The Logarithmic Derivative and Model-Theoretic Analysability in Differentially Closed Fields

by

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AUTHOR'S DECLARATION

I hereby declare that I am the sole author of this thesis. This is a true copy of the thesis, including any required final revisions, as accepted by my examiners.

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Abstract

This thesis deals with internal and analysable types, mainly in the context of the stable theory of differentially closed fields. Two main problems are dealt with: the construction of types analysable in the constants with specific properties, and a criterion for a given analysable type to be actually internal to the constants.

For analysable types, the notion of canonical analyses is introduced. A type has a canonical analysis if all its analyses of shortest length are interalgebraic. Given a finite sequence of ranks, it is constructed, in the theory of differentially closed field, a type analysable in the constants such that it admits a canonical analysis and each step of the analysis is of the given rank. The construction of such a type starts from the well-known example of $\delta(\log\delta x) = 0$, whose generic type is analysable in the constants in 2 steps but is not internal to the constants. Along the way, techniques for comparing analyses in stable theories are developed, including in particular the notions of analyses by reductions and by coreductions.

The property of the $\log\delta$ function is further studied when the following question is raised: given a type internal to the constants, is its preimage under $\log\delta$, which is 2-step analysable in the constants, ever internal to the constants? The question is answered positively, and a criterion for when the preimage is indeed internal is proposed. Partial results are proven for this conjectured criterion, namely the cases where the group of automorphisms (the binding group) of the given internal type is additive, multiplicative, or trivial. In particular, the conjecture is resolved for generic types of equations of the form $\delta x = f(x)$ where f is a rational function over the constants. It is discovered that the related problem where $\log\delta$ is replaced by δ is significantly different, and the analogue of the conjecture fails in this case.

Also included in this thesis are two examples asked for in the literature: internality of a particular twisted D-group, and a 2-step analysable set with independent fibres.

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In memory of 叶敏玮 (Yè Mǐnwěi)*, PhD Candidate, and a friend,*
with whom I have discussed mathematics and everything.

Contents

List of symbols

δ	derivation
$\perp$	forking independence
$\restriction$	restriction of functions or types
$\models$	realization of a type

$\mathrm{acl}(A)$	model-theoretic algebraic closure of the set A
$\mathrm{Aut}(\mathcal{M})$	automorphism group of the structure $\mathcal{M}$
$\mathcal{C}$	field of constants
DCF_0	differentially closed field of characteristic 0
Dom	domain
$\mathrm{dcl}(A)$	definable closure of the set A
F^{alg}	field-theoretic algebraic closure of the field F
id	identity map
$\log\delta$	logarithmic derivative
$p\vert A$	non-forking extension of the stationary type p to the parameter set A
$p \vdash q$	the type p determines the type q
$S_n(A)$	complete n-types over a parameter set A
$\mathrm{stp}(a/A)$	strong type of a over A, i.e., $\mathrm{tp}(a/\mathrm{acl}(A))$
$\mathrm{tp}(a/A)$	type of a over A
Tr.Deg	transcendence degree
$U(a/A)$	U-rank of $\mathrm{tp}(a/A)$
$\mathcal{U}$	saturated model

1 Introduction

This thesis is concerned with the model theory of differentially closed fields (of characteristic 0). A *differential field* is a field F equipped with a derivative $\delta : F \to F$, i.e., a linear operator satisfying the Leibniz Rule $\delta(xy) = x\delta y + y\delta x$. A differential field (F, δ) is *differentially closed* if every system of algebraic differential equations and inequations having a solution in some differential field extension of (F, δ) already has a solution in F. These are the existentially closed differential fields, and they were shown to be axiomatizable by Blum [4]. Its first-order theory, denoted by DCF_0, is ω-stable and thus admits a very tame theory of independence and rank on definable sets.

We will be focusing on definable sets of finite rank. These have played a significant role in the application of model theory to other areas of mathematics. For example, they are central to Hrushovski's [11] renowned proof of the function field Mordell-Lang Conjecture in characteristic 0 — following earlier ideas of Buium [5]. More recently, finite rank definable sets in DCF_0 appear in the work of Freitag and Scanlon on the differential equation satisfied by the j-function, the work of Nagloo and Pillay [24, 25] on functional transcendence of solutions to the Painlevé equations, works of Bell, Launois, León Sánchez and Moosa [2, 3, 16] of the Dixmier–Moeglin equivalence in noncommutative algebra.

The study of DCF_0 can be viewed as an expansion of algebraic geometry into differential-algebraic geometry. We can somehow categorize definable sets by their "distance" from definable sets in the algebraic geometry. In DCF_0 we have the constant field $\{x : \delta x = 0\}$, on which algebraic geometry lives. Indeed, the full induced structure on the constants is that of a pure algebraically closed field. This means that the study of definable sets from the constants is algebraic geometry. A little bit further away we have definable sets that are *internal to the constants*. They are definably isomorphic to a definable set living in (a cartesian power of) the constant field. That this is not the same thing as living in the constants is because we allow the isomorphism to be definable over additional parameters. Even further away we have the notion of *analysability*. An example of an analysable set is illustrated in Example 2.27: it is a definable set admitting a definable surjective map to a set that is internal to the constants, and each fibre of this mapping is also internal. In general, a definable set is analysable in the constants if it admits a finite sequence of definable surjections $X \to X_n \to X_{n-1} \to \cdots \to X_1$ where the fibres at each stage, and X_1, are internal to the constants. If a definable set is not analysable in the constants then it must have some part that has no connection to the constants at all. This thesis has nothing to say about them.

We will now describe the results of the thesis, chapter by chapter. Our original contributions begin in Chapter 3, which is the only chapter that works at the level of a general stable theory. (The rest of the thesis is about DCF_0 in particular.) In that chapter we begin a systematic study of finite rank analyses, introducing the notions of

- *analyses by reductions*: obtained by taking maximal definable images that are internal;
- *analyses by coreductions*: obtained by taking minimal definable images so that the fibres are internal;
- *canonical analyses*: an analysis that has the minimal number of steps and is equivalent up to interalgebraicity with any other analysis with the minimal number of steps.

We prove a number of propositions that begin the foundational study of these notions. Among them are:

- *Any analysable finite rank type has an analysis by reductions.* (Proposition 3.8)
- *If every finite rank type has a coreduction then any analysable finite rank type has an analysis by coreductions.* (Proposition 3.10)

- *Analyses by reductions (or by coreductions) have the minimal number of steps.* (Proposition 3.12)
- *If an analysis by reductions has the same finite rank at each step as an analysis by coreductions, then both analyses are canonical. Moreover, every canonical analysis is of this form.* (Proposition 3.15)
- *Given an analysis where each step is of rank 1, one can produce analyses by reductions (respectively coreductions) where the steps have any given decreasing (respectively increasing) sequence of finite ranks.* (Proposition 3.17)

The proofs of these propositions use only the basic machinery of stability theory.

In Chapter 4 we begin our focus on DCF_0 and in particular the logarithmic derivative. The logarithmic derivative is a group homomorphism from $\mathbb{G}_m$ (the universe viewed as a multiplicative group) to $\mathbb{G}_a$ (the universe viewed as an additive group) with kernel the (multiplicative) constants. There is no such homomorphism in algebraic geometry, but in differential-algebraic geometry there is, namely $\log\delta : x \mapsto \dfrac{\delta x}{x}$. We are able to generate examples of analysable types by simply iterating the logarithmic derivative map (see Corollary 4.2). What's more, by intricately applying the logarithmic derivative map on each level of the analysis, we are able to prove the following main result:

Theorem 4.7. *Given positive integers $n_1, \ldots, n_\ell$, there exists in DCF_0 a type that admits a canonical analysis in the constants with the i-th step having rank n_i.*

The results of Chapters 3 and 4 appeared in [12].

The application of the logarithmic derivative map in the above cases leads to a natural question: for a definable set that is internal to the constants, when is its preimage under the $\log\delta$ map internal instead of merely analysable? This question is also a special case of the following open problem: given a set of differential equations in two variables, if we know in advance that the set of solutions is analysable in the constants, when can we determine that the set of solutions is indeed internal? This problem is the focus of Chapter 5. We have provided a conjecture:

Conjecture 5.4. *Suppose p is a minimal type in $S_1(F)$ that is almost*[1] *internal to the constants, where F is an algebraically closed differential field. Let $q = \log\delta^{-1}(p)$. Then the following are equivalent:*

(1) q is almost internal to the constants,

(2) q is in finite-to-finite definable correspondence with a product of types that are almost internal to the constants.

A third more explicit description of the finite-to-finite correspondence of (2) is also conjectured to be equivalent.

The main results of Chapter 5 are summarized as follows:

Theorems 5.6 and 5.12. *Conjecture 5.4 is true for types p that are not weakly orthogonal to the constants, as well as those p satisfying the following additional condition:*
$(*)$ *For every realization $a \models p$, there exists $v \in F\langle a\rangle \setminus F$ such that $\delta v \in F$ or $\log\delta(v) \in F$.*

Although the condition above seems technical, we show in Corollary 5.13 that it covers the case when p is the generic type of a differential equation $\delta x = f(x)$ where f is a rational function over the constants. So in that case wo obtain a proof of Conjecture 5.4. Moreover, condition $(*)$ can be formulated in terms of a constraint on the binding groups: see the discussion in Section 5.5. In Section 5.3 we exhibit a series of examples illustrating cases where our theorems do apply, and in Section 5.4 an example where they do not.

[1]This is a minor technical weakening of internality — see Definition 2.2.

This (conjectured) behaviour of a clear “split” as in (2) does not hold if $\log\delta$ is replaced by δ, as seen in Section 5.6. If a condition for preimages of internal types under δ to be internal exists, then it must differ significantly from that conjectured for the logarithmic derivative.

A final chapter in this thesis records two specific examples that were asked of me by other researchers and that appear in their publications. The first was requested by Bell, León Sánchez and Moosa, and appeared in [3]. The second was asked for by Haykazyan and Moosa and is referred to in [9]. They are closely related to many of the ideas and techniques developed in Chapters 4 and 5.

2 Preliminaries

We do not, in this thesis, include an introduction to or review of model theory, nor of stability theory. (We have, however, included a glossary at the end and a list of symbols at the beginning.) For one thing, there are many good books on these subjects: we suggest [20] for model theory and [27] for stability theory. Moreover, except for Chapter 3, this thesis is about the model theory of one particular first-order theory — the theory of differentially closed fields of characteristic 0 (DCF_0), a review of which is included in this chapter — so the general machinery is unnecessary for most of this thesis. We will, however, in this chapter, spend some time reviewing the particular notion of internality in stable theories, as this is at the heart of the thesis.

Our model-theoretic notion is standard. In particular, we fix a complete theory T that admits elimination of imaginaries, and a sufficiently saturated model $\mathcal{U} \models T$. All parameter sets and models will be assumed to be small, that is, of cardinality strictly less than $|\mathcal{U}|$. Given $A \subseteq \mathcal{U}$, we let $\mathrm{Aut}_A(\mathcal{U})$ denote the group of automorphisms of $\mathcal{U}$ that fix A pointwise. For each positive integer n, $S_n(A)$ denotes the set of complete n-types in $\mathcal{U}$ over A. We sometimes write $S(A)$ instead of $S_n(A)$ when n is either clear from context or unimportant. Given a tuple a we will write $\mathrm{stp}(a/A)$ for the stationary type $\mathrm{tp}(a/\mathrm{acl}(A))$.

We will assume throughout that T is stable, even though this is not always necessary. Stable theories come with a well-behaved notion of independence, i.e., Shelah's non-forking independence. Indeed, this is their characteristic property. As usual, given a tuple $\bar{a}$ and parameter sets $B \subseteq C$, we will write $\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}_B C$ to mean that $\mathrm{tp}(\bar{a}/C)$ is a non-forking extension of $\mathrm{tp}(\bar{a}/B)$. Associated with non-forking independence are several "ranks"; we will mostly work with the Lascar rank on complete types, referred to here as U-rank. While this rank in a general stable theory is sometimes ordinal-valued and sometimes undefined (so bigger than every ordinal), we are often most interested in types of finite U-rank. The interaction of U-rank with independence is of course central: suppose $B \subseteq C$ and $U(\bar{a}/B)$ is finite, then $\bar{a} \mathop{\smile\!\!\!\!\!\mid}_B C$ iff $U(\bar{a}/C) = U(\bar{a}/B)$.

Finally, a word about tuples. We will often deal with tuples in this thesis, and the following conventions are used: for any n-tuple $\bar{a}$, we use $a_1, a_2, \ldots, a_n$ to denote its elements; for two n-tuples $\bar{a}$ and $\bar{b}$, $\bar{a} \cdot \bar{b}$ (or simply $\bar{a}\bar{b}$ when no confusion arises) is defined as $\displaystyle\sum_{i=1}^{n} a_i b_i$, and $\bar{a}^{\bar{b}} := \displaystyle\prod_{i=1}^{k} a_i^{b_i}$ whenever this makes sense. We sometimes simply write a instead of $\bar{a}$ for a tuple when no confusion arises.

Nothing in this chapter is new, though proofs are sometimes included, either for the sake of completeness or because no appropriate references in the literature were found.

2.1 Internality in stable theories

The following notion expresses a possible interaction between two definable sets.

Definition 2.1. Suppose D and E are A-definable sets. We say that D is *E-internal* if there exists a definable surjective function $f : E^n \to D$.

In other words, D is interpretable in the structure induced by $\mathcal{U}$ on E; D is definably isomorphic to $E^n/\sim$ where $\sim$ is the equivalence relation of being in the same fibre of f. So if the induced structure on E from $\mathcal{U}$ eliminates imaginaries, then D being E-internal is equivalent to D being definably isomorphic to some subset of some cartesian power of E.

It is, however, very important here that f need not be A-definable: additional parameters may be required. All of the internality structure arises from the (potential) need for additional parameters.

This definition does not use stability. But when the theory is stable — as indeed our theory T is — then there is a reformulation of internality for types that is technically very useful even though at first sight much more complicated. The following definition is fundamental to this thesis:

Definition 2.2. Let q be a stationary type over A, and $\mathcal{P}$ be a set of partial types (over different parameter sets) which is invariant under $\mathrm{Aut}_A(\mathcal{U})$. We say that q is *$\mathcal{P}$-internal* if for some (equivalently any) realization a of q, there exists $B \supseteq A$ which is independent from a over A, and $c_1, \ldots, c_n$ realizations of types in $\mathcal{P}$ whose parameter sets are contained in B, such that $a \in \mathrm{dcl}(Bc_1 \cdots c_n)$. We say that q is *almost $\mathcal{P}$-internal* if a is in $\mathrm{acl}(Bc_1 \cdots c_n)$ instead of $\mathrm{dcl}(Bc_1 \cdots c_n)$.

The following explains the correlation with Definition 2.1.

Proposition 2.3. *Suppose D, E are A-definable sets and $\varphi(x)$ is an L_A-formula defining E. The following are equivalent:*

(1) D is E-internal.

(2) For every $d \in D$, $\mathrm{stp}(d/A)$ is $\{\{\varphi(x)\}\}$-internal.

Proof. Suppose D is E-internal witnessed by a B-definable surjective function $f : E^n \to D$ for some $B \supseteq A$. Given $d \in D$ let $d' \models \mathrm{stp}(d/A)$ but with $d' \mathop{\smile\!\!\!\!\!\!\,|}_A B$. Then $d' = f(\bar{c})$ for some $\bar{c} = (c_1, \ldots, c_n) \in E^n$, so $d' \in \mathrm{dcl}(Bc_1 \cdots c_n)$. That is, $\mathrm{stp}(d'/A) = \mathrm{stp}(d/A)$ is $\{\{\varphi(x)\}\}$-internal.

For the converse we use the following fact about $\mathcal{P}$-internality in stable theories.

Fact 2.4 (Lemma 4.2 of Chapter 7 in [27]). *Let A be a small set of parameters, and $q \in S(A)$ be a stationary $\mathcal{P}$-internal type. Then there exists a partial A-definable function $f(y_1, \ldots, y_m, z_1, \ldots, z_n)$, a sequence of realizations $a_1, \ldots, a_m$ of q, and $p_1, \ldots, p_n \in \mathcal{P}$, such that any realization a of q satisfies $a = f(a_1, \ldots, a_m, c_1, \ldots, c_n)$ for some $c_1 \models p_1, \ldots, c_n \models p_n$.*

Suppose now that any complete type extending D over $\mathrm{acl}(A)$ is $\{\{\varphi\}\}$-internal. For each such $q \in S(\mathrm{acl}(A))$, Fact 2.4 gives us a surjective definable function

$$f_q : X_q \to Y_q$$

where $X_q \subseteq E^{n_q}$ and Y_q are definable over some $B_q \supseteq A$ and $q(\mathcal{U}) \subseteq Y_q \subseteq D$. Letting B be the union of all these B_q (which is still small by stability since $S(\mathrm{acl}(A))$ is small), we have that the set of all such Y_q forms a B-definable cover of D. By saturation, $D = Y_{q_1} \cup \cdots \cup Y_{q_\ell}$ for some finite set of complete types $q_1, \ldots, q_\ell \in S(\mathrm{acl}(A))$. Letting X be the disjoint union of $X_{q_1}, \ldots, X_{q_n}$ in some fixed E^n, we get a definable surjection $f_0 : X \to D$. Fixing some $d \in D$, we further extend f_0 to f with domain E^n by defining $f(x) = d$ for all $x \in E^n \backslash X$. The definable surjection f witnesses E-internality of D. $\square$

Remark 2.5. In Fact 2.4 we can actually take $\{a_1, \ldots, a_m\}$ to be a *Morley sequence* in q, i.e., to be a sequence of independent realizations of q (see proof of Lemma 4.2 of Chapter 7 in [27]).

Here are some basic properties of almost internal types.

Lemma 2.6. *(1) If* $\mathrm{stp}(a/A)$ *is almost* $\mathcal{P}$*-internal and* $b \in \mathrm{acl}(Aa)$*, then* $\mathrm{stp}(b/A)$ *is almost* $\mathcal{P}$*-internal.*

(2) If $\mathrm{stp}(a_1/A)$ *and* $\mathrm{stp}(a_2/A)$ *are almost* $\mathcal{P}$*-internal, then* $\mathrm{stp}(a_1a_2/A)$ *is almost* $\mathcal{P}$*-internal.*

(3) If q *is almost* $\mathcal{P}$*-internal then every stationary extension of* q *is almost* $\mathcal{P}$*-internal.*

Proof. (1) Since $\mathrm{stp}(a/A)$ is almost $\mathcal{P}$-internal, there exists $B \supseteq A$ and $c_1, \ldots, c_n$ realizations of types in $\mathcal{P}$ such that $a \perp_A B$ and $a \in \mathrm{acl}(Bc_1 \cdots c_n)$. As $b \in \mathrm{acl}(Aa)$, we have $b \perp_A B$ and $b \in \mathrm{acl}(Bc_1 \cdots c_n)$, so $\mathrm{stp}(b/A)$ is almost $\mathcal{P}$-internal.

(2) Since $\mathrm{stp}(a_1/A)$ and $\mathrm{stp}(a_2/A)$ are almost $\mathcal{P}$-internal, suppose for $i = 1, 2$ we have $B_i \supseteq A$ and $c_{i1}, \ldots, c_{in_i}$ realizations of types in $\mathcal{P}$ such that $a_i \perp_A B_i$ and $a_i \in \mathrm{acl}(B_i c_{i1} \cdots c_{in_i})$. Let $B_1' \models \mathrm{stp}(B_1/Aa_1)$ but with $B_1' \perp_{Aa_1} a_2$, and let $B_2' \models \mathrm{stp}(B_2/Aa_2)$ but with $B_2' \perp_{Aa_2} a_1 B_1'$. Then we have $B_1' B_2' \perp_A a_1 a_2$. Moreover, as types in $\mathcal{P}$ are invariant under $\mathrm{Aut}_A(\mathcal{U})$, for $i = 1, 2$ there exists $c_{i1}', \ldots, c_{in_i}'$ realizations of types in $\mathcal{P}$ such that $a_i \in$

$\mathrm{acl}(B_i'c_{i1}'\cdots c_{in_i}')$. Thus $a_1a_2 \in \mathrm{acl}(B_1'B_2'c_{11}'\cdots c_{1n_1}'c_{21}'\cdots c_{2n_2}')$, and therefore $\mathrm{stp}(a_1a_2/A)$ is almost $\mathcal{P}$-internal.

