \sigma'_t$, and hence the associated endomorphisms are the automorphisms σ'_{ik}. ◻

Suppose now that X is an irreducible affine variety over an algebraically closed $\mathcal{D}$-field K. For each $i = 0, \dots, t$, we have the abstract prolongations with the induced morphisms as constructed in §4 of [13]:

$$\tau(X, \mathcal{D}, e) \xrightarrow{\widehat{\theta}_i} \tau(X, \mathcal{D}_i, e_i) \xrightarrow{\widehat{\rho}_i} \tau(X, \mathcal{E}_i, \sigma_i)$$

But we also have, for each fixed $k = 1, \dots, d_i$, the morphism

$$\tau(X, \mathcal{E}_i, \sigma_i) \longrightarrow X^{\sigma_{ik}}$$

Indeed, the kth factor projection $\mathcal{E}_i(K) = K^{d_i} \to K$ induces a map from $X\big(\mathcal{E}_i(K)\big)$, where X is viewed as a scheme over $\mathcal{E}_i(K)$ via base change coming from $\sigma_i : K \to$

$\mathcal{E}_i(K)$, to $X^{\sigma_{ik}}(K)$.[5] Composing, we have for each i and k the morphism

$$\tau(X,\mathcal{D},e) \xrightarrow{\widehat{\pi_{ik}}} X^{\sigma_{ik}}$$

Moreover, if we set $F := A(b_{ik})_{1\leq i\leq t, 1\leq k\leq d_i}$, and X moves uniformly within an F-definable family of varieties, then so do the $\widehat{\pi_{ik}} : \tau(X,\mathcal{D},e) \to X^{\sigma_{ik}}$.

Now we can state the version of Theorem 4.6 without Assumption 4.1(ii).

Theorem 7.6. *Drop Assumptions 4.1, and assume only that A is a field of characteristic zero. Then $(K,\partial) \in \mathcal{K}$ is existentially closed if and only if*

I. *K is an algebraically closed field.*

II. *There exist distinct roots $b_{i,1},\ldots,b_{i,d_i}$ of P_i in K such that $\sigma_{ik} := \sum_{j=0}^{d_i-1} b_{ik}^j \alpha_{ij}$ is an automorphism of K, for all $i = 1,\ldots,t$ and $k = 1,\ldots,d_i$.*

III. *There exist distinct roots $b_{i,1},\ldots,b_{i,d_i}$ of P_i in K such that if X is an irreducible affine variety over K and $Y \subseteq \tau(X,\mathcal{D},e)$ is an irreducible subvariety over K such that $\widehat{\pi_{ik}}(Y)$ is Zariski dense in $X^{\sigma_{ik}}$ for all $i = 0,\ldots,t$ and $k = 1,\ldots,d_i$, then there exists $a \in X(K)$ with $\nabla(a) \in Y(K)$.*

The theory of $\mathcal{D}$-fields of characteristic zero thus admits a model companion.

Theorem 7.6 is proved just as Theorem 4.6 was, using Lemmas 7.3(a), 7.3(b), and 7.5 in place of 4.9, 4.10, and 4.11. That the given axioms are first-order also follows as before. We omit the details.

On the face of it, Assumption 4.1(i) is more serious than Assumption 4.1(ii), but we may reduce to the case where it holds. Indeed, if (K,∂) is a $\mathcal{D}$-field for which A is not necessarily a field, then by regarding $\mathcal{D}$ as a ring scheme over the field of fractions of the image of A in K, we may see (K,∂) as a $\mathcal{D}$-field in which Assumption 4.1(i) holds. That is, if we consider each possible way in which $\mathcal{D}(K)$ may split as a product of local rings via maps defined by linear equations defined over the algebraic closure of the field of fractions of the image of A, then we see that in every $\mathcal{D}$-field one of these splittings must hold. We obtain an axiomatisation in the absence of Assumptions 4.1 by taking each such possible form of the linear maps used for a splitting as an antecedent and then relativising the axiomatisation of Theorem 7.6.

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[5] This morphism does not follow formally from the comparing of prolongations done in [13] because the identification $\mathcal{E}_i(K) = K^{d_i}$ is over B, not A. To fit into the formalism of [13] we would thus require σ_i to be a B_i-algebra homomorphism, which it need not be as it may move the roots of P_i.

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