(3) Suppose $p \in S(A')$ extends $q \in S(A)$ and is stationary. Fix some $a \models p$. Since $a \models q$, there exists $B \supseteq A$ and $c_1, \dots, c_n$ realizations of types in $\mathcal{P}$ such that $a \mathop{\smile\!\!\!\!\!\!\mid\;}_A B$ and $a \in \mathrm{acl}(Bc_1\cdots c_n)$. Let $B' \models \mathrm{tp}(B/Aa)$ but with $B' \mathop{\smile\!\!\!\!\!\!\mid\;}_{Aa} A'$, and let $\alpha \in \mathrm{Aut}_{Aa}(\mathcal{U})$ be such that $\alpha(B) = B'$. Then $a \mathop{\smile\!\!\!\!\!\!\mid\;}_{A'} B'$, and $a \in \mathrm{acl}(A'B'\alpha(c_1)\cdots\alpha(c_n))$, and therefore $p = \mathrm{tp}(a'/A')$ is almost $\mathcal{P}$-internal. $\square$

The following lemma shows that, for any almost $\mathcal{P}$-internal type q, there exists a $\mathcal{P}$-internal type which is interalgebraic with q. Hence almost internality is not so far from internality.

Lemma 2.7. *If a stationary type q over A is almost $\mathcal{P}$-internal, then for any $a \vDash q$, there exists a tuple a_0 such that $\mathrm{tp}(a_0/A)$ is $\mathcal{P}$-internal and $\mathrm{acl}(Aa) = \mathrm{acl}(Aa_0)$.*

Proof. Given any realization $a \vDash q$, let n be the least number such that there exists an L_A-formula $\varphi(x, y, z)$, a tuple b independent from a over A and a tuple c realizing types in $\mathcal{P}$ such that $\vDash \varphi(a, b, c)$ and $\varphi(\mathcal{U}, b, c)$ is of size n. We fix these b, c, and φ that satisfy $|\varphi(\mathcal{U}, b, c)| = n$.

Step 1. We prove that $\varphi(\mathcal{U}, b, c) \subseteq \mathrm{acl}(Aa)$.

Let $a = a_1, a_2, \dots, a_n$ be the elements of $\varphi(\mathcal{U}, b, c)$. Towards a contradiction, suppose without loss of generality that $a_2 \notin \mathrm{acl}(Aa)$. Then there are a_2', b' and c' such that $\mathrm{tp}(a_2'b'c'/Aa) = \mathrm{tp}(a_2bc/Aa)$ and $a_2'b' \mathop{\smile\!\!\!\!\!\!\mid\;}_{Aa} a_2\cdots a_nb$. Since $a_2' \notin \mathrm{acl}(Aa)$ and $a_2' \mathop{\smile\!\!\!\!\!\!\mid\;}_{Aa} a_2\cdots a_nb$, $a_2' \notin \mathrm{acl}(Aaa_2\cdots a_nb)$. In particular, $a_2' \neq a_i$ for $i = 1, 2, \dots, n$. Also, since $a \mathop{\smile\!\!\!\!\!\!\mid\;}_A b$ and $b \mathop{\smile\!\!\!\!\!\!\mid\;}_{Aa} b'$, we have $b \mathop{\smile\!\!\!\!\!\!\mid\;}_A ab'$, and therefore $b \mathop{\smile\!\!\!\!\!\!\mid\;}_{Ab'} a$. As $\mathrm{tp}(b'/Aa) = \mathrm{tp}(b/Aa)$ and $b \mathop{\smile\!\!\!\!\!\!\mid\;}_A a$, we have $b' \mathop{\smile\!\!\!\!\!\!\mid\;}_A a$, which, together with $b \mathop{\smile\!\!\!\!\!\!\mid\;}_{Ab'} a$, yields $bb' \mathop{\smile\!\!\!\!\!\!\mid\;}_A a$. Now the fact that q is almost $\mathcal{P}$-internal is witnessed by $a \vDash \varphi(x, b, c) \wedge \varphi(x, b', c')$, and the size of $\varphi(\mathcal{U}, b, c) \wedge \varphi(\mathcal{U}, b', c')$ is smaller than n (notice that $|\varphi(\mathcal{U}, b, c)| =$

$|\varphi(\mathcal{U}, b', c')| = n$, but the two sets are not the same), contradicting minimality of n.

Step 2. Let d be the *code* of the set $\varphi(\mathcal{U}, b, c)$. That is, d is a tuple with the property that $\sigma \in \mathrm{Aut}(\mathcal{U})$ fixes d pointwise iff σ fixes $\varphi(\mathcal{U}, b, c)$ setwise. Then $\mathrm{tp}(d/A)$ is $\mathcal{P}$-internal and $\mathrm{acl}(Aa) = \mathrm{acl}(Ad)$.

We have $a \in \mathrm{acl}(d) \subseteq \mathrm{acl}(Ad)$ by the definition of a code, and $d \in \mathrm{dcl}(aa_2 \cdots a_n) \subseteq \mathrm{acl}(Aa)$. Moreover, as $a \mathop{\smile\!\!\!\!\!\!\mid}_A b$, we have $d \mathop{\smile\!\!\!\!\!\!\mid}_A b$. Since d is the code of $\varphi(\mathcal{U}, b, c)$ where φ is an L_A-formula, $d \in \mathrm{dcl}(Abc)$. Therefore $\mathrm{tp}(d/A)$ is $\mathcal{P}$-internal. $\square$

When q is $\mathcal{P}$-internal, the following group measures, to some extent, how many additional parameters are needed to witness the internality.

Definition 2.8. For a $\mathcal{P}$-internal type q over A, the *binding group* of q is $\mathrm{Aut}_A(q/\mathcal{P}) := \{\sigma\restriction_{q(\mathcal{U})} : \sigma \in \mathrm{Aut}_A(\mathcal{U}) \text{ and } \sigma\restriction_{p(\mathcal{U})} = \mathrm{id} \text{ for all } p \in \mathcal{P}.\}$.

Note that $\mathrm{Aut}_A(q/\mathcal{P})$ acts naturally on the type-definable set $q(\mathcal{U})$. A central theorem in stability theory, going back to Zilber [32] and developed further by Poizat [29], is that when q is $\mathcal{P}$-internal, this group is definable.

Fact 2.9 (Theorem 4.8 of Chapter 7 in [27])**.** *If the types in $\mathcal{P}$ are over A, and q is a stationary type over A that is $\mathcal{P}$-internal, then both $\mathrm{Aut}_A(q/\mathcal{P})$ and its action on $q(\mathcal{U})$ are type-definable. That is, there exists a type-definable group G over A, acting type-definably over A on $q(\mathcal{U})$, such that G together with its action on $q(\mathcal{U})$ is isomorphic to $\mathrm{Aut}_A(q/\mathcal{P})$ acting naturally on $q(\mathcal{U})$.*

While the group G given by the binding group theorem need only be type-definable in general, if T is ω-stable (as it will be in our intended application) then G will in fact be a definable group. This is because in an ω-stable theory all type-definable groups are in fact definable (see Theorem 7.5.3 of [20]).

Lemma 2.10. *Let $q \in S(A)$ be stationary and $\mathcal{P}$-internal. Then there exists an integer n and $a_1, \ldots, a_n \models q$ such that if $\alpha_1, \alpha_2 \in \mathrm{Aut}_A(q/\mathcal{P})$ satisfies $\alpha_1(a_i) = \alpha_2(a_i)$ for $i = 1, 2, \ldots, n$, then $\alpha_1 = \alpha_2$.*

Proof. This is because, by Fact 2.4, there exist $a_1, \dots, a_n \models q$ such that any $a \models q$ satisfies $a \in \operatorname{dcl}(A\mathcal{P}(\mathcal{U})a_1 \cdots a_n)$. $\square$

Remark 2.11. This property of a binding group is known as *finite faithfulness* (see, for example, Definition 3.9 of [9]). In addition, by Remark 2.5, we may choose $a_1, \dots, a_n$ to be any Morley sequence of p of length n.

As in the beginning of our discussion of internality, we are often interested in the case that $\mathcal{P} = \{\{\varphi(x)\}\}$ where $\varphi(x)$ is an L_A-formula. Writing $E := \varphi(\mathcal{U})$ we will say that q is *E-internal* or *almost E-internal* instead of $\mathcal{P}$-internal and almost $\mathcal{P}$-internal, and we write $\operatorname{Aut}_A(q/E)$ for the binding group.

We conclude this section by exploring a little bit what happens when q is E-internal but the internality really requires new parameters. But first recall the following useful fact about stable theories.

Fact 2.12 (See, for example, Corollary 8.3.3 of [31])**.** *Suppose $E \subseteq U^n$ is A-definable. Then E is* stably embedded *in $\mathcal{U}$. That is, if $X \subseteq \mathcal{U}^n$ is any definable set then $X \cap E$ is definable with parameters from $A \cup E$.*

The following lemma, essentially contained in the appendix to [8], captures the usefulness of stable embeddedness.

Lemma 2.13. *Suppose E is a stably embedded A-definable set.*

(a) For any tuple a, $\operatorname{tp}(a/\operatorname{dcl}(Aa) \cap \operatorname{dcl}(A \cup E)) \vdash \operatorname{tp}(a/A \cup E)$.

(b) Given tuples a_1, a_2, if $\operatorname{tp}(a_1/A \cup E) = \operatorname{tp}(a_2/A \cup E)$ *then there is* $\alpha \in \operatorname{Aut}_A(\mathcal{U})$ *with* $\alpha\restriction_E = \operatorname{id}$ *and* $\alpha(a_1) = a_2$.

Proof. Part (a) is the (5)$\Rightarrow$(1) direction of Lemma 1 of the appendix to [8].

Part (b) can be deduced from the proof of (2)$\Rightarrow$(6) in that lemma. But we give some details.

Claim. Given A_1, A_2 subsets of $\mathcal{U}$ of cardinality less than $|\mathcal{U}|$, and a partial elementary map $\tau : A \cup E \cup A_1 \to A \cup E \cup A_2$ such that $\tau\restriction_{A\cup E} = \operatorname{id}$ and $\tau(A_1) = A_2$, if $b_1 \in \mathcal{U}$ then there exists $b_2 \in \mathcal{U}$ realizing $\tau(\operatorname{tp}(b_1/A \cup E \cup A_1))$.

Proof of Claim. By Part (a) applied to all finite tuples from $A_1 \cup \{b_1\}$, there is $E_0 \subseteq E$ such that $|E_0| < |\mathcal{U}|$ and $\mathrm{tp}(A_1b_1/A \cup E_0) \vdash \mathrm{tp}(A_1b_1/A \cup E)$. By saturation there is b_2 realizing $\tau(\mathrm{tp}(b_1/A \cup E_0 \cup A_1))$. But $\tau(\mathrm{tp}(b_1/A \cup E_0 \cup A_1)) \vdash \tau(\mathrm{tp}(b_1/A \cup E \cup A_1))$. This proves the claim.

Using this claim it is easy to build α by a familiar back-and-forth construction. $\square$

Fix an A-definable set E and a stationary type $q \in S(A)$.

Definition 2.14. We say that q is *weakly orthogonal to* E if for some (equivalently any) $a \models q$ and any finite tuple $\bar{e}$ from E, $a \mathop{\smile\!\!\!\!\!\!\mid}_A \bar{e}$.

So in some sense this says that q has nothing to do with E — but only if you fix the parameter set A. It leaves open the possibility of a lot of interaction if you pass to more parameters. So, in fact, a can be both E-internal and weakly orthogonal to E. We will see natural examples of this in the context of differentially closed fields.

Lemma 2.15. *Suppose q is E-internal. Then q is weakly orthogonal to E iff $\mathrm{Aut}_A(q/E)$ acts transitively on $q(\mathcal{U})$.*

Proof. We first show that if q is weakly orthogonal to E then q has a unique extension to $A \cup E$. Indeed, for any $a_1, a_2 \models q$ and $\bar{e}$ a tuple from E, we have $a_i \mathop{\smile\!\!\!\!\!\!\mid}_A \bar{e}$ for $i = 1, 2$, so $a_1, a_2 \models q|\bar{e}$. Now, as q is stationary, this implies $\mathrm{tp}(a_1/A \cup \bar{e}) = \mathrm{tp}(a_2/A \cup \bar{e})$. This shows that there is a unique extension of q to $A \cup \bar{e}$. Since $\bar{e}$ is chosen arbitrarily, we get that there is a unique extension of q to $A \cup E$.

Suppose $a_1, a_2 \models q$. Then as we have just seen, we have $\mathrm{tp}(a_1/A \cup E) = \mathrm{tp}(a_2/A \cup E)$. Since E is stably embedded in $\mathcal{U}$, by Lemma 2.13, there exists $\alpha \in \mathrm{Aut}_A(\mathcal{U})$ such that $\alpha\restriction_E = \mathrm{id}$ and $\alpha(a_1) = a_2$. That is, $\mathrm{Aut}_A(q/E)$ acts transitively on $q(\mathcal{U})$.

For the converse, suppose $\mathrm{Aut}_A(q/E)$ acts transitively on $q(\mathcal{U})$. Let $a \models q$ and $\bar{e}$ be a tuple from E. Let $a' \models q$ with $a' \mathop{\smile\!\!\!\!\!\!\mid}_A \bar{e}$. By assumption there exists $\alpha \in \mathrm{Aut}_A(q/E)$ such that $\alpha(a') = a$. Since $\alpha\restriction_{A\cup E} = \mathrm{id}$, we get $a \mathop{\smile\!\!\!\!\!\!\mid}_A \bar{e}$ as desired. $\square$

Lemma 2.16. *Suppose $q \in S(A)$ is minimal, i.e., of U-rank 1. Then q is not weakly orthogonal to E iff $q(\mathcal{U}) \subseteq \mathrm{acl}(AE)$.*

Proof. If q is not weakly orthogonal to E, for some $a \models q$ and some $\bar{e}$ from the set E, we have $a \not\mathop{\smile\!\!\!\!\!\!\mid}_A \bar{e}$. As q is of $\mathcal{U}$-rank 1, $a \in \mathrm{acl}(A\bar{e}) \subseteq \mathrm{acl}(AE)$. By an automorphism argument we get $q(\mathcal{U}) \subseteq \mathrm{acl}(AE)$.

Conversely, suppose $q(\mathcal{U}) \subseteq \mathrm{acl}(AE)$. Let $a \models q$. Then $a \in \mathrm{acl}(A\bar{e})$ for some $\bar{e}$ from E. As q is not algebraic, $a \notin \mathrm{acl}(A)$. Hence $a \not\mathop{\smile\!\!\!\!\!\!\mid}_A \bar{e}$ and we see that q is not weakly orthogonal to E. $\square$

2.2 Differentially closed fields

We begin with a review of differential algebra. A *differential ring* is a commutative unitary ring R equipped with an additional function $\delta : R \to R$ that satisfies the Leibniz rule $\delta(xy) = x\delta y + y\delta x$. A *differential field* is a differential ring whose underlying ring is is a field. If (F, δ) is a differential field then by the *field of constants* we mean the subfield $\mathcal{C}_F := \{x \in F : \delta x = 0\}$.

In this thesis all differential fields will be of characteristic zero.

Given a differential field (F, δ), the *ring of δ-polynomials in x* is the differential ring $F\{x\} := F[x = x^{(0)}, x^{(1)}, \ldots]$ where $\delta x^{(i)} = x^{(i+1)}$. It is not hard to check that this uniquely determines a differential ring structure on $F\{x\}$ that extends (F, δ). One can, of course, in the obvious way, consider the δ-polynomial ring $F\{\bar{x}\}$ in a tuple of variables $\bar{x} = (x_1, \ldots, x_n)$. The fraction field of $F\{\bar{x}\}$, denoted by $F\langle\bar{x}\rangle$, is the differential field of *δ-rational functions.*

The following useful fact is deduced by a straightforward Leibniz rule computation.

Fact 2.17. *Suppose (F, δ) is a differential field, $f \in F[\bar{x}]$, and $\bar{a} = (a_1, \ldots, a_n)$ is a tuple from some differential field extension $(K, \delta) \supseteq (F, \delta)$. Then*

$\delta f(\bar{a}) = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(\bar{a})\delta a_i + f^\delta(\bar{a})$, *where* $f^\delta \in F[\bar{x}]$ *is obtained by applying* δ *to the coefficients of* f*.*

Corollary 2.18. *Let* $L \supseteq K$ *be two differential fields. Then*

$$\mathcal{C}_K^{\mathrm{alg}} \cap L = K^{\mathrm{alg}} \cap \mathcal{C}_L.$$

In particular, taking $L = K$*, we see that the field of constants is relatively algebraically closed in a differential field.*

Proof. Suppose $a \in L$ is algebraic over $\mathcal{C}_K$. Then $a \in \mathcal{C}_K^{\mathrm{alg}} \subseteq K^{\mathrm{alg}}$. Now let $P \in \mathcal{C}_K[x]$ be the minimal polynomial of a over $\mathcal{C}_K$. Then $0 = \delta P(a) = \frac{dP}{dx}(a)\delta a + P^\delta(a) = \frac{dP}{dx}(a)\delta a$ by Fact 2.17. Since $P(x)$ is the minimal polynomial of a over $\mathcal{C}_K$, $\frac{dP}{dx}(a) \neq 0$, so $\delta a = 0$. Therefore $a \in \mathcal{C}_L$.

Suppose now that $a \in \mathcal{C}_L$ is algebraic over K. Then $a \in \mathcal{C}_L \subseteq L$. Let $P \in K[x]$ be the minimal polynomial of a over K. Then $0 = \delta P(a) = \frac{dP}{dx}(a)\delta a + P^\delta(a) = P^\delta(a)$ by Fact 2.17. Note that P^δ is either the zero polynomial or a polynomial of strictly smaller degree than P (since P is monic). As $P^\delta(a) = 0$, P^δ must be the zero polynomial, so $a \in \mathcal{C}_K^{\mathrm{alg}}$. □

For a differential field extension $F \subseteq K$, if $\alpha \in K$, then we use $F\langle\alpha\rangle$ to denote the *differential field generated by* F *and* α. Note that if α is *differentially transcendental* over F (meaning $\{\alpha, \delta\alpha, \delta^2\alpha, \ldots\}$ is algebraically independent over F), then $F\langle\alpha\rangle$ is isomorphic to the differential rational function field $F\langle x\rangle$. Similarly, for a set $A \subseteq K$, we use $F\langle A\rangle$ to denote the differential field generated by F and A.

We study differential fields model theoretically in the language $L = \{0, 1, +, -, \times, \delta\}$ of rings together with a unary function symbol for the derivation. The class of differential fields of characteristic 0 is axiomatizable in the natural way by the theory denoted DF_0. That this theory has a model completion was shown by Blum [4] who gave differential-algebraic axioms, but see also the geometric axioms of Pierce and Pillay [26]. That model completion is the theory of *differentially closed fields in characteristic zero*, denoted by DCF_0. It is the theory of existentially closed differential fields. That is, a differential field (K, δ) is *differentially closed* if every system of differential polyomial equations with a solution in some differential field extension already has a solution in K. DCF_0 enjoys a number of model-theoretic tameness properties including

- Quantifier elimination,
- Elimination of imaginaries, and
- ω-stability.

See, for example, [19] for a survey of this theory.

We work as usual in a sufficiently saturated model $(\mathcal{U}, \delta) \models \mathrm{DCF}_0$, and denote its constant field by $\mathcal{C}$. Model-theoretic definable closure in this theory is given by differential field generation, and model-theoretic algebraic closure is given by field-theoretic algebraic closure. That is, if $A \subseteq \mathcal{U}$ then $\mathrm{dcl}(A) = \mathbb{Q}\langle A\rangle$ and $\mathrm{acl}(A) = \mathbb{Q}\langle A\rangle^{\mathrm{alg}}$. Shelah's non-forking independence has the following algebraic characterization: for $\bar{a}$ a tuple and $B \subseteq C$ sets of parameters, given that $\bar{a}$ is not differentially transcendental over B, then $\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}_B C$ iff

$$\mathrm{Tr.Deg}(\mathbb{Q}\langle C, \bar{a}\rangle / \mathbb{Q}\langle C\rangle) = \mathrm{Tr.Deg}(\mathbb{Q}\langle B, \bar{a}\rangle / \mathbb{Q}\langle B\rangle).$$

Equivalently, $\mathbb{Q}\langle B, \bar{a}\rangle^{\mathrm{alg}}$ is algebraically disjoint from $\mathbb{Q}\langle C\rangle^{\mathrm{alg}}$ over $\mathbb{Q}\langle B\rangle^{\mathrm{alg}}$.

Let us now review the basics of differential-algebraic geometry.

The *Kolchin topology* on $\mathcal{U}^n$ is defined as follows: we say that a definable set $A \subseteq \mathcal{U}^n$ is *Kolchin closed* if there exist $f_1, \ldots, f_m \in \mathcal{U}\{\bar{x}\}$ such that $A = \{\bar{x} \in \mathcal{U}^n : f_1(\bar{x}) = \cdots = f_m(\bar{x}) = 0\}$. Theorem 1.16 of [18] shows that this topology is Noetherian: there is no descending chain of closed sets. The *generic type* of an irreducible Kolchin closed set B over a δ-field k is the type which says that x is in B but not in any k-definable Kolchin closed proper subset of B. A definable set is *irreducible* if its Kolchin closure is. By the generic type of an irreducible definable set, we mean the generic type of its Kolchin closure. Note that this does not always coincide with the type of greatest U-rank: over the empty set, the generic type p of $\mathcal{C}$ is of U-rank 1, which is the type of greatest U-rank in the set defined by $x\delta^2 x = \delta x$, as shown in Corollary 5.17 of [18], but p is not generic in this definable set. The term "generic solution" is similarly defined: a *generic solution* of a set of equations over a given parameter set is a solution that realizes the generic type over that parameter set of the definable set defined by the set of equations.

As mentioned above, the generic type of the field of constants $\mathcal{C}$ has U-rank 1. In fact it is strongly minimal and the full induced structure in $\mathcal{C}$ from $(\mathcal{U}, \delta)$ is that of a pure algebraically closed field. It is in this sense that we view the study of DCF_0 as an expansion of algebraic geometry: algebraic geometry lives as the structure induced by DCF_0 in the constant field.

The following consequence of strong minimality of $\mathcal{C}$ will be useful:

Lemma 2.19. *Let $F \subseteq \mathcal{U}$ be an algebraically closed differential field with $\mathcal{C}_F$ being its field of constants. For any $f(\bar{x}) \in F(\bar{x})$, if $f(\bar{x}) = 0$ has a solution $\bar{c} \in \mathcal{C}$, then it has a solution in $\mathcal{C}_F$.*

Proof. Assume the conclusion does not hold. Let i be the maximal possible number such that the first i coordinates of a solution of the equation $f(\bar{x}) = 0$ are in $\mathcal{C}_F$, i.e., there exists a solution $\bar{e} = (e_1, \ldots, e_k) \in \mathcal{C}$ of $f(\bar{x}) = 0$ such that $e_1, \ldots, e_i \in \mathcal{C}_F$, and for any solution $\bar{c}$, at least one of $c_1, \ldots, c_{i+1}$ is not in $\mathcal{C}_F$. In particular, $e_{i+1} \notin \mathcal{C}_F$.

Let $f_i(x_{i+1}, \ldots, x_k) := f(e_1, \ldots, e_i, x_{i+1}, \ldots, x_k)$. Note that $(e_{i+1}, \ldots, e_k)$ is a solution of $f_i(x_{i+1}, \ldots, x_k) = 0$, which is an equation over F. Since $\mathcal{C}$ is strongly minimal (of dimension 1 in DCF_0), and $e_{i+1} \notin F = \mathrm{acl}(F)$, the formula $\exists (x_{i+2}, \ldots, x_k) \in \mathcal{C}_F (f_i(x_{i+1}, \ldots, x_k) = 0)$ has co-finitely many realizations in $\mathcal{C}$. Since $\mathcal{C}_F$ is infinite, Let $e^*_{i+1} \in C_F$ be a realization of this formula witnessed by $(e^*_{i+2}, \ldots, e^*_k) \in \mathcal{C}$, i.e., $(e_1, \ldots, e_i, e^*_{i+1}, e^*_{i+2}, \ldots, e^*_k)$ is a solution of $f(\bar{x}) = 0$. Then the first $i+1$ coordinates of this solution are in C_F, contradicting the definition of i. $\square$

Let us now look at $\mathcal{C}$-internality and weak orthogonality to $\mathcal{C}$ in DCF_0. The following characterization improves upon Lemma 2.16 of the previous section.

Lemma 2.20. *For F a differential field and $p \in S(F)$ minimal, p is not weakly orthogonal to $\mathcal{C}$ iff any realization $\bar{a}$ of p is interalgebraic over F with some $c \in \mathcal{C}$.*

Proof. If some $\bar{a}$ realizing p satisfies that $\bar{a}$ is interalgebraic over F with some $c \in \mathcal{C}$, then $\bar{a} \not\perp_F c$, so p is not weakly orthogonal to $\mathcal{C}$.

For the converse, suppose p is not weakly orthogonal to $\mathcal{C}$. Since any minimal type in $\mathcal{C}$ is interalgebraic with the type of a singleton, it suffices to find a tuple in $\mathcal{C}$ with which $\bar{a} \models p$ is interalgebraic over F. By not weak orthogonality, there exists a tuple $\bar{c}$ from $\mathcal{C}$ such that $\bar{a} \in \mathrm{acl}(F\bar{c})$. Let $\varphi(\bar{x}, \bar{y})$ be an F-formula such that $\bar{a}$ is one of n possible solutions to $\varphi(\bar{c}, \bar{y})$. Consider the $F\bar{a}$-formula $\psi(\bar{x})$ given by $\exists^{\leq n}\bar{y}\varphi(\bar{x}, \bar{y}) \wedge \varphi(\bar{x}, \bar{a})$. Then $\psi(\mathcal{C})$ is nonempty (it contains $\bar{c}$) and by stable embeddedness is defined over $\mathcal{C}_{F\langle \bar{a} \rangle}$ in the language of rings. It therefore has a solution in $\mathcal{C}^{\mathrm{alg}}_{F\langle \bar{a} \rangle}$, say $\bar{c}^*$. It is then clear that $\bar{c}^* \in \mathrm{acl}(F\bar{a})$ and $\bar{a} \in \mathrm{acl}(F\bar{c}^*)$. $\square$

We now point out one more equivalent condition of p being weakly orthogonal if we know in addition that p is $\mathcal{C}$-internal.

Lemma 2.21. *Let F be an algebraically closed differential field. Suppose $p \in S(F)$ is a minimal $\mathcal{C}$-internal type. Then p is weakly orthogonal to $\mathcal{C}$ iff p is isolated.*

Proof. If p is isolated, then it is not interalgebraic with a constant over F (as the only isolated types in the constants are algebraic), so p is weakly orthogonal to $\mathcal{C}$ by Lemma 2.20.

If p is weakly orthogonal to $\mathcal{C}$, then by Lemma 2.15, $p(\mathcal{U})$ is a definable set as it is the orbit of $a \models p$ under the action of the definable group $\mathrm{Aut}_F(p/\mathcal{C})$. It follows that p is isolated. $\square$

If we drop the minimality assumption. we get the full characterization for weak orthogonality.

Lemma 2.22. *Let F be an algebraically closed differential field, and $p \in S(F)$. Then p is weakly orthogonal to $\mathcal{C}$ iff $\mathcal{C}_F = \mathcal{C}_{F\langle a\rangle}$ for $a \models p$.*

Proof. If $\mathcal{C}_F \neq \mathcal{C}_{F\langle a\rangle}$ for some $\bar{a} \models p$, then there is some $c \in \mathcal{C}_{F\langle \bar{a}\rangle} \setminus \mathcal{C}_F$, so $\bar{a} \not\downarrow_F c$, witnessing that p is not weakly orthogonal to $\mathcal{C}$.

If p is not weakly orthogonal to $\mathcal{C}$, then for any $\bar{a} \models p$, $\bar{a} \not\downarrow_F \bar{c}$ for some constant tuple $\bar{c}$. Assume without loss of generality that $\bar{c}$ is one of the constant tuples with the minimal length that satisfies $\bar{a} \not\downarrow_F \bar{c}$. Suppose $\bar{c} = (c_1, \ldots, c_n)$. Then $\bar{a} \downarrow_F c_1 \cdots c_{n-1}$, so $\bar{a} \not\downarrow_{Fc_1\cdots c_{n-1}} c_n$, which means $c_n \in \mathrm{acl}(Fac_1 \cdots c_{n-1})$. Therefore, there exists a formula $\varphi(x_1, \ldots, x_{n-1}, y)$ over Fa such that c_n is one of m solutions of $\varphi(c_1, \ldots, c_{n-1}, y)$. As $\mathcal{C}$ is stably embedded, we may assume φ is in fact a formula over $\mathcal{C}_{F\langle a\rangle}$. Suppose for a contradiction that $\mathcal{C}_F = \mathcal{C}_{F\langle a\rangle}$. Then φ is over $\mathcal{C}_F$, so $c_n \in \mathrm{acl}(Fc_1 \cdots c_{n-1})$. This, together with $\bar{a} \downarrow_F c_1 \cdots c_{n-1}$, yields $\bar{a} \downarrow_F \bar{c}$, a contradiction. We therefore get that $\mathcal{C}_F \neq \mathcal{C}_{F\langle a\rangle}$. $\square$

Remark 2.23. We may also prove this lemma from the fact that $\mathcal{C}$ is stably embedded. By Lemma 2.13, since $\mathcal{C}$ is stably embedded, for any a, $\mathrm{tp}(a/\mathrm{dcl}(a) \cap \mathcal{C})$ determines $\mathrm{tp}(a/\mathcal{C})$, so $\mathrm{dcl}(a) \cap \mathcal{C}$ is the smallest set B (up to interalgebraicity) that satisfies $a \downarrow_B \mathcal{C}$. Name all the elements in F, and we have that p is weakly orthogonal to $\mathcal{C}$ iff $a \downarrow \mathcal{C}$ iff $a \downarrow \mathrm{dcl}(a) \cap \mathcal{C}$ iff $\mathrm{dcl}(a) \cap \mathcal{C} \subseteq \mathrm{acl}(F)$ iff $\mathcal{C}_{F\langle a\rangle} = \mathcal{C}_F$.

We end this section with examples of $\mathcal{C}$-internal types (and an example where the type is not $\mathcal{C}$-internal).

Example 2.24. Let F be a differential field. Suppose D is the solution space to a set of linear differential equations over F. If D is of finite dimension then D is $\mathcal{C}$-internal.

Proof. Let $\{\bar{b}_1, \ldots, \bar{b}_k\}$ be a $\mathcal{C}$-basis of D. Then $f : (c_1, \ldots, c_k) \mapsto c_1\bar{b}_1 + \cdots + c_k\bar{b}_k$ gives a definable map from $\mathcal{C}^k$ to D, as D is a vector space of finite dimension over $\mathcal{C}$. Therefore D is $\mathcal{C}$-internal.

If the differential equations are not all homogeneous, then let $\bar{b}_0$ be a solution to the inhomogeneous system, and $\bar{b}_1, \ldots, \bar{b}_k$ be a $\mathcal{C}$-basis of the set of solutions of the corresponding homogeneous linear differential equations. The function $f : (c_1, \ldots, c_k) \mapsto c_1\bar{b}_1 + \cdots + c_k\bar{b}_k + \bar{b}_0$ then gives a definable map from $\mathcal{C}^k$ to D, so D is again $\mathcal{C}$-internal. $\square$

Example 2.25. Let D be the strongly minimal set defined by the equation $\delta x = ax$ for some element $a \in \mathcal{U}$. Then D is $\mathcal{C}$-internal. Moreover, suppose F is an algebraically closed differential field that contains a and $D \cap F = \{0\}$. Let p be the generic type of D over F. Then p is weakly orthogonal to $\mathcal{C}$ and $\mathrm{Aut}_F(p/\mathcal{C}) = \mathbb{G}_m(\mathcal{C})$.

Proof. By Example 2.24, since $\delta x = ax$ is linear, D is $\mathcal{C}$-internal.

We now prove that p is weakly orthogonal to $\mathcal{C}$. By Lemma 2.21, we only need to prove that p is isolated. The set of realizations of p is those elements in D which are not inside any proper F-definable Kolchin closed subset of D. Since D is strongly minimal, any proper F-definable Kolchin closed subset of it is finite. Therefore $p(\mathcal{U}) = D \backslash F^{\mathrm{alg}} = D \backslash F = D \backslash \{0\}$, so p is isolated.

Finally, we compute the binding group $\mathrm{Aut}_F(p/\mathcal{C})$. We first prove that any element $\alpha \in \mathrm{Aut}_F(p/\mathcal{C})$ acts as multiplication by a constant. Let $b_0 \models p$ and set $c = \dfrac{\alpha(b_0)}{b_0}$. Since for any $b \models p$,

$$\begin{aligned} \alpha(b) &= \alpha(\frac{b}{b_0}b_0) \\ &= \alpha(\frac{b}{b_0})\alpha(b_0) \end{aligned}$$

$$
\begin{aligned}
&= \frac{b}{b_0}\alpha(b_0) \\
&= cb,
\end{aligned}
$$

we get that α is indeed multiplication by c. We now prove that multiplication by any $c \in \mathbb{G}_m(\mathcal{C})$ is an element of $\mathrm{Aut}_F(p/\mathcal{C})$. Given any $b \models p$, since $p(\mathcal{U}) \subseteq \mathrm{dcl}(F\mathcal{C}b)$, we only need to prove that $\mathrm{tp}(b/F\mathcal{C}) = \mathrm{tp}(bc/F\mathcal{C})$. Since b and bc are both in $D\backslash\{0\}$, we have $\mathrm{tp}(b/F) = \mathrm{tp}(bc/F) = p$. This implies $\mathrm{tp}(b/F\mathcal{C}) = \mathrm{tp}(bc/F\mathcal{C})$ as p is weakly orthogonal to $\mathcal{C}$. $\square$

In the last example the *logarithmic derivative* was already implicit. Much of this thesis is about the logarithmic derivative. If we denote by $\mathbb{G}_m$ the multiplicative group $\mathcal{U}\backslash\{0\}$ and by $\mathbb{G}_a$ the additive group $\mathcal{U}$, then the logarithmic derivative is the definable group homomorphism $\log\delta : \mathbb{G}_m \to \mathbb{G}_a$ given by $\log\delta u = \dfrac{\delta u}{u}$. Its kernel is $\mathbb{G}_m(\mathcal{C})$. The final example we wish to discuss is the equation $\delta(\log\delta x) = 0$. This is the definable subgroup $\log\delta^{-1}(\mathcal{C}) \leq \mathbb{G}_m$. But first let us record an algebraic fact about $\log\delta$ that will be of use later:

Lemma 2.26. *If $f \in F(\bar{x})$ is a rational function (viewed as a partial differential rational function on $\mathcal{U}^n$), then there is $g \in F(\bar{x}, \bar{y})$ such that for any $\bar{u} \in \mathrm{Dom} f$ such that $f(\bar{u}) \neq 0$, $\log\delta f(\bar{u}) = g(\bar{u}, \delta\bar{u})$. In particular, if $f \in F(\bar{x})$, then there exists $g \in F(\bar{x})$ such that $\log\delta f\restriction_{\mathcal{C}^n} = g\restriction_{\mathcal{C}^n}$.*

Proof. Indeed, suppose $f(\bar{x}) = \dfrac{f_1(\bar{x})}{f_2(\bar{x})}$ where $f_1, f_2 \in F[\bar{x}]$, then $\log\delta f(\bar{u}) = \dfrac{f_2(\bar{u})\delta f_1(\bar{u}) - f_1(\bar{u})\delta f_2(\bar{u})}{f_1(\bar{u})f_2(\bar{u})}$. Now, by Fact 2.17, $\delta f_i(\bar{u}) = g_i(\bar{u}, \delta\bar{u})$ for some $g_1, g_2 \in F(\bar{x}, \bar{y})$.

The “in particular” clause is because $\delta u = 0$ for any $u \in \mathcal{C}$. $\square$

Finally, the promised example.

Example 2.27 (See, for example, Fact 4.2 of [7]). Let G be the differential algebraic subgroup of $\mathbb{G}_m$ defined by $\{x : \delta(\log\delta x) = 0\}$. The generic type q of G (over $F := \mathbb{Q}^{\mathrm{alg}}$) is not almost $\mathcal{C}$-internal.

Proof. A proof is given in [7]. Here we give another more elementary proof.

Suppose for a contradiction that q is almost $\mathcal{C}$-internal. Let $(u_1, u_2, \ldots)$ be a Morley sequence of q, and $a_i = \mathrm{log}\delta u_i$ for $i = 1, 2, \ldots$. Note that $\delta a_i = 0$, so $a_i \in \mathcal{C}$. Note also that $(a_1, a_2, \ldots)$ is algebraically independent as it is a Morley sequence. Almost $\mathcal{C}$-internality of q implies that $u_1 \in \mathrm{acl}(F\mathcal{C}u_2u_3 \cdots u_n) = \mathcal{C}\langle u_2, \ldots, u_n\rangle^{\mathrm{alg}}$ for some $n > 0$. As $\delta u_i = a_i u_i$ and $a_i \in \mathcal{C}$, $\mathcal{C}\langle u_2, \ldots, u_n\rangle = \mathcal{C}(u_2, \ldots, u_n)$. Hence $\{u_1, \ldots, u_n\}$ is algebraically dependent over $\mathcal{C}$. Let $f \in \mathcal{C}[x_1, x_2, \ldots, x_n]$ be a nonzero polynomial such that $f(u_1, \ldots, u_n) = 0$. Suppose

$$f = \sum_{\bar{k}\in I} g_{\bar{k}} \bar{x}^{\bar{k}}$$

where I is a finite set of non-negative integer n-tuples, and $g_{\bar{k}} \in \mathcal{C}$ nonzero for $\bar{k} \in I$. Let $\bar{u} = (u_1, \ldots, u_n)$ and $\bar{a} = (a_1, \ldots, a_n)$. Then

$$0 = \sum_{\bar{k}\in I} g_{\bar{k}} \bar{u}^{\bar{k}}.$$

View δ as a $\mathcal{C}$-linear operator on the $\mathcal{C}$-vector space $\mathcal{U}$. Notice that for each $\bar{k} \in I$, $\mathrm{log}\delta(\bar{u}^{\bar{k}}) = \bar{k} \cdot \mathrm{log}\delta\bar{u} = \bar{k} \cdot \bar{a}$, so $\delta(\bar{u}^{\bar{k}}) = (\bar{k} \cdot \bar{a})\bar{u}^{\bar{k}}$. Hence, $\bar{u}^{\bar{k}}$ is an eigenvector of δ with eigenvalue $\bar{k} \cdot \bar{a}$. As $\{a_1, \ldots, a_n\}$ is F-linearly independent, $\bar{k}_1 \cdot \bar{a} \neq \bar{k}_2 \cdot \bar{a}$ if $\bar{k}_1 \neq \bar{k}_2$. That is, the eigenvalues for $\bar{u}^{\bar{k}}$ are different for different $\bar{k} \in I$. Therefore $\{\bar{u}^{\bar{k}} : \bar{k} \in I\}$ is $\mathcal{C}$-linearly independent, so $g_{\bar{k}} = 0$ for all $\bar{k} \in I$, a contradiction. $\square$

Note that for $u \models q$ and $a := \mathrm{log}\delta u$, $\mathrm{tp}(u/Fa)$ is $\mathcal{C}$-internal (as u is a solution to $\delta x = ax$, a linear differential equation) and $\mathrm{tp}(a/F)$ is also $\mathcal{C}$-internal (as a itself is inside $\mathcal{C}$). We see that the type q, although not $\mathcal{C}$-internal, is somehow "2-step $\mathcal{C}$-internal". Types like this are said to be $\mathcal{C}$-*analysable*. In the next chapter, we formalize the definition of $\mathcal{C}$-analysable types and discuss their properties.

2.3 An intuition from meromorphic functions

When working in the theory of differentially closed fields, it is difficult to get an intuitive idea as we do not have a concrete model. Often the differential field M of meromorphic functions on the complex plane can serve as a helpful tool: although not differentially closed, it is a relatively rich differential field that enables us to see, among other things, the behaviour of sets that are internal to or analysable in the constants.

The structure $\mathcal{M}$ is defined as follows. The domain M is the set of meromorphic functions on the complex plane, with the usual addition, multiplication, and derivation. The field of constants in $\mathcal{M}$ is the field of complex numbers $\mathbb{C}$. We will use $'$ to denote the derivation in $\mathcal{M}$. We let $t \in M$ be the identity function, so that $t' = 1$.

Remark 2.28. $\mathcal{M} \not\models \mathrm{DCF}_0$.

Proof. Let M_0 be the field of meromorphic functions on $\mathbb{C}\backslash\mathbb{R}^-$, the complex plane with the negative half of the real line removed. Restriction induces an embedding of M into M_0 as a differential field. Note that the differential equation $x' = \dfrac{1}{t}$ has a solution in M_0, namely $x = \mathrm{Log} t$, but does not have a solution in M. This shows that $\mathcal{M}$ is not a differentially closed field. $\square$

We now illustrate how the structure $\mathcal{M}$ helps us understand the behaviour of internal and analysable sets in DCF_0 by analysing Example 2.27 using our intuitive tool $\mathcal{M}$. The equation $\left(\dfrac{x'}{x}\right)' = 0$ has as its set of solutions in $\mathcal{M}$ the multiplicative group $G = \{c_1 e^{c_2 t} : c_1, c_2 \in \mathbb{C}, c_1 \neq 0\}$. We can guess that the equation is analysable in the constants by observing that $\log\delta : G \to \mathbb{C}$ is given by $c_1 e^{c_2 t} \mapsto c_2$ and has fibres $G_{c_2} := \left\{c_1 e^{c_2 t} : c_1 \in \mathbb{C}\backslash\{0\}\right\}$, which is a definable copy of $\mathbb{C}\backslash\{0\}$. On the other hand, the equation should not be internal because the parametrization $\mathbb{C}\backslash\{0\} \times \mathbb{C} \to G$ given by $(c_1, c_2) \mapsto c_1 e^{c_2 t}$ is not definable in the differential field $\mathcal{M}$ even with parameters (it requires exponentiation).

3 Analysability

A notion similar to but weaker than internality is that of *analysability*. Instead of types that are internal, we explore types that are "internal in several steps", i.e., types that are built up through a finite sequence of fibrations whose fibres are internal. This is central to this thesis. It appears first in [10] though a form of it was implicit in the earlier work of Baldwin and Lachlan [1].

As a general setting, we work in a saturated model $\mathcal{U}$ of a complete stable theory T that eliminates imaginaries.

Definition 3.1. Let $\mathcal{P}$ be a set of partial types (over possibly different parameter sets) which is invariant under $\mathrm{Aut}_A(\mathcal{U})$, and q be a stationary type over a parameter set A. We say that q is *$\mathcal{P}$-analysable* if for some (equivalently any) realization a of q, there are $a_0 = \varnothing, a_1, \ldots, a_n$ such that for all $i = 1, 2, \ldots, n$, $a_{i-1} \in \mathrm{dcl}(Aa_i)$, $\mathrm{stp}(a_i/Aa_{i-1})$ is almost $\mathcal{P}$-internal, and $\mathrm{acl}(Aa) = \mathrm{acl}(Aa_n)$. The sequence $(a_i)_{i=1}^n$ mentioned above is called a *$\mathcal{P}$-analysis of q* or a *$\mathcal{P}$-analysis of a over A*.

In this chapter we begin with a finite U-rank type q that is $\mathcal{P}$-analysable and study the structure of the various possible analyses that might witness this. In particular, we introduce a notion of equivalence of analyses and produce extremal analyses (by "reductions" or by "coreductions"). We show that the analyses by reductions always exist (Proposition 3.8) and discuss certain conditions for analysis by coreductions to exist (Proposition 3.10). When analyses by reductions and coreductions exist and are equivalent, every analysis of q of the shortest possible length is equivalent; we call this unique analysis of shortest length the *canonical analysis*. This is Proposition 3.15 below. We also give criteria to determine if a given analysis is one of these extremals, in Proposition 3.16.

As a general reference for analysability we suggest Chapter 8 of [27]. We have provided proofs where explicit references were not available.

The results presented in this chapter appeared in [12].

3.1 Basic notions

For notational convenience, for any analysis $(a_i)_{i=1}^n$ we use a_0 to denote the empty tuple. We call n the *length* of the analysis. Note that an algebraic type has a $\mathcal{P}$-analysis of length zero, and an almost $\mathcal{P}$-internal type has a $\mathcal{P}$-analysis of length 1.

Definition 3.1 looks more like what might be called *almost analysable*, and we may instead say that a type is *strictly $\mathcal{P}$-analysable* if $\mathrm{stp}(a_i/a_{i-1})$ is internal (rather than almost internal) to $\mathcal{P}$. Indeed, this is closer to the original definition appearing in [10]. The following proposition proves that these two definitions are in fact equivalent. This is well-known but we include a proof here for completeness.

Proposition 3.2. *A stationary type q over A is $\mathcal{P}$-analysable iff it is strictly $\mathcal{P}$-analysable.*

Proof. The nontrivial direction is from left to right. Suppose $(b_1, \ldots, b_n)$ is an analysis of a over A. For convenience, let a_0 be the empty tuple. We now construct the sequence $(a_1, \ldots, a_n)$.

Suppose we already have a sequence $(a_1, \ldots, a_{i-1})$ where $1 \leq i \leq n$ such that $\mathrm{stp}(a_j/Aa_{j-1})$ is $\mathcal{P}$-internal, $a_{j-1} \in \mathrm{dcl}(Aa_j)$, and $\mathrm{acl}(Aa_j) = \mathrm{acl}(Ab_j)$ for $j = 1, 2, \ldots, i-1$. Then as $\mathrm{stp}(b_i/Ab_{i-1})$ is almost $\mathcal{P}$-internal and $\mathrm{acl}(Aa_{i-1}) = \mathrm{acl}(Ab_{i-1})$, we have that $\mathrm{stp}(b_i/Aa_{i-1})$ is almost $\mathcal{P}$-internal, so by Lemma 2.7, there exists a^* such that $\mathrm{acl}(Aa_{i-1}a^*) = \mathrm{acl}(Aa_{i-1}b_i)$ and $\mathrm{stp}(a^*/Aa_{i-1})$ is $\mathcal{P}$-internal. Let $a_i = a_{i-1}a^*$. Then we have $a_{i-1} \in \mathrm{dcl}(Aa_i)$, $\mathrm{acl}(Aa_i) = \mathrm{acl}(Aa_{i-1}b_i) = \mathrm{acl}(Ab_{i-1}b_i) = \mathrm{acl}(Ab_i)$, and $\mathrm{stp}(a_i/Aa_{i-1})$ is $\mathcal{P}$-internal.

The sequence $(a_1, \ldots, a_n)$ then witnesses the fact that $\mathrm{tp}(a/A)$ is strictly analysable. $\square$

Here are some elementary properties of analysability analogous to Lemma 2.6 about almost internality.

Lemma 3.3. *(1) If* $\mathrm{tp}(a/A)$ *is* $\mathcal{P}$*-analysable and* $b \in \mathrm{acl}(Aa)$ *then* $\mathrm{stp}(b/A)$ *is* $\mathcal{P}$*-analysable.*

(2) If $\mathrm{tp}(a_1/A)$ *and* $\mathrm{tp}(a_2/A)$ *are* $\mathcal{P}$*-analysable, then* $\mathrm{stp}(a_1a_2/A)$ *is* $\mathcal{P}$*-analysable.*

(3) If q *is* $\mathcal{P}$*-analysable, then every stationary extension of* q *is also* $\mathcal{P}$*-analysable.*

Proof. (1) In contrast to what one might expect, this does not follow immediately from the analogous property for almost internality (Lemma 2.6(1)). Let $(a_1, \ldots, a_n)$ be a $\mathcal{P}$-analysis of a over A. Let $b_i = a_i$ for $i = 1, 2, \ldots, n-1$, and $b_n = b$. Then $(b_1, \ldots, b_n)$ satisfies that $\mathrm{tp}(b_i/Ab_{i-1})$ is almost $\mathcal{P}$-internal (but $b_n \notin \mathrm{acl}(Ab_{n-1})$, so this is not a $\mathcal{P}$-analysis). This shows that $\mathrm{stp}(b/A)$ is what Hrushovski calls "externally $\mathcal{P}$-analysable" in [10]. However, in Remark 2.7(d) of that paper he explains that externally $\mathcal{P}$-analysable implies $\mathcal{P}$-analysable.

(2) Let $(a_{11}, \ldots, a_{1m})$ and $(a_{21}, \ldots, a_{2n})$ be $\mathcal{P}$-analyses of a_1 and a_2 over A, respectively. Without loss of generality, suppose $m \leq n$. Set $a_{1,m+1}, \ldots, a_{1n}$ to be all equal to a_{1m}. Then note that $(a_{11}, \ldots, a_{1n})$ is still a $\mathcal{P}$-analysis of a_1 over A. Using Lemma 2.6 we see that $(a_{11}a_{21}, \ldots, a_{1n}a_{2n})$ is a $\mathcal{P}$-analysis of a_1a_2 over A.

(3) This is a direct consequence of part (3) of Lemma 2.6. □

The U*-type* of an analysis $(a_1, \ldots, a_n)$ is the sequence $(U(a_i/Aa_{i-1}))_{i=1}^n$ of U-ranks. Note that U-ranks may be an ordinal or even ∞. We are mainly interested in the finite U-rank case, although results in this chapter work generally. We say the analysis is *non-degenerate* if each entry of the U-type is nonzero. Note that every analysis can be made non-degenerate by simply dropping those a_i such that $a_i \in \mathrm{acl}(Aa_{i-1})$.

We use the following definitions in order to better talk about analysable types and their analyses.

Definition 3.4. We say that the type q is *n-step $\mathcal{P}$-analysable*, or *$\mathcal{P}$-analysable in n-steps*, if there exists a $\mathcal{P}$-analysis of q of length n. A $\mathcal{P}$-analysis of q is *minimal* if there is no $\mathcal{P}$-analysis of q of strictly shorter length. A $\mathcal{P}$-analysis $(a_i)_{i=1}^{n}$ is said to be *incompressible* if $\operatorname{stp}(a_{i+1}/Aa_{i-1})$ is not almost $\mathcal{P}$-internal for all $i = 1, 2, \ldots, n-1$.

While a minimal analysis is clearly incompressible, the converse does not generally hold.

Example 3.5. Let $\operatorname{stp}(a)$ be 2-step $\mathcal{P}$-analysable with an incompressible $\mathcal{P}$-analysis (a_1, a). Now let (b_1, b) be such that $bb_1 \downarrow aa_1$ and $\operatorname{stp}(bb_1) = \operatorname{stp}(aa_1)$. Let $c = ab$. Then c is 3-step $\mathcal{P}$-analysable, with an analysis $(a_1, ab_1, c = ab)$. This analysis is incompressible: $\operatorname{stp}(ab_1)$ is not almost $\mathcal{P}$-internal because $\operatorname{stp}(a)$ is not almost $\mathcal{P}$-internal and $\operatorname{stp}(ab/a_1)$ is not almost $\mathcal{P}$-internal because $\operatorname{stp}(b)$ is not almost $\mathcal{P}$-internal, and $\operatorname{stp}(b/a_1)$ is its non-forking extension. But c is 2-step $\mathcal{P}$-analysable by $(a_1b_1, c = ab)$, so the $\mathcal{P}$-analysis $(a_1, ab_1, c = ab)$ is not minimal despite being incompressible.

Nonetheless, the following lemma shows that incompressibility implies minimality if the U-type of an analysis is $(1, 1, \ldots, 1)$.

Lemma 3.6. *Let $(a_1, \ldots, a_n)$ be an incompressible $\mathcal{P}$-analysis of a over A of U-type $\underbrace{(1, 1, \ldots, 1)}_{n}$. Then the analysis is minimal, i.e.,* $\operatorname{tp}(a/A)$ *is not $\mathcal{P}$-analysable in $n-1$ steps.*

Proof. For $n = 2$, the only possibility that the analysis is not minimal is that $\operatorname{stp}(a/A)$ is 1-step $\mathcal{P}$-analysable, i.e., almost $\mathcal{P}$-internal, which contradicts the fact that (a_1, a_2) is an incompressible analysis.

Assume we have proved the conclusion for $n < k$. Suppose towards a contradiction that $(a_1, \ldots, a_k)$ is an incompressible $\mathcal{P}$-analysis of a over A of

U-type $\underbrace{(1,1,\ldots,1)}_{k}$ which is not minimal. Let $(c_1,\ldots,c_{k-1})$ be another $\mathcal{P}$-analysis of a over A. Note that $(a_1c_1, a_2c_2, \ldots, a_{k-1}c_{k-1})$ is also a $\mathcal{P}$-analysis of a over A. Let $b_1,\ldots,b_\ell$ be a subsequence of $(a_ic_i)_{i=1}^{k-1}$ such that $(b_j)_{j=1}^{\ell}$ is a non-degenerate $\mathcal{P}$-analysis of p. This can be done by taking away all elements a_ic_i in $(a_ic_i)_{i=1}^{k-1}$ such that $U(a_ic_i/Aa_{i-1}c_{i-1}) = 0$. Let $b_j = a$ for $\ell+1 \leq j \leq k-1$. Then the only zero entries of the U-type of $(b_j)_{j=1}^{k-1}$ (if any) are at the end of the sequence.

If $U(b_1/A) = 1$, then $\operatorname{acl}(Ab_1) = \operatorname{acl}(Aa_1)$, and $\operatorname{stp}(a/Aa_1) = \operatorname{stp}(a/Ab_1)$. But then $(a_2,\ldots,a_k)$ is a $(k-1)$-step incompressible $\mathcal{P}$-analysis of a over Aa_1 of U-type $\underbrace{(1,1,\ldots,1)}_{k-1}$, while $(b_2,\ldots,b_{k-1})$ is a $(k-2)$-step $\mathcal{P}$-analysis of the same type with shorter length, contradicting our induction hypothesis.

Now suppose $U(b_1/A) \geq 2$. If the U-type of $(b_j)_{j=1}^{k-1}$ is degenerate then $U(b_{k-1}/b_{k-2}) = 0$, and we have $U(b_{k-2}/A) = U(a/A) = k$. If $(b_j)_{j=1}^{k-1}$ is non-degenerate, then $U(b_j/Ab_{j-1}) \geq 1$ for any $j = 1,\ldots,k-2$ which gives us $U(b_j/A) \geq j+1$ for any $j = 1,\ldots,k-2$. In both cases $U(b_{k-2}/A) \geq k-1$. By the induction hypothesis, $\operatorname{acl}(Ab_{k-2}) \neq \operatorname{acl}(Aa_{k-1})$: otherwise, $(a_i)_{i=1}^{k-1}$ is a $(k-1)$-step incompressible $\mathcal{P}$-analysis of a_{k-1} over A of U-type $\underbrace{(1,1,\ldots,1)}_{k-1}$, while $(b_i)_{i=1}^{k-2}$ is a $(k-2)$-step $\mathcal{P}$-analysis of the same type, contradicting our induction hypothesis. Similarly, $\operatorname{acl}(Ab_{k-2}) \supsetneq \operatorname{acl}(Aa_{k-1})$ does not hold: otherwise $U(b_{k-2}/Aa_{k-1}) \geq 1$, and since $b_{k-2} \in \operatorname{acl}(Aa)$ and $U(a/Aa_{k-1}) = 1$, we have $\operatorname{acl}(Ab_{k-2}) = \operatorname{acl}(Aa)$; therefore $(a_2,\ldots,a_k)$ is a $(k-1)$-step incompressible $\mathcal{P}$-analysis of $\operatorname{stp}(a/Aa_1)$ of U-type $\underbrace{(1,1,\ldots,1)}_{k-1}$, while $(b_1,\ldots,b_{k-2})$ is a $(k-2)$-step $\mathcal{P}$-analysis of the same type, contradicting our induction hypothesis. Hence $\operatorname{acl}(Ab_{k-2}) \supseteq \operatorname{acl}(Aa_{k-1})$ does not hold, i.e., $a_{k-1} \notin \operatorname{acl}(Ab_{k-2})$. We have $k = U(a/A) \geq U(a_{k-1}b_{k-2}/A) = U(b_{k-2}/A) + U(a_{k-1}b_{k-2}/Ab_{k-2}) \geq (k-1)+1 = k$, so $\operatorname{acl}(Ab_{k-2}a_{k-1}) = \operatorname{acl}(Aa)$. But then since $\operatorname{stp}(b_{k-2}/Aa_1)$ and $\operatorname{stp}(a_{k-1}/Aa_1)$ are $(k-2)$-step $\mathcal{P}$-analysable, so is $\operatorname{stp}(a/Aa_1)$, while $(a_2,\ldots,a_k)$ is a $(k-1)$-step incompressible $\mathcal{P}$-analysis of a over Aa_1 of U-

type $\underbrace{(1, 1, \ldots, 1)}_{k-1}$, contradicting our induction hypothesis. $\square$

3.2 Reductions and coreductions

As shown in Example 3.5, Lemma 3.6 does not hold if the entries of the U-type are not all 1. In the higher U-rank case, incompressibility will have to be replaced by some maximality or minimality property. We will use the notions of $\mathcal{P}$-reduction and $\mathcal{P}$-coreduction.

Definition 3.7 (See, for example, Section 4 of [22])**.** Suppose a is a tuple and A is a parameter set. We say a tuple b is a *$\mathcal{P}$-reduction of a over A* if b is maximally almost $\mathcal{P}$-internal over A in $\operatorname{acl}(Aa)$, i.e., $\operatorname{stp}(b/A)$ is almost $\mathcal{P}$-internal, $b \in \operatorname{acl}(Aa)$, and if $c \in \operatorname{acl}(Aa)$ with $\operatorname{stp}(c/A)$ almost $\mathcal{P}$-internal then $c \in \operatorname{acl}(Ab)$. We say a non-degenerate $\mathcal{P}$-analysis $(a_1, \ldots, a_n)$ of a over A is a *$\mathcal{P}$-analysis by reductions* if a_i is the $\mathcal{P}$-reduction of a over Aa_{i-1} for $i = 1, 2, \ldots, n$.

Note that by definition $\mathcal{P}$-reductions are unique up to interalgebraicity over the parameter set, i.e., if b and c are both $\mathcal{P}$-reductions of a over A, then $\operatorname{acl}(Ab) = \operatorname{acl}(Ac)$. We may therefore call b *the* $\mathcal{P}$-reduction of a over A.

Proposition 3.8. *Every $\mathcal{P}$-analysable type of finite U-rank has a $\mathcal{P}$-analysis by reductions.*

Proof. We first show that if $U(a/A) < \omega$ then a $\mathcal{P}$-reduction of a over A exists. Indeed, let b be a tuple that has maximal U-rank over A satisfying the condition that $\operatorname{stp}(b/A)$ is almost $\mathcal{P}$-internal and $b \in \operatorname{acl}(Aa)$. Now, if $c \in \operatorname{acl}(Aa)$ and $\operatorname{stp}(c/A)$ is almost $\mathcal{P}$-internal, then $\operatorname{stp}(bc/A)$ is almost $\mathcal{P}$-internal and $bc \in \operatorname{acl}(Aa)$, so $U(bc/A) = U(b/A)$, which means $c \in \operatorname{acl}(Ab)$. So b is the $\mathcal{P}$-reduction of a over A.

Now suppose $q = \operatorname{tp}(a/A)$ is a stationary type that is of finite U-rank and is $\mathcal{P}$-analysable. Let $a_0 = \varnothing$, and define $a_1, a_2, \ldots$ recursively so that

a_i is the $\mathcal{P}$-reduction of a over Aa_{i-1} and $a_{i-1} \in \mathrm{dcl}(Aa_i)$. By definition we will have that $\mathrm{stp}(a_i/Aa_{i-1})$ is almost $\mathcal{P}$-internal. If $a \notin \mathrm{acl}(Aa_{i-1})$ then, as $\mathrm{stp}(a/Aa_{i-1})$ is also $\mathcal{P}$-analysable by Lemma 3.3, there must exist $b \in \mathrm{acl}(Aa)\backslash\mathrm{acl}(Aa_{i-1})$ such that $\mathrm{stp}(b/Aa_{i-1})$ is $\mathcal{P}$-internal. Hence $U(a/Aa_i) < U(a/Aa_{i-1})$. So this process must stop with $a \in \mathrm{acl}(Aa_n)$, and we have a $\mathcal{P}$-analysis of a over A by reductions. $\square$

Definition 3.9 (See, for example, Definition 4.1 of [22])**.** Suppose a is a tuple and A is a parameter set. We say a tuple b is a *$\mathcal{P}$-coreduction of a over A* if b is minimal in $\mathrm{acl}(Aa)$ such that a is almost $\mathcal{P}$-internal over Ab, i.e., $\mathrm{stp}(a/Ab)$ is almost $\mathcal{P}$-internal, $b \in \mathrm{acl}(Aa)$, and if $c \in \mathrm{acl}(aA)$ with $\mathrm{stp}(a/Ac)$ almost $\mathcal{P}$-internal then $b \in \mathrm{acl}(Ac)$. We say a non-degenerate $\mathcal{P}$-analysis $(a_1, \ldots, a_n)$ of a over A is a *$\mathcal{P}$-analysis by coreductions* if a_{i-1} is a $\mathcal{P}$-coreduction of a_k over A for $i = 1, \ldots, n$.

By definition $\mathcal{P}$-coreductions are unique up to interalgebraicity over the parameter set. We may therefore call b *the* $\mathcal{P}$-coreduction of a over A.

However, $\mathcal{P}$-coreductions do not automatically exist. The analogue of Proposition 3.8 becomes:

Proposition 3.10. *Suppose that T has the property that every finite U-rank type has a $\mathcal{P}$-coreduction. Then every finite U-rank $\mathcal{P}$-analysable type has a $\mathcal{P}$-analysis by coreductions.*

Proof. Suppose $q = \mathrm{tp}(a/A)$ is stationary, of finite U-rank, and $\mathcal{P}$-analysable. We prove by induction on $U(q)$ that it has a $\mathcal{P}$-analysis by coreductions. If $U(q) = 0$ then the 0-step $\mathcal{P}$-analysis is vacuously by coreductions. Suppose $U(q) > 0$. Since q is $\mathcal{P}$-analysable, there is $b \in \mathrm{acl}(Aa)$ with $a \notin \mathrm{acl}(Ab)$ such that $\mathrm{stp}(a/Ab)$ is almost $\mathcal{P}$-internal. Hence if we let $\tilde{b}$ be the $\mathcal{P}$-coreduction of a over A then $\tilde{b} \in \mathrm{acl}(Aa)$, so $\mathrm{stp}(\tilde{b}/A)$ is $\mathcal{P}$-analysable, and $a \notin \mathrm{acl}(A\tilde{b})$ which implies $U(\tilde{b}/A) < U(q)$. By induction we have a $\mathcal{P}$-analysis of $\tilde{b}$ over A, $(b_1, \ldots, b_n)$, that is by coreductions. Then $(b_1, \ldots, b_n, a)$ is a $\mathcal{P}$-analysis of a over A by coreductions. $\square$

One context in which $\mathcal{P}$-coreductions always exist is when $\mathcal{P}$ is the set of all nonmodular minimal types and T has the CBP. Recall that T has the *canonical base property* (CBP) if whenever $U(a/b) < \omega$ and $\operatorname{acl}(b) = \operatorname{acl}(\operatorname{Cb}(a/b))$, then $\operatorname{stp}(b/a)$ is almost internal to the set of all nonmodular minimal types. See, for example, Section 1 of [23]. It is a fact that if T has the CBP then $\mathcal{P}$-coreduction exists for any finite-rank type (see Theorem 2.4 of [6]). So by Proposition 3.10, if $\mathcal{P}$ is the set of nonmodular minimal types and T has the CBP, then every $\mathcal{P}$-analysable type of finite U-rank has a $\mathcal{P}$-analysis by coreductions.

Proposition 3.11. *In* DCF_0 *every* $\mathcal{C}$*-analysable type of finite* U*-rank has a* $\mathcal{C}$*-analysis by coreductions.*

Proof. By Theorem 1.1 of [28], DCF_0 has the CBP. Let $\mathcal{P}$ be the set of all nonmodular minimal types. Therefore, if $\operatorname{tp}(a/A)$ is of finite U-rank then there exists b which is the $\mathcal{P}$-coreduction of a over A. We want to show that b is the $\mathcal{C}$-coreduction of a over A. Recall that $\mathcal{C}$ denotes the field of constants of the differential field $\mathcal{U}$, and that by $\mathcal{C}$-coreduction we mean of course the $\{\{\delta x = 0\}\}$-coreduction. We only need to show that if a type is almost $\mathcal{P}$-internal then it is almost $\mathcal{C}$-internal. Suppose $\operatorname{tp}(e/D)$ is $\mathcal{P}$-internal. Then for some $B \supset D$ such that $B \underset{D}{\downarrow} e$ and a tuple c consisting of realizations of types in $\mathcal{P}$ with bases in B, $e \in \operatorname{acl}(Bc)$. Since every minimal nonmodular type in DCF_0 is almost $\mathcal{C}$-internal, there exist $F \supset B$ such that $F \underset{B}{\downarrow} ec$ and $c \in \operatorname{acl}(F\mathcal{C})$. Now $e \in \operatorname{acl}(Bc) \subseteq \operatorname{acl}(F\mathcal{C})$, and since $e \underset{B}{\downarrow} F$ and $e \underset{D}{\downarrow} B$, we have $e \underset{D}{\downarrow} F$. This shows that $\operatorname{tp}(e/D)$ is almost $\mathcal{C}$-internal. So every finite U-rank type has a $\mathcal{C}$-coreduction. The proposition now follows by Proposition 3.10. $\square$

It is not hard to see that analyses by reductions or coreductions are incompressible. If $(a_1, \ldots, a_n)$ is a $\mathcal{P}$-analysis by reductions of $\operatorname{tp}(a/A)$ and $\operatorname{stp}(a_{i+1}/Aa_{i-1})$ is almost $\mathcal{P}$-internal for some $i = 1, 2, \ldots, n-1$, then since a_i

is the $\mathcal{P}$-reduction of a over Aa_{i-1}, $a_{i+1} \in \operatorname{acl}(Aa_i)$ which implies $\operatorname{acl}(Aa_i) = \operatorname{acl}(Aa_{i+1})$. Now for any $j > i$, assume that $\operatorname{acl}(Aa_j) = \operatorname{acl}(Aa_i)$. Then since a_{j+1} is the $\mathcal{P}$-reduction of a over Aa_j and $\operatorname{acl}(Aa_j) = \operatorname{acl}(Aa_i)$, a_{j+1} is the $\mathcal{P}$-reduction of a over Aa_i, so $\operatorname{acl}(Aa_{j+1}) = \operatorname{acl}(Aa_{i+1}) = \operatorname{acl}(Aa_i)$. Thus $a_i, \ldots, a_n$ are all the same up to interalgebraicity over A, and this is possible only if $i = n$, contradicting the fact that $i \leq n-1$. Similarly, if $(a_1, \ldots, a_n)$ is a $\mathcal{P}$-analysis by coreductions of $\operatorname{tp}(a/A)$ and $\operatorname{stp}(a_{i+1}/Aa_{i-1})$ is almost $\mathcal{P}$-internal for some $i = 1, 2, \ldots, n-1$, then since a_i is the $\mathcal{P}$-coreduction of a_{i+1} over A, $a_i \in \operatorname{acl}(Aa_{i-1})$ which implies a_i and a_{i-1} are interalgebraic over A. An inductive argument similar to the reduction case shows that $a_0, \ldots, a_i$ are all the same up to interalgebraicity over A, and this is possible only if $i = 0$, contradicting the fact that $i \geq 1$.

More is true: they are actually minimal.

Proposition 3.12. *Analyses by reductions are minimal and analyses by coreductions are minimal.*

Proof. Let $(a_1, \ldots, a_n)$ and $(c_1, \ldots, c_\ell)$ be $\mathcal{P}$-analyses of a over A with the analysis $(a_1, \ldots, a_n)$ being by reductions. We shall prove that $n \leq \ell$. We show that $c_i \in \operatorname{acl}(Aa_i)$ for $i = 1, 2, \ldots, \min(n, \ell)$. For $i = 1$, since $\operatorname{stp}(c_1/A)$ is almost $\mathcal{P}$-internal and a_1 is the $\mathcal{P}$-reduction of a over A, $c_1 \in \operatorname{acl}(Aa_1)$. Now if $c_{i-1} \in \operatorname{acl}(Aa_{i-1})$, then $\operatorname{stp}(c_i/a_{i-1})$ is almost $\mathcal{P}$-internal, and as a_i is the $\mathcal{P}$-reduction of a over Aa_{i-1}, $c_i \in \operatorname{acl}(Aa_i)$ as desired. Suppose $\ell < n$. Then $\operatorname{acl}(Aa_\ell) \subsetneq \operatorname{acl}(Aa_n)$ since $(a_1, \ldots, a_n)$ is incompressible, so $\operatorname{acl}(Aa) = \operatorname{acl}(Ac_\ell) \subseteq \operatorname{acl}(Aa_\ell) \subsetneq \operatorname{acl}(Aa_n) = \operatorname{acl}(Aa)$, a contradiction.

Now suppose $(b_1, \ldots, b_m)$ is a $\mathcal{P}$-analysis by coreductions of a over A. We shall prove that $m \leq \ell$. We show that $b_{m-j} \in \operatorname{acl}(Ac_{\ell-j})$ for $j = 0, 1, \ldots, \min(m, \ell) - 1$. For $j = 0$, notice that b_m, c_ℓ are both interalgebraic over A with a. Now if $b_{m-j+1} \in \operatorname{acl}(Ac_{\ell-j+1})$, then $\operatorname{stp}(b_{m-j+1}/c_{\ell-j})$ is almost $\mathcal{P}$-internal, and as b_{m-j} is the $\mathcal{P}$-coreduction of b_{m-j+1} over A, $b_{m-j} \in \operatorname{acl}(Ac_{\ell-j})$ as desired. Assume towards a contradiction that $\ell < m$. Then $\operatorname{acl}(Ab_{m-\ell+1}) \subseteq \operatorname{acl}(Ac_1)$. Since $m - \ell + 1 \geq 2$, $\operatorname{stp}(b_{m-\ell+1}/A)$ is not almost $\mathcal{P}$-internal because $(b_1, \ldots, b_m)$ is incompressible, but $\operatorname{stp}(c_1/A)$ is almost $\mathcal{P}$-internal, a contradiction. $\square$

So analyses by reductions and coreductions are of the same length. However, analyses by reductions and coreductions do not always have to agree (even up to interalgebraicity).

Definition 3.13. We say that two $\mathcal{P}$-analyses $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_m)$ of a over A are *interalgebraic over* A if $n = m$ and $\operatorname{acl}(Aa_i) = \operatorname{acl}(Ab_i)$ for $i = 1, 2, \ldots, n$. We call an analysis *canonical* if it is minimal and interalgebraic with every other minimal analysis.

Example 3.14. Using the notation of Example 3.5, the $\mathcal{P}$-analysis by reductions of ab_1 over $\varnothing$ is $(a_1 b_1, ab_1)$, while the $\mathcal{P}$-analysis by coreductions of ab_1 is (a_1, ab_1). But $(a_1 b_1, ab_1)$ and (a_1, ab_1) are not interalgebraic. In particular, $\operatorname{stp}(ab_1)$ does not have a canonical $\mathcal{P}$-analysis.

The following theorem points out, however, that if an analysis by reductions has the same U-type as one by coreductions, then they are interalgebraic and are in fact canonical $\mathcal{P}$-analyses.

Proposition 3.15. *Let $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ be $\mathcal{P}$-analyses by reductions and coreductions of a over A, respectively. Suppose $U(a/A) \neq \infty$. If the U-types of $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ are the same, then $(a_1, \ldots, a_n)$ is interalgebraic with $(b_1, \ldots, b_n)$ over A. Moreover, if $(c_1, \ldots, c_n)$ is another $\mathcal{P}$-analysis of a over A, then $(c_1, \ldots, c_n)$ is also interalgebraic with both $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ over A.*

In particular, if p has an analysis by reductions and an analysis by coreductions of the same U-type, then these analyses are canonical. Conversely, any canonical analysis of p is an analysis by reductions, and if p has an analysis by coreductions, then the canonical analysis is also an analysis by coreductions.

Proof. Having the same U-type implies that $U(a_i/A) = U(b_i/A)$ for $i = 1, 2, \ldots, n$. Let $(c_1, \ldots, c_n)$ be another $\mathcal{P}$-analysis of a over A, We have seen in the proof of Proposition 3.12 that $c_i \in \operatorname{acl}(Aa_i)$ and $b_i \in \operatorname{acl}(Ac_i)$ for $i = 1, 2, \ldots, n$. Therefore $U(a_i/A) = U(b_i/A) = U(c_i/A)$ and $\operatorname{acl}(Aa_i) = \operatorname{acl}(Ab_i) = \operatorname{acl}(Ac_i)$ for $i = 1, 2, \ldots, n$, as desired.

The "in particular" clause now follows by Proposition 3.12. For the converse, let $(a_i)_{i=1}^n, (b_i)_{i=1}^n, (c_i)_{i=1}^n$ be $\mathcal{P}$-analyses of a over A, which are an analysis by reductions, an analysis by coreductions, and a canonical analysis, respectively. We have that a_i is the $\mathcal{P}$-reduction of a over Aa_{i-1}, $\operatorname{acl}(Aa_i) = \operatorname{acl}(Ac_i)$, and $\operatorname{acl}(Aa_{i-1}) = \operatorname{acl}(Ac_{i-1})$, so c_i is the $\mathcal{P}$-reduction of a over Ac_{i-1}. Thus $(c_i)_{i=1}^n$ is a $\mathcal{P}$-analysis by reductions. Similarly, we have that b_i is the $\mathcal{P}$-coreduction of b_{i+1} over A, $\operatorname{acl}(Ab_i) = \operatorname{acl}(Ac_i)$, and $\operatorname{acl}(Ab_{i+1}) = \operatorname{acl}(Ac_{i+1})$, so c_i is the $\mathcal{P}$-coreduction of a over Ac_{i-1}. Thus $(c_i)_{i=1}^n$ is a $\mathcal{P}$-analysis by coreductions. $\square$

To make use of the above result we will need, both here and in Chapter 4, a way of determining if a given analysis is an analysis by reductions or coreductions. The following is a useful "local" criterion for when an analysis is by reductions or by coreductions.

Proposition 3.16. *Let $(a_1, \ldots, a_n)$ be a $\mathcal{P}$-analysis of a over A. Then it is a $\mathcal{P}$-analysis by reductions iff a_i is a $\mathcal{P}$-reduction of a_{i+1} over Aa_{i-1} for $i = 1, \ldots, n-1$; it is a $\mathcal{P}$-analysis by coreductions iff a_i is a $\mathcal{P}$-coreduction of a_{i+1} over Aa_{i-1} for $i = 1, \ldots, n-1$.*

Proof. Suppose $(a_1, \ldots, a_n)$ is a $\mathcal{P}$-analysis by reductions of a over A. For any $k = 1, 2, \ldots, n-1$, a_k is a $\mathcal{P}$-reduction of a over Aa_{k-1}, i.e., for any $a'_k \in \operatorname{acl}(Aa)$, if $\operatorname{stp}(a'_k/Aa_{k-1})$ is almost $\mathcal{P}$-internal, then $a'_k \in \operatorname{acl}(a_k)$. In particular, for any $a'_k \in \operatorname{acl}(Aa_{k+1})$, if $\operatorname{stp}(a'_k/Aa_{k-1})$ is almost $\mathcal{P}$-internal, then $a'_k \in \operatorname{acl}(a_k)$. Note that $a_k \in \operatorname{acl}(Aa_{k+1})$, so a_k is a $\mathcal{P}$-reduction of a_{k+1} over Aa_{k-1}.

Now suppose $(a_1, \ldots, a_n)$ is a $\mathcal{P}$-analysis of a over A such that a_i is a $\mathcal{P}$-reduction of a_{i+1} over Aa_{i-1} for $i = 1, \ldots, n-1$. We need to check that a_k is the $\mathcal{P}$-reduction of a over Aa_{k-1}. In fact, let a'_k be the $\mathcal{P}$-reduction of a over Aa_{k-1}, then we only need to show that $a'_k \in \operatorname{acl}(Aa_k)$.

We know $a'_k \in \operatorname{acl}(Aa_n)$. Suppose $a'_k \in \operatorname{acl}(Aa_i)$ for some i such that $k < i \leq n$. Since a'_k is almost $\mathcal{P}$-internal over Aa_{k-1} and $k-1 < i-1$, a'_k is almost $\mathcal{P}$-internal over Aa_{i-2}. Now a_{i-1} is a $\mathcal{P}$ reduction of a_i over Aa_{i-2}, $a'_k \in \operatorname{acl}(Aa_i)$, and a'_k is almost $\mathcal{P}$-internal over Aa_{i-2}, so $a'_k \in \operatorname{acl}(Aa_{i-1})$. By induction we get $a'_k \in \operatorname{acl}(Aa_k)$.

We now turn to the coreduction part of this proposition. Suppose that $(a_1, \ldots, a_n)$ is a $\mathcal{P}$-analysis by coreductions of a over A. For any $k = 1, 2, \ldots, n-1$, a_k is a $\mathcal{P}$-coreduction of a_{k+1} over A, i.e., for any $a'_k \in \operatorname{acl}(Aa_{k+1})$, if $\operatorname{stp}(a_{k+1}/Aa'_k)$ is $\mathcal{P}$-internal, then $a_k \in \operatorname{acl}(Aa'_k)$. In particular, for any $a'_k \in \operatorname{acl}(Aa_{k+1})$, if $\operatorname{stp}(a_{k+1}/Aa_{k-1}a'_k)$ is $\mathcal{P}$-internal, then $a_k \in \operatorname{acl}(Aa_{k-1}a'_k)$. So we have that a_k is a reduction of a_{k+1} over Aa_{k-1}.

Now suppose $(a_1, \ldots, a_n)$ is a $\mathcal{P}$-analysis of a over A such that a_i is a $\mathcal{P}$-coreduction of a_{i+1} over Aa_{i-1} for $i = 1, \ldots, n-1$. Fixing a $k \in \{1, 2, \ldots, n-1\}$, we need to check that a_k is the $\mathcal{P}$-coreduction of a_{k+1} over A. In fact, let a' be be such that $\operatorname{stp}(a_{k+1}/Aa')$ is almost $\mathcal{P}$-internal. We need to prove that $a_k \in \operatorname{acl}(Aa')$.

We know that $a_1 \in \operatorname{acl}(Aa')$. This is because a_1 is the $\mathcal{P}$-coreduction of a_2 over A, and $\operatorname{stp}(a_2/Aa')$ is almost $\mathcal{P}$-internal (since $a_2 \in \operatorname{dcl}(Aa_{k+1})$).

Suppose $a_{i-1} \in \operatorname{acl}(Aa')$ for some i such that $1 < i \leq k$. Since a_{i+1} is almost $\mathcal{P}$-internal over Aa' (as $i+1 \leq k+1$, $a_{i+1} \in \operatorname{acl}(Aa_{k+1})$), and a_i is the $\mathcal{P}$-coreduction of a_{i+1} over Aa_{i-1}, we have that $a_i \in \operatorname{acl}(Aa')$. By induction we get $a_k \in \operatorname{acl}(Aa')$. $\square$

It follows from the above lemma that an incompressible analysis of U-type $(1, 1, \ldots, 1)$ is canonical. Indeed, for such an analysis $(a_1, \ldots, a_n)$ of a over A, as $\operatorname{stp}(a_{i+1}/Aa_{i-1})$ is not almost $\mathcal{P}$-internal, by rank consideration, a_i must be both the $\mathcal{P}$-reduction and the $\mathcal{P}$-coreduction of a_{i+1} over Aa_{i-1}

for $i = 1, 2, \ldots, n-1$.

We end this section by pointing out that once we have a type with an incompressible analysis of U-type $\underbrace{(1, 1, \ldots, 1)}_{n}$ — as for example we will prove in Corollary 4.2 below that we do in DCF_0 — then every decreasing sequence of positive integers of length n appears as the U-type of the $\mathcal{P}$-analysis by reductions of some other type in this theory. A similar statement holds for increasing sequences and $\mathcal{P}$-analyses by coreductions provided that every finite U-rank type has a $\mathcal{P}$-coreduction. For convenience we work over the empty set.

Proposition 3.17. *Suppose $(a_1, \ldots, a_n)$ is an incompressible $\mathcal{P}$-analysis of U-type $(1, 1, \ldots, 1)$.*

(a) Given positive integers $s_1 \geq \cdots \geq s_n$, there exists a tuple whose $\mathcal{P}$-analysis by reductions is of U-type $(s_1, \ldots, s_n)$.

(b) Suppose every type of finite U-rank has a $\mathcal{P}$-coreduction. Given positive integers $s_1 \leq \cdots \leq s_n$, there exists a tuple whose $\mathcal{P}$-analysis by coreductions is of U-type $(s_1, \ldots, s_n)$.

Proof. (a) Let $\bar{a}^{(j)} = (a_1^{(j)}, \ldots, a_n^{(j)})$, $j = 1, 2, \ldots$ be tuples such that $(\bar{a}^{(1)}, \bar{a}^{(2)}, \ldots)$ is a Morley sequence of $\mathrm{tp}(a_1, \ldots, a_n)$. Let $\alpha_i = (a_i^{(1)}, \ldots, a_i^{(s_i)})$ and $\beta_i = (\alpha_1, \ldots, \alpha_i)$. Note that $a_i^{(j)} \in \beta_i$ for $j = 1, 2, \ldots, s_i$. We claim the tuple β_n is $\mathcal{P}$-analysable and its $\mathcal{P}$-analysis by reductions is of U-type $(s_1, \ldots, s_n)$. To show this, since $(\bar{a}^{(j)})_j$ is a Morley sequence, we have

$$\begin{aligned}
U(\beta_i/\beta_{i-1}) &= U(\alpha_i/\beta_{i-1}) \\
&= U(a_i^{(1)} \cdots a_i^{(s_i)}/\beta_{i-1}) \\
&= U(a_i^{(1)} \cdots a_i^{(s_i)}/a_{i-1}^{(1)} \cdots a_{i-1}^{(s_i)}) \\
&= s_i,
\end{aligned}$$

so we only need to prove that the $\mathcal{P}$-analysis by reductions of β is $(\beta_1, \beta_2, \ldots, \beta_n)$.

We simply check the definition in this case. Let b_i be the $\mathcal{P}$-reduction of β_n over β_{i-1}. We claim, and this will suffice, that b_i is interalgebraic with β_i. Since $a^{(j)}_{i-1} \in \mathrm{dcl}(\beta_{i-1})$ for $j = 1, 2, \ldots, s_i$ (since $s_{i-1} \geq s_i$), $\mathrm{stp}(a^{(j)}_i/\beta_{i-1})$ is almost $\mathcal{P}$-internal for $j = 1, 2, \ldots, s_i$, so $\mathrm{stp}(\alpha_i/\beta_{i-1})$ is almost $\mathcal{P}$-internal. Since $\beta_i \in \mathrm{dcl}(\alpha_i, \beta_{i-1})$, $\mathrm{stp}(\beta_i/\beta_{i-1})$ is almost $\mathcal{P}$-internal, so $\beta_i \in \mathrm{acl}(b_i)$. We now need to show that $U(b_i/\beta_i) = 0$. Toward a contradiction, suppose $U(b_i/\beta_i) > 0$.

Set $B = \beta_i$, which is the collection of elements of the form $a^{(q)}_p$ where $1 \leq p \leq i$ and $1 \leq q \leq s_i$. Now we add elements of the form $a^{(q)}_p$ one by one into B according to lexicographic order of (p, q) where $i+1 \leq p \leq n$ and $1 \leq q \leq s_i$ as long as $U(b_i/B)$ remains unchanged. Since $b_i \in \beta_n$, $U(b_i/\beta_n) = 0$, so this process will terminate for some $a^{(q)}_p$ where $U(b_i/Ba^{(q)}_p) < U(b_i/B)$.

Now B contains elements of the form $a^{(q')}_{p'}$ where $(p', q') < (p, q)$ by lexicographic order. We have $a^{(q)}_p \not\mathop{\smile\!\!\!\!\!\!|}_B b_i$. As $a^{(q)}_{p-1} \in B$ and $a^{(q)}_p \mathop{\smile\!\!\!\!\!\!|}_{a^{(q)}_{p-1}} B$, $U(a^{(q)}_p/B) = 1$, so $a^{(q)}_p \in \mathrm{acl}(Bb_i)$. Let $C = \{a^{(j)}_i : a^{(j)}_{i+1} \in \mathrm{dcl}(B)\}$. Then $\mathrm{stp}(B/C)$ is almost $\mathcal{P}$-internal as $\mathrm{stp}(a^{(j)}_{i+1}/a^{(j)}_i)$ is almost internal for any i, j, and $\mathrm{stp}(b_i/C)$ is almost $\mathcal{P}$-internal because $\beta_{i-1} \in \mathrm{dcl}(C)$. Since $a^{(q)}_p$ is in $\mathrm{acl}(Bb_i)$, this yields that $\mathrm{stp}(a^{(q)}_p/C)$ is almost $\mathcal{P}$-internal. However, the latter is impossible since $a^{(q)}_{p-1} \notin \mathrm{acl}(C)$, which is because $a^{(q)}_{p-1} \notin \mathrm{acl}(a^{(q)}_{p-2})$ and $a^{(q)}_{p-1} \mathop{\smile\!\!\!\!\!\!|}_{a^{(q)}_{p-2}} C$.

(b) Let $\bar{a}^{(j)} = (a^{(j)}_1, \ldots, a^{(j)}_n)$, $j = 1, 2, \ldots$ be tuples such that $(\bar{a}^{(1)}, \bar{a}^{(2)}, \ldots)$ is a Morley sequence of $\mathrm{tp}(a_1, \ldots, a_n)$. Let $\beta_i = (a^{(1)}_1, \ldots, a^{(s_i)}_1, a^{(1)}_2, \ldots, a^{(s_{i-1})}_2, \ldots, a^{(1)}_i, \ldots, a^{(s_1)}_i)$. Let $f(j) = \min\{k : j \leq s_k\}$, and let $f(j)$ be infinity if it is not defined. Then $a^{(j)}_k \in \mathrm{acl}(\beta_i)$ iff $k \leq i - f(j) + 1$, and $\beta_i = \bigcup_{j=1}^{s_i} a^{(j)}_{i+1-f(j)}$. We claim the tuple β_n is $\mathcal{P}$-analysable and its $\mathcal{P}$-analysis by coreductions is of U-type $(s_1, \ldots, s_n)$. Since $\beta_i = \bigcup_{j=1}^{s_i} a^{(j)}_{i+1-f(j)}$

and $\beta_{i-1} = \bigcup_{j=1}^{s_i} a^{(j)}_{i-f(j)}$ (as $i - f(j) = 0$ for $s_{i-1} < j \leq s_i$, we may set the upper bound as s_i), we have

$$\begin{aligned} U(\beta_i/\beta_{i-1}) &= U(\bigcup_{j=1}^{s_i} a^{(j)}_{i+1-f(j)}/\beta_{i-1}) \\ &= \sum_{j=1}^{s_i} U(a^{(j)}_{i+1-f(j)}/a^{(j)}_{i-f(j)}) \\ &= s_i \end{aligned}$$

as $(\bar{a}^{(j)})_j$ is a Morley sequence. Thus we only need to prove that the $\mathcal{P}$-analysis by coreductions of β is $(\beta_1, \beta_2, \ldots, \beta_n)$.

Again, we check the definition. Fix $i = 0, 1, \ldots, n-1$. Suppose b is the $\mathcal{P}$-coreduction of β_{i+1} over the empty set. We claim that $\mathrm{acl}(b) = \mathrm{acl}(\beta_i)$. Note that $\mathrm{stp}(\beta_{i+1}/\beta_i)$ is almost $\mathcal{P}$-internal, so $b \in \mathrm{acl}(\beta_i)$. Take any $a^{(k)}_j \in \mathrm{dcl}(\beta_i)$. Since $a^{(k)}_{j+1} \in \mathrm{dcl}(\beta_{i+1})$ and $\mathrm{stp}(\beta_{i+1}/b)$ is almost $\mathcal{P}$-internal, $\mathrm{stp}(a^{(k)}_{j+1}/b)$ is almost $\mathcal{P}$-internal, so $a^{(k)}_j \in \mathrm{acl}(b)$ since $a^{(k)}_j$ is the $\mathcal{P}$-coreduction of $a^{(k)}_{j+1}$. We therefore have that $\beta_i \in \mathrm{acl}(b)$. $\square$

4 Some constructions of analysability in DCF_0

Probably the best known example of an analysable but not internal to the constants Kolchin closed set is the one defined by the equation $\delta\left(\frac{\delta x}{x}\right) = 0$. It decomposes as an extension of the additive group of constants by the multiplicative group of constants, without itself being almost internal to the constants. Our first observation is to generalize this construction by iterating the logarithmic derivative. Writing $\log\delta x := \frac{\delta x}{x}$ and $\log\delta^{(m)} = \underbrace{\log\delta \cdots \log\delta}_{m}$ we consider the equation $\log\delta^{(m)} x = 0$, and show in Section 4.1 that while it is analysable in the constants in m steps, it is not analysable in $m-1$ steps. This is done by essentially reducing to the $m = 2$ case.

Furthermore, we want analyses of a type p that are *canonical*. Not every finite rank type in DCF_0 admits a canonical analysis (see Example 3.5). However, we show in Section 4.2 that given any sequence of positive integers $(n_1, \dots, n_m)$ there exists in DCF_0 a type that has a canonical analysis in the constants with ith step having U-rank n_i. Unlike in the logarithmic derivative case, these examples are not differential algebraic groups, and hence that theory is not directly available to us. Our proofs involve a careful algebraic analysis of the equations that arise. Note that the situation is very different for differential algebraic groups; in [3] it is shown that every differential algebraic group over the constants is analysable in at most 3 steps.

The results presented in this chapter appeared in [12].

4.1 Iterated logarithmic derivative

We work in a saturated model $\mathcal{U} = (U, 0, 1, +, \times, \delta)$ of DCF_0. We often omit $0, 1, +, \times$ and write $\mathcal{U} = (U, \delta)$.

We focus on types which are almost $\mathcal{C}$-internal or $\mathcal{C}$-analysable in DCF_0, where $\mathcal{C} = \{x : \delta x = 0\}$ is the field of constants of $\mathcal{U}$.

We will be considering iterated logarithmic derivatives. For any $n \geq 1$ we set $\log\delta^{(n)}(x) := \underbrace{\log\delta \log\delta \ \cdots \ \log\delta(x)}_{n \text{ times}}$. Note that $\log\delta^{(n)}(x)$ is only defined at x if $\log\delta^{(i)}(x) \neq 0$ for $i = 0, 1, \ldots, n-1$ where $\log\delta^{(0)}(x) = x$. Whenever we write $\log\delta^{(n)}(x)$ it is always assumed that x is in this domain of definition. Note that for any $h \in \mathcal{U}$, the equation $\log\delta^{(n)}(x) = h$ defines an irreducible Kolchin constructible subset B of $\mathcal{U}$. Indeed, B is isomorphic to

$$\begin{aligned} B^* &= \{(x, \log\delta(x), \ldots, \log\delta^{(n-1)}(x)) : x \in B\} \\ &= \{(x_1, \ldots, x_n) : x_i \neq 0; \frac{\delta x_i}{x_i} = x_{i+1}, i = 1, 2, \ldots, n-1; \frac{\delta x_n}{x_n} = h\} \end{aligned}$$

whose Kolchin closure is

$$\{(x_1, \ldots, x_n) : \delta x_i = x_i x_{i+1}, i = 1, 2, \ldots, n-1; \delta x_n = h x_n\},$$

which is irreducible since it is the set of D-points (or "sharp" set) corresponding to the irreducible D-variety $(\mathbb{A}^n, s)$ where $s(x_1, \ldots, x_{n-1}, x_n) = (x_1 x_2, \ldots, x_{n-1} x_n, h x_n)$. (For details on D-varieties see [15].)

In particular, $\{x : \log\delta^{(2)}(x) = h\}$ is irreducible. Note also that the generic type of $\log\delta^{(2)}(x) = 0$ is the same as that of G which is defined in Example 2.27. So the following proposition is in fact a generalisation of Example 2.27.

Proposition 4.1. *Let $h \in U$ and consider $B = \{x : \log\delta^{(2)}(x) = h\}$. Let k be a δ-field containing h, and p be the generic type of B over k. Then p is not almost $\mathcal{C}$-internal.*

Proof. We may assume that k contains an element of the form $a = \log\delta g_0$ where $g_0 \in B$. Indeed, this follows from the fact that for any $g_0 \in B$, p is almost $\mathcal{C}$-internal iff the non-forking extension of p to $k\langle g_0\rangle$ is, and $p|k\langle g_0\rangle$ is the generic type of B over $k\langle g_0\rangle$.

We now construct a new model $\mathcal{V} = (U, D)$ of DCF_0 as follows. The set U and the interpretation of $0, 1, +$ and $\times$ remain the same, while $Dg := \dfrac{\delta g}{a}$

for all $g \in \mathcal{U}$. Notice that $\mathcal{V}$ is also a model of DCF_0 with the same field of constants as $\mathcal{U}$, and any definable set in one model is definable in the other, with the same set of parameters, as long as the parameter set contains a. Now let q be a type in the model $\mathcal{V}$ over k so that q and p have the same set of realizations in U. This can be done by replacing each occurrence of δ in formulas in p by aD.

Assume towards a contradiction that p is almost $\mathcal{C}$-internal. Hence, for any $g \models p$, there is $B \supset k$ such that $g \downarrow_k B$ and $g \in \mathrm{acl}(BC)$, in the model $\mathcal{U}$. Replacing δ by aD in the formulas witnessing this fact, we have that $g \in \mathrm{acl}(BC)$ in $\mathcal{V}$ as well. Moreover, $g \downarrow_k B$ holds in $\mathcal{V}$ because U-ranks of types are the same in $\mathcal{U}$ and $\mathcal{V}$ if the parameter set contains a. We get that q is almost $\mathcal{C}$-internal in $\mathcal{V}$.

However, q is the generic type of B, since Kolchin closed sets definable over k (which contains a) are the same in $\mathcal{U}$ and $\mathcal{V}$. The set B is defined in $\mathcal{U}$ by the formula $\log\delta(\log\delta x) = h$, which is just $a \log D(a\log Dx) = h$, which is equivalent to $\log D(\log Dx) = 0$. So q is the generic type of $B = \{x : \log D(\log Dx) = 0\}$, which is not almost $\mathcal{C}$-internal in $\mathcal{V}$ by Example 2.27, a contradiction. $\square$

We can now show that the iterated logarithmic derivatives give rise to n-step $\mathcal{C}$-analysable types that are not $(n-1)$-step $\mathcal{C}$-analysable.

Corollary 4.2. *In* DCF_0*, let* $D = \{x \in U : \underbrace{\log\delta \log\delta \cdots \log\delta}_{n} x = 0\}$. *Then the generic type* p *of* D *is* n*-step* $\mathcal{C}$*-analysable but not* $(n-1)$*-step* $\mathcal{C}$*-analysable.*

Proof. Let $a \in D$ be generic. Let $a_n = a$, $a_k = \log\delta a_{k+1}$ for $k = 0, 1, \ldots, n-1$. Note that $a_0 = 0$, $a_k \in \mathrm{dcl}(a_{k+1})$ for $k = 0, 1, \ldots, n-1$, and a is interdefinable with $(a_1, \ldots, a_n)$.

As a is generic in D, $a_{i+1} \notin \mathrm{acl}(a_i)$ for each $i = 0, 1, \ldots, n-1$. By additivity of U-rank, for each $i = 0, 1, \ldots, n-1$, $U(a_{i+1}/a_i) = 1$. Hence, $\mathrm{stp}(a_{i+1}/a_i)$ is the generic type over a_i of $\log\delta(x) = a_i$. The latter equation defines a multiplicative translation of $\mathbb{G}_m(\mathcal{C}) = \ker(\log\delta)$, so $\mathrm{stp}(a_{i+1}/a_i)$ is almost $\mathcal{C}$-internal of U-rank 1. That is, $(a_1, a_2, \ldots, a_n = a)$ is a $\mathcal{C}$-analysis of p of U-type $\underbrace{(1, 1, \ldots, 1)}_{n}$.

For each $i = 1, 2, \ldots, n-1$, $\mathrm{stp}(a_{i+1}/a_{i-1})$ is the generic type of $\log\delta^{(2)}x = a_{i-1}$ over a_{i-1}. Proposition 4.1 tells us that this type is not almost $\mathcal{C}$-internal. That is, $(a_1, a_2, \ldots, a_n)$ is an incompressible $\mathcal{C}$-analysis.

Hence, by Lemma 3.6, p is not $\mathcal{C}$-analysable in $n-1$ steps. $\square$

4.2 A construction of a canonical $\mathcal{C}$-analysis with given U-type

In this section we show that in DCF_0 we can do better than the conclusions of Proposition 3.17. Given any sequence of positive integers we provide a type which has a canonical $\mathcal{C}$-analysis with that U-type. Our strategy is to build an example with a $\mathcal{C}$-analysis of the desired U-type, use Proposition 3.16 to check that it is an analysis by both reductions and coreductions, and then use Proposition 3.15 to conclude that it is canonical. Throughout we use the fact proven in Proposition 3.11 that any finite rank type has a $\mathcal{C}$-coreduction.

Suppose $n_1, \ldots, n_\ell$ are positive integers. We want to construct a type admitting a $\mathcal{C}$-analysis in ℓ steps where the ith step has U-rank n_i, and such that the analysis is canonical. Here is our construction.

For convenience, we name everything in $\mathbb{Q}^{\mathrm{alg}}$ in the language. Let $c_{ij} \in \mathbb{Q}^{\mathrm{alg}}$ be algebraic numbers for $i = 1, 2, \ldots, \ell$ and $1 \leq j \leq n_i$ such that $\{c_{ij}\}_{j=1}^{n_i}$ is $\mathbb{Q}$-linearly independent for $i = 1, 2, \ldots, \ell$.

We inductively define (D_i, e_i) for $i = 1, 2, \ldots, \ell$ as follows:

Set $D_1 := \delta$ and let e_1 be a generic solution over $\varnothing$ to

$$(D_1 - c_{11})(D_1 - c_{12}) \cdots (D_1 - c_{1n_1})x = 0. \tag{E$_1$}$$

For $i > 1$ set $D_i := \dfrac{\delta}{\prod_{j=1}^{i-1} e_j}$ and let e_i be a generic solution over $\{e_1, \ldots e_{i-1}\}$ to

$$(D_i - c_{i1})(D_i - c_{i2}) \cdots (D_i - c_{in_i})x = 0. \tag{E$_i$}$$

The notation $D_i - c_{ij}$ here represents a linear operator which sends y to $D_i y - c_{ij} y$, so equation (E$_i$) is a linear differential equation over $\{e_1, \ldots e_{i-1}\}$ of order n_i.

Now let $a_i = (e_1, \ldots, e_i)$ for $i = 1, 2, \ldots, n$, and $a_0 = \varnothing$. We will show that $(a_1 \cdots a_\ell)$ is a canonical $\mathcal{C}$-analysis of a_ℓ of U-type $(n_1, \ldots, n_\ell)$.

Since e_i is a generic solution of (E$_i$), an order n_i linear differential equation over a_{i-1}, we have $U(a_i/a_{i-1}) = n_i$, and $\mathrm{stp}(a_i/a_{i-1})$ is almost $\mathcal{C}$-internal. So this is a $\mathcal{C}$-analysis of the correct U-type. We need to show it is by $\mathcal{C}$-reductions and $\mathcal{C}$-coreduction.

Fixing $i \in \{1, 2, \ldots, \ell\}$, the following coordinatisation of solutions of (E$_i$) is a useful tool that we will apply often.

Lemma 4.3. *If f is any solution to (E$_i$) then we can decompose $f = \sum_{j=1}^{n_i} f_j$ such that each f_j is a solution to $D_i x - c_{ij} x = 0$ and f is interdefinable with $(f_1, \ldots, f_{n_i})$ over a_{i-1}.*

Proof. Indeed, let g_j be a generic solution of $D_i x - c_{ij} x = 0$. The set $\{g_j : j = 1, 2, \ldots, n_i\}$ is $\mathcal{C}$-linearly independent because g_j's are nonzero eigenvectors of different eigenvalues under the $\mathcal{C}$-linear operator D_i. Note that since $(D_i - c_{ij})$ commutes with $(D_i - c_{ij'})$ for any j, j', each g_j is a solution to (E$_i$). Since (E$_i$) is an order n_i linear differential equation and $\{g_j : j = 1, 2, \ldots, n_i\}$ is a set of $\mathcal{C}$-linearly independent solutions of (E$_i$), any solution of (E$_i$) is a $\mathcal{C}$-linear combination of g_j's. In particular, f is of the form $\sum_{j=1}^{n_i} u_j g_j$ where $u_j \in \mathcal{C}$ for

$j = 1, \ldots, n_i$. Let $f_j = u_j g_j$, so $f = \sum_{j=1}^{n_i} f_j$, and $f \in \mathrm{dcl}(f_1, \ldots, f_{n_i})$. Also,

$$D_i f_j - c_{ij} f_j = u_j (D_i g_j - c_{ij} g_j) = 0,$$

so f_j is a solution to $D_i x - c_{ij} x = 0$.

We still need to verify that $(f_1, \ldots, f_{n_i}) \in \mathrm{dcl}(a_{i-1} f)$. Indeed, suppose $(f_j^*)_{j=1}^{n_i}$ and $(f_j)_{j=1}^{n_i}$ have the same type over $a_{i-1} f$. Then in particular f_j^* is a solution to $D_i x - c_{ij} x = 0$, and

$$\sum_{j=1}^{n_i} f_j = f = \sum_{j=1}^{n_i} f_j^*$$

which gives us $\sum_{j=1}^{n_i} (f_j - f_j^*) = 0$. As $\{f_j - f_j^* : j = 1, 2, \ldots, n_i\}$ is a set of eigenvectors of different eigenvalues under the $\mathcal{C}$-linear operator D_i, we then have $f_j - f_j^* = 0$ for all $j = 1, 2, \ldots, n_i$, so $(f_j^*)_{j=1}^{n_i} = (f_j)_{j=1}^{n_i}$. $\square$

Lemma 4.4. *If f is a generic solution to (E_i) over a_{i-1}, then $\{f_1, \ldots, f_{n_i}\}$ obtained in Lemma 4.3 is independent over a_{i-1} and each f_j is a generic solution to $D_i x - c_{ij} x = 0$.*

Proof. Since f is a generic solution over a_{i-1} to (E_i), which is a linear differential equation of order n_i, we have $U(f/a_{i-1}) = n_i$ Since f_j is a solution for $D_i x - c_{ij} x = 0$, $U(f_{ij}/a_{i-1}) \leq 1$. But

$$\begin{aligned}
n_i &= U(f/a_{i-1}) \\
&= U(f_1 f_2 \cdots f_{n_i}/a_{i-1}) \\
&= U(f_1/a_{i-1}) + U(f_2/a_{i-1} f_1) + \cdots + U(f_{n_i}/a_{i-1} f_1 f_2 \cdots f_{n_i - 1}) \\
&\leq U(f_1/a_{i-1}) + U(f_2/a_{i-1}) + \cdots + U(f_{n_i}/a_{i-1}) \\
&\leq n_i.
\end{aligned}$$

So $U(f_j/a_{i-1}) = 1$ and $U(f_j/a_{i-1}f_1f_2\cdots f_{j-1}) = 1$ for $j = 1, 2, \ldots, n_i$. This means that $\{f_1, \ldots, f_{n_i}\}$ is independent over a_{i-1} and each f_j is a generic solution to $D_ix - c_{ij}x = 0$. $\square$

Lemma 4.5. *Let f be a generic solution over $\mathbb{Q}^{\text{alg}}$ to (E$_1$). Then* $\operatorname{acl}(f) \cap \mathcal{C} = \mathbb{Q}^{\text{alg}}$.

Proof. Let $m = n_1$. Let $(f_1, \ldots, f_m)$ be the decomposition of f by Lemma 4.3 with respect to (E$_1$). Since f is generic, $f_j \neq 0$ for $j = 1, 2, \ldots, m$. Suppose the conclusion is false and there exists some c such that $c \in (\operatorname{acl}(f) \cap \mathcal{C}) \setminus \mathbb{Q}^{\text{alg}}$. Note that $\operatorname{acl}(f) = \mathbb{Q}(f_1, \ldots, f_m)^{\text{alg}}$ since $\delta f_j = c_{1j}f_j \in \mathbb{Q}^{\text{alg}}(f_j)$.

For simplicity, let $\bar{f} = (f_1, \ldots, f_m)$, and $\bar{y} = (y_1, \ldots, y_m)$. Let $F(x, \bar{y})$ be a polynomial with coefficients in $\mathbb{Q}^{\text{alg}}$ such that $F(c, \bar{f}) = 0$ and $F(x, \bar{f}) \neq 0$. Since $c \notin \mathbb{Q}^{alg}$, $F(c, \bar{y}) \neq 0$. Let $G(\bar{y})$ be a nonzero polynomial over $\mathcal{C}$ with minimal number of terms such that $G(\bar{f}) = 0$. Since $F(c, \bar{y}) \neq 0$ and $F(c, \bar{f}) = 0$, $F(c, \bar{y})$ satisfies all conditions of G except for the minimality, so such a G exists.

Let
$$G(\bar{y}) = \sum_{\bar{r} \in I} s_{\bar{r}} \bar{y}^{\bar{r}},$$
where I is a set of m-tuples of nonnegative integers, and $s_{\bar{r}} \in \mathcal{C}$. Let $\bar{c} = (c_{11}, \ldots, c_{1m})$, and set $\bar{f}\bar{c} := \sum_{j=1}^{m} f_j c_{1j}$.

We claim that
$$\bar{r}^{(1)}\bar{c} = \bar{r}^{(2)}\bar{c}$$
for all $\bar{r}^{(1)}, \bar{r}^{(2)} \in I$. Indeed, otherwise, fixing any $\bar{r}^* \in I$, we have
$$\begin{aligned} G^*(\bar{y}) &:= \bar{r}^*\bar{c}G(\bar{y}) - \delta(G(\bar{y})) \\ &= \sum_{\bar{r} \in I} (\bar{r}^*\bar{c}) s_{\bar{r}} \bar{y}^{\bar{r}} - \sum_{\bar{r} \in I} s_{\bar{r}} \delta \bar{y}^{\bar{r}} \\ &= \sum_{\bar{r} \in I} (\bar{r}^*\bar{c} - \bar{r}\bar{c}) s_{\bar{r}} \bar{y}^{\bar{r}} \end{aligned}$$

is a polynomial with fewer terms than G (since the term with index $\bar{r}^*$ is cancelled) such that its coefficients are in $\mathcal{C}$, $G^*(\bar{f}) = 0$ as $G(\bar{f}) = \delta(G(\bar{f})) = 0$, and $G^*(\bar{y}) \neq 0$ as there exist $\bar{r} \in I$ such that $\bar{r}\bar{c} \neq \bar{r}^*\bar{c}$. This contradicts the minimality of G.

We now have $\bar{r}^{(1)}\bar{c} = \bar{r}^{(2)}\bar{c}$ for all $\bar{r}^{(1)}, \bar{r}^{(2)} \in I$, i.e., $(\bar{r}^{(1)} - \bar{r}^{(2)})\bar{c} = 0$ for all $\bar{r}^{(1)}, \bar{r}^{(2)} \in I$. But $\{c_{11}, \ldots, c_{1m}\}$ is $\mathbb{Q}$-linearly independent, so in fact $\bar{r}^{(1)} = \bar{r}^{(2)}$ for all $\bar{r}^{(1)}, \bar{r}^{(2)} \in I$. Therefore there is only one element $\bar{r}$ in I, and $G(\bar{f}) = s_{\bar{r}}\bar{f}^{\bar{r}}$. Since all f_j's are nonzero, $s_{\bar{r}} = 0$, so G is the zero polynomial, a contradiction. $\square$

The following lemma is the technical heart of the construction.

Lemma 4.6. *Fix $i \in \{1, 2, \ldots, \ell - 1\}$, and for notational convenience, let $m := n_i$ and $L := \operatorname{acl}(a_{i-1})$. Then the following are true:*

(i) Suppose f is a solution of (E$_i$) and $(f_1, \ldots, f_m)$ is the decomposition of f by Lemma 4.3. Then f is generic over L iff all the f_j are nonzero.

(ii) Suppose f is a generic solution to (E$_i$) over L, $\alpha \in \mathbb{Q}^{\mathrm{alg}}$ is nonzero, and h is a nonzero solution of $D_i x - \alpha f x = 0$. Then f is the $\mathcal{C}$-coreduction of h over L.

(iii) The $\mathcal{C}$-coreduction of a_{i+1} over a_{i-1} is a_i.

(iv) The $\mathcal{C}$-reduction of a_{i+1} over a_{i-1} is a_i.

Proof. We use induction on i.

(i) Suppose the conclusion is true for $i - 1$.

By Lemma 4.4, if f is a generic solution to (E$_i$) over L, then for any $j \in \{1, 2, \ldots, m\}$, f_j is a generic solution to $D_i x - c_{ij} x = 0$. In particular, $f_j \neq 0$.

Now suppose $f_j \neq 0$ for all $j = 1, 2, \ldots, m$, but f is not generic, i.e., $U(f/L) < m$. Since

$$\begin{aligned} U(f/a_{i-1}) &= U(f_1 f_2 \cdots f_m/a_{i-1}) \\ &= U(f_1/a_{i-1}) + U(f_2/a_{i-1}f_1) + \cdots + U(f_m/a_{i-1}f_1 f_2 \cdots f_{m-1}), \end{aligned}$$

$U(f_j/a_{i-1}f_1 f_2 \cdots f_{j-1}) < 1$ for some j, and hence $f_j \in L\langle\bigcup_{k\neq j} f_k\rangle^{\mathrm{alg}}$ for that j. Note that

$$\delta f_k = (D_i f_k)\prod_{j=1}^{i-1} e_j = c_{ik} f_k \prod_{j=1}^{i-1} e_j \in L(f_k),$$

so $f_j \in L\langle\bigcup_{k\neq j} f_k\rangle^{\mathrm{alg}} = L(\bigcup_{k\neq j} f_k)^{\mathrm{alg}}$, which means that $\{f_1, \ldots, f_m\}$ is algebraically dependent over L in the field theoretic sense.

Let $\bar{f} = (f_1, \ldots, f_m)$, $\bar{y} = (y_1, \ldots, y_m)$, and $\bar{c} = (c_{i1}, \ldots, c_{im})$. Let $G(\bar{y})$ be a nonzero polynomial with minimal number of terms such that its coefficients are in L and $G(\bar{f}) = 0$. We will use a minimality argument similar to that in the proof of Lemma 4.5. Suppose

$$G(y_1, \ldots, y_m) = \sum_{\bar{r}\in I} s_{\bar{r}} \bar{y}^{\bar{r}},$$

where I is a set of m-tuples of nonnegative integers, and $s_{\bar{r}} \in L$ for $\bar{r} \in I$. Now

$$\begin{aligned} D_i(G(\bar{f})) &= D_i \sum_{\bar{r}\in I} s_{\bar{r}} \bar{f}^{\bar{r}} \\ &= \sum_{\bar{r}\in I} (\bar{f}^{\bar{r}} D_i s_{\bar{r}} + s_{\bar{r}} D_i \bar{f}^{\bar{r}}) \\ &= \sum_{\bar{r}\in I} (\log D_i s_{\bar{r}} + \bar{r}\bar{c}) s_{\bar{r}} \bar{f}^{\bar{r}}. \end{aligned}$$

We claim that

$$\log D_i s_{\bar{r}^{(1)}} + \bar{r}^{(1)}\bar{c} = \log D_i s_{\bar{r}^{(2)}} + \bar{r}^{(2)}\bar{c}$$

for all $\bar{r}^{(1)}, \bar{r}^{(2)} \in I$. Indeed, otherwise, fixing any $\bar{r}^* \in I$, we have

$$\begin{aligned} G^*(\bar{y}) &:= (\log D_i s_{\bar{r}^*} + \bar{r}^*\bar{c})G(\bar{y}) - D_i(G(\bar{y})) \\ &= \sum_{\bar{r}\in I} (\log D_i s_{\bar{r}^*} + \bar{r}^*\bar{c} - \log D_i s_{\bar{r}} - \bar{r}\bar{c}) s_{\bar{r}} \bar{y}^{\bar{r}} \end{aligned}$$

is a polynomial with fewer terms than G (since the term with index $\bar{r}^*$ is cancelled) such that its coefficients are in L, $G^*(\bar{f}) = 0$ as $G(\bar{f}) = D_i(G(\bar{f})) = 0$, and $G^*(\bar{y}) \neq 0$ as there exist $\bar{r}$ in I such that $\log D_i s_{\bar{r}} + \bar{r}\bar{c} \neq \log D_i s_{\bar{r}^*} + \bar{r}^*\bar{c}$. This contradicts the minimality of G.

There are at least two terms in $G(\bar{y})$. Indeed, if there is only one term, then $G(\bar{y}) = s_{\bar{r}}\bar{y}^{\bar{r}}$ for the unique $\bar{r} \in I$. Since $f_j \neq 0$ for $j = 1, 2, \ldots, m$ and $G(\bar{f}) = 0$, we have $s_{\bar{r}} = 0$, so $G(\bar{y}) = 0$, contradicting the fact that G is nonzero.

We now have

$$\log D_i s_{\bar{r}^{(1)}} + \bar{r}^{(1)}\bar{c} = \log D_i s_{\bar{r}^{(2)}} + \bar{r}^{(2)}\bar{c}$$

for all $\bar{r}^{(1)}, \bar{r}^{(2)} \in I$. Note that $\log D_i s_{\bar{r}} + \bar{r}\bar{c} = \log D_i(s_{\bar{r}}\bar{f}^{\bar{r}})$ for any $\bar{r} \in I$. Therefore, fixing $\bar{r}^{(1)} \neq \bar{r}^{(2)}$ in I, we get $s_{\bar{r}^{(1)}}\bar{f}^{\bar{r}^{(1)}} = c s_{\bar{r}^{(2)}}\bar{f}^{\bar{r}^{(2)}}$ for some nonzero $c \in \mathcal{C}$. This means that

$$c\bar{f}^{\bar{r}^{(2)} - \bar{r}^{(1)}} = s_{\bar{r}^{(1)}} s_{\bar{r}^{(2)}}^{-1}. \tag{$*$}$$

Note that as all $f_j \neq 0$, $\bar{f}^{\bar{r}^{(2)} - \bar{r}^{(1)}}$ makes sense and is nonzero. Let $h = c\bar{f}^{\bar{r}^{(2)} - \bar{r}^{(1)}}$. Then h is a nonzero solution to

$$\log D_i x = (\bar{r}^{(2)} - \bar{r}^{(1)})\bar{c}. \tag{$**$}$$

When $i = 1$, right side of $(*)$ is in $\mathrm{acl}(a_0) = \mathbb{Q}^{\mathrm{alg}} \subset \mathcal{C}$, so h is also a constant, but then it is not a solution for $(**)$. When $i > 1$, we apply part (ii) of the lemma for $i-1$ with e_{i-1} a generic solution of (E_{i-1}) over a_{i-2}, $\alpha = (\bar{r}^{(2)} - \bar{r}^{(1)})\bar{c}^* \neq 0$, and h a nonzero solution of $D_{i-1}x - dx = 0$. We get that e_{i-1} is the coreduction of h over a_{i-2}. In particular, since $e_{i-1} \notin \mathrm{acl}(a_{i-2})$, we have that $\mathrm{stp}(h/a_{i-2})$ is not almost $\mathcal{C}$-internal. But the right side of $(*)$ is in L which is almost $\mathcal{C}$-internal over a_{i-2}, a contradiction.

(ii) Suppose the conclusion is true for $i-1$, and (i) is true for i.

We use induction on m, the order of the differential equation (E_i).

If $m = n_i = 1$, we have that $\log D_i h = \alpha f$ and $\log D_i(\alpha f) = c_{i1}$. Let h^* be a generic solution of $\log D_i x = \alpha f$ over Lf. Since f is a generic solution of $\log D_i(x) = c_{i1}$ over L, αf is also a generic solution of $\log D_i(x) = c_{i1}$ over L, and therefore h^* is a generic solution of $\log D_i^{(2)} x = c_{i1}$ over a_{i-1}. Thus $\mathrm{stp}(h^*/L)$ is not almost $\mathcal{C}$-internal by Proposition 4.1. Since h^* is a constant multiple of h, $\mathrm{stp}(h/L)$ is also not almost $\mathcal{C}$-internal. Note that (f, h) is a $\mathcal{C}$-analysis of h over L, and as it is incompressible of U-type $(1,1)$, we have that f is the $\mathcal{C}$-coreduction of h over L.

Now suppose the conclusion of (ii) is proven if the order of the equation (E_i) is less than or equal to $m-1$.

Let β be the $\mathcal{C}$-coreduction of h over L. Since $\mathrm{stp}(h/Lf)$ is almost $\mathcal{C}$-internal, we only need to show that $f \in \mathrm{acl}(L\beta)$. Let $(f_1, \ldots, f_m)$ be the decomposition of f by Lemma 4.3. By Lemma 4.4, f_j is a generic solution of $D_i x - c_{ij}x = 0$ for $j = 1, 2, \ldots, m$. Suppose towards a contradiction that $f \notin \mathrm{acl}(L\beta)$. We may, without loss of generality, suppose $f_1, \ldots, f_s \notin \mathrm{acl}(L\beta)$ and $f_{s+1}, \ldots, f_m \in \mathrm{acl}(L\beta)$ for some $1 \leq s \leq m$.

In the rest of the proof we seek a contradiction to the above assumption.

We prove first that $s = m$. Suppose not, so $f_m \in \mathrm{acl}(L\beta)$. Let h_m be a nonzero solution to $D_i x - \alpha f_m x = 0$. We have that $\mathrm{stp}(h_m/Lf_m)$ is almost $\mathcal{C}$-internal. Since $f_m \in \mathrm{acl}(L\beta)$, $\mathrm{stp}(h_m/L\beta)$ is almost $\mathcal{C}$-internal. Let

$h^* = hh_m^{-1}$. Then

$$\begin{aligned}\log D_i(h^*) &= \log D_i(h) - \log D_i(h_m)\\ &= \alpha(f_1 + \cdots + f_{m-1} + f_m) - \alpha f_m\\ &= \alpha(f_1 + \cdots + f_{m-1}).\end{aligned}$$

Let $f^* = f_1 + \cdots + f_{m-1}$. Then h^* is a nonzero solution to $D_i x - \alpha f^* x = 0$. From (i), since $f_1, \ldots, f_{m-1}$ are all nonzero, f^* is a generic solution over L to

$$(D_i - c_{i1}) \cdots (D_i - c_{i,m-1})x = 0.$$

By the induction hypothesis, we conclude that the $\mathcal{C}$-coreduction of h^* over L is f^*. Since h and h_m are almost $\mathcal{C}$-internal over $L\beta$ and $h^* = hh_m^{-1}$, we get that $f^* \in \operatorname{acl}(L\beta)$. As f^* is interdefinable with $(f_1, \ldots, f_{m-1})$ over L, $f_1 \in \operatorname{acl}(L\beta)$, contradicting our assumption.

Let $g_{t1} = tf_1$ for $t = 1, 2, \ldots$. We show that $\operatorname{stp}(g_{t1}/L\beta) = \operatorname{stp}(f_1/L\beta)$. Since

$$D_i g_{t1} - c_{i1} g_{t1} = tD_i f_1 - tc_{i1} f_1 = 0, \tag{4.1}$$

we have that $g_{t1} \in \{x : D_i x - c_{i1} x = 0\}$, a strongly minimal set. Thus in order to prove $\operatorname{stp}(g_{t1}/L\beta) = \operatorname{stp}(f_1/L\beta)$ we only need to show that $g_{t1} \notin \operatorname{acl}(L\beta)$, which follows from $f_1 \notin \operatorname{acl}(L\beta)$.

For each integer $t \geq 1$, let η_t be an automorphism fixing $\operatorname{acl}(L\beta)$ and taking f_1 to g_{t1}. Set $g_{tj} := \eta_t(f_j)$ for all $j = 1, 2, \ldots, m$, $g_t := \eta_t(f)$, and $h_t := \eta_t(h)$. So $\operatorname{stp}(h_t, g_t, g_{t1}, \ldots, g_{tm}/L\beta) = \operatorname{stp}(h, f, f_1, \ldots, f_m/L\beta)$ for all $t \geq 1$. In particular, g_t is a generic solution to (E_i) over L, h_t is a nonzero solution to $D_i x - \alpha g_t x = 0$, $g_t = \sum_{j=1}^{m} g_{tj}$ is the decomposition by Lemma 4.3, and $\operatorname{stp}(h_t/\beta)$ is almost $\mathcal{C}$-internal.

We next show that $g_{tj} = tf_j$ for all $t \geq 1$ and all j.

Towards a contradiction, suppose that $g_{tj} \neq tf_j$ for some t and j. Fix this t. We argue first that $g_{tj} - tf_j \in \operatorname{acl}(L\beta)$. Let $H = h_t h^{-t}$, and let

$I = \{j : 2 \leq j \leq m, g_{tj} - tf_j \neq 0\}$ (note that $g_{t1} = tf_1$, so we only need $j \geq 2$; also note that I is nonempty since $g_{tj} \neq tf_j$ for some j by assumption). We have that

$$\begin{aligned} D_i H &= (\log D_i H) H \\ &= (\log D_i h_t - t \log D_i h) H \\ &= (\alpha g_t - t\alpha f) H \\ &= (\alpha \sum_{j=1}^{m} (g_{tj} - tf_j)) H, \\ &= (\alpha \sum_{j \in I} (g_{tj} - tf_j)) H. \end{aligned}$$

So H is a nonzero solution of $D_i x - (\alpha \sum_{j \in I} (g_{tj} - tf_j)) x = 0$.

Note that $\sum_{j \in I} (g_{tj} - tf_j)$ is a solution to

$$\left(\prod_{j \in I} (D_i - c_{ij}) \right) (x) = 0. \tag{4.2}$$

This is because (4.2) is linear, and for each $j \in I$,

$$(D_i - c_{ij})(g_{tj} - tf_j) = (D_i - c_{ij}) g_{tj} - (D_i - c_{ij}) tf_j = 0.$$

The decomposition of $\sum_{j \in I} (g_{tj} - tf_j)$ by Lemma 4.3 with respect to (4.2) is $(g_{tj} - tf_j)_{j \in I}$, and $g_{tj} - tf_j \neq 0$ for every $j \in I$. Therefore, applying part (i) where we replace (E_i) with (4.2), we get that $\sum_{j \in I} (g_{tj} - tf_j)$ is a generic solution to (4.2) over L.

Now, since (4.2) is of order less than m and H is a nonzero solution of $D_i x - (\alpha \sum_{j \in I} (g_{tj} - tf_j)) x = 0$, by the induction hypothesis, the coreduction of H over L is $\sum_{j\in I}(g_{tj}-tf_j)$. Since $H=h_th^{-t}$ and both h and h_t are almost $\mathcal{C}$-internal over $L\beta$, we have $\mathrm{stp}(H/L\beta)$ is almost $\mathcal{C}$-internal. Therefore, for any $j\in I$, $g_{tj}-tf_j\in\mathrm{acl}(L\beta)$. We now fix some $j\in I$.

Let $\gamma=\dfrac{g_{tj}}{f_j}-t=\dfrac{g_{tj}-tf_j}{f_j}\neq 0$. Then γ is a constant in $\mathrm{acl}(LF)\backslash\mathrm{acl}(L\beta)$. Indeed, γ is a constant because g_{tj} and f_j are both solutions to $D_ix-c_{ij}x=0$, and hence $\dfrac{g_{tj}}{f_j}\in\mathcal{C}$. We get $\gamma\in\mathrm{acl}(Lf)$ by the fact that $g_{tj}-tf_j\in\mathrm{acl}(L\beta)\subseteq\mathrm{acl}(Lf)$. And $\gamma\notin\mathrm{acl}(L\beta)$ because if it were, then so would $f_j=\dfrac{g_{tj}-tf_j}{\gamma}$, but we know that is not the case.

When $i=1$ this is impossible, since $\mathrm{acl}(Lf)=\mathrm{acl}(f)$, and Lemma 4.5 tells us that $\mathrm{acl}(f)\cap\mathcal{C}=\mathbb{Q}^{\mathrm{alg}}$.

Suppose $i>1$. We apply part (iv) of the lemma for $i-1$ and get that the $\mathcal{C}$-reduction of a_i over a_{i-2} is a_{i-1}. As f is a generic solution of (E_i) over L, $\mathrm{stp}(f/L)=\mathrm{stp}(e_i/L)$, so the $\mathcal{C}$-reduction of f over a_{i-2} is a_{i-1}. Since $\gamma\in\mathrm{acl}(Lf)\backslash\mathrm{acl}(L\beta)$, $\gamma\notin L=\mathrm{acl}(a_{i-1})$. So $\mathrm{stp}(\gamma/a_{i-2})$ is not almost $\mathcal{C}$-internal. On the other hand, γ is a constant, a contradiction.

What we have actually shown is that for any $t\geq 1$, $\mathrm{stp}(tf_1/L\beta)=\mathrm{stp}(f_1/L\beta)$, and if $\mathrm{stp}(\tilde{f}_1,\tilde{f}_2,\ldots,\tilde{f}_m/L\beta)=\mathrm{stp}(f_1,\ldots,f_m/L\beta)$ and $\tilde{f}_1=tf_1$, then $\tilde{f}_j=tf_j$ for $j=2,3,\ldots,m$. In particular, $\mathrm{stp}(tf_1,\ldots,tf_m/L\beta)=\mathrm{stp}(f_1,\ldots,f_m/L\beta)$ holds for all t. In addition, the case of $t=1$ tells us that $f_j\in\mathrm{dcl}(f_1\mathrm{acl}(L\beta))$ for $j=2,3,\ldots,m$.

We now show that $\dfrac{f_j}{f_1}\in\mathrm{acl}(L\beta)$ for $j=2,3,\ldots,m$. Fix some j. Since $f_j\in\mathrm{dcl}(f_1\mathrm{acl}(L\beta))$, there exists a formula $\varphi_1(x,y)$ over $\mathrm{acl}(L\beta)$ such that $\varphi_1(\mathcal{U},f_1)=\{f_j\}$. Since $\mathrm{stp}(tf_1,tf_j/L\beta)=\mathrm{stp}(f_1,f_j/L\beta)$, we have $\varphi_1(\mathcal{U},tf_1)=\{tf_j\}$ for all t. Now set $\varphi_2(x,y):=\forall z(\varphi_1(z,y)\rightarrow x=\dfrac{z}{y})$. Then $\varphi_2(\mathcal{U},tf_1)=\left\{\dfrac{f_j}{f_1}\right\}$ for all t. So we have

$$\{tf_1:t\geq 1\}\subseteq\left\{b\in\mathcal{U}:\log D_ib=c_{i1}\text{ and }\varphi_2(\mathcal{U},b)=\left\{\frac{f_j}{f_1}\right\}\right\}.$$

Since $\log D_i x = c_{i1}$ is strongly minimal, it must be that for all but finitely many solutions to $\log D_i x = c_{i1}$, $\varphi_2(\mathcal{U}, b) = \left\{\dfrac{f_j}{f_1}\right\}$. It follows that $\dfrac{f_j}{f_1} \in \operatorname{acl}(L\beta)$.

Let g_{01} be a generic solution over Lh to $D_i x - c_{i1} x = 0$, and $g_{0j} = g_{01}\dfrac{f_j}{f_1}$ for $j = 2, 3, \ldots, m$. We have shown that each $\dfrac{f_j}{f_1}$ is in $\operatorname{acl}(L\beta)$, so $(g_{01}, \ldots, g_{0m}) \in \operatorname{acl}(L\beta g_{01})$. Let $c_{01} = \dfrac{f_1}{g_{01}} \in \mathcal{C}$. Now,

$$\begin{aligned}\log D_i^{(2)}(h) &= \log D_i(\alpha f)\\ &= \log D_i(\alpha(f_1 + \cdots + f_m))\\ &= \log D_i(\alpha c_{01}(g_{01} + \cdots + g_{0m}))\\ &= \log D_i(g_{01} + \cdots + g_{0m}) =: \epsilon.\end{aligned}$$

Hence h is a solution to $\log D_i^{(2)}(x) = \epsilon$ which is over $\operatorname{acl}(L\beta g_{01})$, which implies that $U(h/L\beta g_{01}) \leq 2$. Note that $U(h/L\beta) \geq 2$ since h is a generic solution to $\log D_i x = \alpha f$ and $U(f/L\beta) \geq 1$. But we also have $h \underset{L\beta}{\downarrow} g_{01}$ (recall that $\beta \in \operatorname{acl}(Lh)$), so $U(h/L\beta g_{01}) = U(h/L\beta) \geq 2$. Thus $U(h/L\beta g_{01}) = 2$, and h is a generic solution to $\log D_i^{(2)}(x) = \epsilon$ over $\operatorname{acl}(L\beta g_{01})$. Hence $\operatorname{stp}(h/L\beta g_{01})$ is not almost $\mathcal{C}$-internal by Proposition 4.1, and therefore $\operatorname{stp}(h/L\beta)$ is not almost $\mathcal{C}$-internal, contradicting the definition of β.

(iii) Assume part (ii) of the lemma is true for i.

Let $e_{i+1} = \displaystyle\sum_{j=1}^{n_{i+1}} b_{i+1,j}$ be the decomposition by Lemma 4.3 with respect to (E_{i+1}). We have that $\operatorname{stp}(a_{i+1}/a_i)$ is almost $\mathcal{C}$-internal. Also, by part (ii) applied to $f = e_i$ and $h = b_{i+1,1}$, the $\mathcal{C}$-coreduction of $b_{i+1,1}$ over a_{i-1} is e_i, which is interdefinable over a_{i-1} with a_i. Since $b_{i+1,1} \in \operatorname{dcl}(a_i e_{i+1}) = \operatorname{dcl}(a_{i+1})$, the $\mathcal{C}$-coreduction of a_{i+1} over a_{i-1} is a_i.

(iv) Assume parts (i) and (ii) of the lemma are true for i. For simplicity, we use n to denote n_{i+1}. Let $K = \operatorname{acl}(a_i)$. Let $\bar{b}_{i+1} = (b_{i+1,1}, \ldots, b_{i+1,n})$.

We already know that $\operatorname{stp}(a_i/a_{i-1})$ is $\mathcal{C}$-internal. Suppose $\beta \in \operatorname{acl}(a_{i+1})$ is almost $\mathcal{C}$-internal over a_{i-1} and $\beta \notin \operatorname{acl}(a_i)$. Since e_{i+1} is interalgebraic with $\bar{b}_{i+1}$ over a_i, $\beta \in \operatorname{acl}(a_i\bar{b}_{i+1})$, which means $\beta \in K\langle\bar{b}_{i+1}\rangle^{\mathrm{alg}}$. Since $\delta b_{i+1,j} = c_{i+1,j}b_{i+1,j}\prod_{k=1}^{i} e_k \in K(b_{i+1,j})$ for $j = 1, 2, \dots, n$, we have $K\langle\bar{b}_{i+1}\rangle = K(\bar{b}_{i+1})$, so $\beta \in K(\bar{b}_{i+1})^{\mathrm{alg}}$. Thus there exist a polynomial $F(x, y_1, \dots, y_n)$ with coefficients in K so that $F(\beta, b_{i+1,1}, \dots, b_{i+1,n}) = 0$ and $F(x, b_{i+1,1}, \dots, b_{i+1,n}) \neq 0$. Also, $F(\beta, y_1, \dots, y_n) \neq 0$ since $\beta \notin K$.

Suppose $G(y_1, \dots, y_n)$ is a nonzero polynomial with minimal number of terms such that the coefficients of G are almost $\mathcal{C}$-internal over a_{i-1} and $G(\bar{b}_{i+1}) = 0$. Note that this is well-defined because $F(\beta, y_1, \dots, y_n)$ satisfies all the conditions except for the minimality, as K and β are both almost $\mathcal{C}$-internal over a_{i-1}.

Let

$$G(y_1, \dots, y_n) = \sum_{\bar{r}\in I} s_{\bar{r}}\bar{y}^{\bar{r}},$$

where I is a set of n-tuples of nonnegative integers, and $\operatorname{stp}(s_{\bar{r}}/a_{i-1})$ is almost $\mathcal{C}$-internal. Let $\bar{c}_{i+1} = (c_{i+1,1}, \dots, c_{i+1,n})$. Arguing exactly as in the proof of part (i) of the lemma, we get by minimality of G that

$$\log D_i s_{\bar{r}^{(1)}} + \bar{r}^{(1)}\bar{c}_{i+1}e_i = \log D_i s_{\bar{r}^{(2)}} + \bar{r}^{(2)}\bar{c}_{i+1}e_i \tag{4.3}$$

for any $r^{(1)}, r^{(2)} \in I$. Indeed,

$$\begin{aligned} D_i(G(\bar{b}_{i+1})) &= \sum_{\bar{r}\in I}(\bar{b}_{i+1}^{\bar{r}} D_i s_{\bar{r}} + s_{\bar{r}} D_i \bar{b}_{i+1}^{\bar{r}}) \\ &= \sum_{\bar{r}\in I}(\bar{b}_{i+1}^{\bar{r}} D_i s_{\bar{r}} + s_{\bar{r}}\bar{r}\bar{c}_{i+1}e_i\bar{b}_{i+1}^{\bar{r}}) \\ &= \sum_{\bar{r}\in I}(\log D_i s_{\bar{r}} + \bar{r}\bar{c}_{i+1}e_i)s_{\bar{r}}\bar{b}_{i+1}^{\bar{r}}, \end{aligned}$$

where the second equality is by the fact that

$$\begin{aligned} D_i \bar{b}_{i+1}^{\bar{r}} &= \bar{r}\bar{b}_{i+1}^{\bar{r}-\bar{1}} D_i \bar{b}_{i+1} \\ &= \bar{r}\bar{b}_{i+1}^{\bar{r}-\bar{1}} e_i D_{i+1} \bar{b}_{i+1} \\ &= \bar{r}\bar{b}_{i+1}^{\bar{r}-\bar{1}} e_i \bar{c}_{i+1} \bar{b}_{i+1} \\ &= \bar{r} e_i \bar{c}_{i+1} \bar{b}_{i+1}^{\bar{r}}. \end{aligned}$$

Now if (4.3) failed, then fixing any $\bar{r}^* \in I$ we see that

$$\begin{aligned} G^*(\bar{y}) :&= (\log D_i s_{\bar{r}^*} + \bar{r}^* \bar{c}_{i+1} e_i) G(\bar{y}) - D_i G(\bar{y}) \\ &= \sum_{\bar{r} \in I} (\log D_i s_{\bar{r}^*} + \bar{r}^* \bar{c}_{i+1} e_i - \log D_i s_{\bar{r}} - \bar{r}\bar{c}_{i+1} e_i) s_{\bar{r}} \bar{y}^{\bar{r}} \end{aligned}$$

whose coefficients are again almost $\mathcal{C}$-internal over a_{i-1}, would contradict the minimal choice of G.

If G has only one term, then for the only $\bar{r} \in I$, $G(\bar{b}_{i+1}) = s_{\bar{r}} \bar{b}_{i+1}^{\bar{r}}$. Since $b_{i+1,j} \neq 0$ for $j = 1, 2, \ldots, n$, $s_{\bar{r}} = 0$, which means $G(\bar{y}) = 0$, a contradiction. Now fix $r^{(1)} \neq r^{(2)}$ in I. Since $\log D_i s_{\bar{r}} + \bar{r}\bar{c}_{i+1} e_i = \log D_i (s_{\bar{r}} \bar{b}_{i+1}^{\bar{r}})$ for any $\bar{r} \in I$, we have $s_{\bar{r}^{(1)}} \bar{b}_{i+1}^{\bar{r}^{(1)}} = c s_{\bar{r}^{(2)}} \bar{b}_{i+1}^{\bar{r}^{(2)}}$ for some $c \in \mathcal{C}$. This means that $\bar{b}_{i+1}^{\bar{r}^{(1)}-\bar{r}^{(2)}} = c s_{\bar{r}^{(2)}} s_{\bar{r}^{(1)}}^{-1}$. So $\bar{b}_{i+1}^{\bar{r}^{(1)}-\bar{r}^{(2)}}$ is almost $\mathcal{C}$-internal over a_{i-1}.

On the other hand, as $D_{i+1} \bar{b}_{i+1}^{\bar{r}^{(1)}-\bar{r}^{(2)}} = (\bar{r}^{(1)} - \bar{r}^{(2)}) \bar{c}_{i+1} \bar{b}_{i+1}^{\bar{r}^{(1)}-\bar{r}^{(2)}}$, $\bar{b}_{i+1}^{\bar{r}^{(1)}-\bar{r}^{(2)}}$ is a solution of $(D_{i+1} - (\bar{r}^{(1)} - \bar{r}^{(2)}) \bar{c}_{i+1}) x = 0$, with $(\bar{r} - \bar{r}^*) \bar{c}_{i+1} \neq 0$ since $\{c_{i+1,j} : j = 1, 2, \ldots, n\}$ is $\mathbb{Q}$-linearly independent. By part (ii) of the lemma with $f = e_i$, $h = \bar{b}_{i+1}^{\bar{r}^{(1)}-\bar{r}^{(2)}}$, and $\alpha = (\bar{r}^{(1)} - \bar{r}^{(2)}) \bar{c}_{i+1}$, e_i is a $\mathcal{C}$-coreduction of $\bar{b}_{i+1}^{\bar{r}^{(1)}-\bar{r}^{(2)}}$ over a_{i-1}. In particular, $\bar{b}_{i+1}^{\bar{r}^{(1)}-\bar{r}^{(2)}}$ is not almost $\mathcal{C}$-internal over a_{i-1}. This contradiction proves part (iv) of the lemma. $\square$

We have accomplished the desired construction:

Theorem 4.7. *Given positive integers $n_1, \ldots, n_\ell$, there exists in* DCF_0 *a type over $\mathbb{Q}^{\mathrm{alg}}$ that admits a canonical $\mathcal{C}$-analysis of U-type $(n_1, \ldots, n_\ell)$.*

Proof. Let $(a_1, \ldots, a_\ell)$ be as in the above construction. We have seen that

$(a_1, \ldots, a_\ell)$ is a $\mathcal{C}$-analysis of $p = \mathrm{stp}(a_\ell)$ of U-type $(n_1, \ldots, n_\ell)$. By Proposition 3.16, parts (iii) and (iv) of Lemma 4.6 prove that it is a $\mathcal{C}$-analysis by reductions and coreductions. The result now follows from the "in particular" clause of Proposition 3.15. $\square$

5 Pullbacks under the logarithmic derivative map

Since $\log\delta : \mathbb{G}_m \to \mathbb{G}_a$ is a definable group homomorphism with kernel $\mathbb{G}_m(\mathcal{C})$, whenever $D \subseteq \mathbb{G}_a$ is a definable set with generic type almost $\mathcal{C}$-internal, the generic type of $\log\delta^{-1}(D)$ will be analysable in $\mathcal{C}$ in at most 2 steps. When is it in fact already almost $\mathcal{C}$-internal?

A rephrasing of Fact 2.27 is that if $D = \mathcal{C}$ then $\log\delta^{-1}(D)$ is *not* almost $\mathcal{C}$-internal. A rephrasing of Proposition 4.1 is that if D is defined by $\delta x = hx$ for any $h \in \mathcal{U}$, then again $\log\delta^{-1}(D)$ is *not* almost $\mathcal{C}$-internal. One might guess after seeing the above examples that in fact $\log\delta^{-1}(D)$ is never almost $\mathcal{C}$-internal, but this is false.

Example 5.1. Let $D = \{x : \delta(\frac{1}{x}) = 1\}$. Then $\log\delta^{-1}(D)$ is $\mathcal{C}$-internal.

Proof. Let $E = \{x : \frac{1}{x} \in D\} = \{x : \delta x = 1\}$, which is 0-definably isomorphic to D by multiplicative inverse. Note that if $u \in \log\delta^{-1}(D)$ then $\frac{1}{\log\delta u} \in E$. On the other hand, $\delta^2 u = 0$. Indeed, $\frac{\delta u}{u} \in D$, so $1 = \delta\left(\frac{u}{\delta u}\right) = \frac{(\delta u)^2 - u\delta^2 u}{(\delta u)^2}$, and as $u \neq 0$ this implies $\delta^2 u = 0$. That is, $\delta u \in \mathcal{C}\backslash\{0\}$. So we have a (0-definable) map $i : \log\delta^{-1}(D) \to E \times \mathcal{C}\backslash\{0\}$ defined by

$$u \mapsto (\frac{1}{\log\delta u}, \delta u).$$

We now show that i is a bijection. For injectivity, suppose $i(u) = i(v)$ for some $u, v \in \log\delta^{-1}(D)$. Then $\frac{1}{\log\delta u} = \frac{1}{\log\delta v}$ and $\delta u = \delta v$, so $u = \frac{\delta u}{\log\delta u} = \frac{\delta v}{\log\delta v} = v$. For surjectivity, if $e \in E$ and $c \in \mathcal{C}\backslash\{0\}$, then $u = ce$ satisfies that $\delta u = c\delta e = c$ and $\frac{1}{\log\delta u} = \frac{u}{\delta u} = \frac{u}{c} = e$, and $\log\delta u = \frac{1}{e} \in D$.

Since E and $\mathcal{C}\backslash\{0\}$ are $\mathcal{C}$-internal, this means that $\log\delta^{-1}(D)$ is $\mathcal{C}$-internal. $\square$

Note that in the above example, the $\mathcal{C}$-internality of $\log\delta^{-1}(D)$ was witnessed by a 0-definable decomposition into a product of strongly minimal $\mathcal{C}$-internal definable sets. In fact, we conjecture that if D is strongly minimal and $\log\delta^{-1}(D)$ is almost $\mathcal{C}$-internal, then it must be for a very strong reason similar to the above example.

To clearly state the conjecture, it is more convenient for us to work with types rather than definable sets.

Definition 5.2. Let $p \in S_1(F)$ be a complete type where F is an algebraically closed differential field. We say that $q \in S_1(F)$ is the *logarithmic inverse of* p, denoted $q = \log\delta^{-1}(p)$, if for some realization u of q, $\log\delta u$ realizes p and $u \notin \operatorname{acl}(F, \log\delta u)$.

Proposition 5.3. *If $p \in S_1(F)$, then $\log\delta^{-1}(p)$ exists and is unique. Moreover, $U(\log\delta^{-1}(p)) = U(p) + 1$.*

Proof. Let a be a realization of p. Note that $\log\delta x = a$ has a solution not in $\operatorname{acl}(Fa)$. Indeed, by saturation of $\mathcal{U}$ we only need to find a solution to $(\log\delta x = a) \wedge (p(x) \neq 0)$ for any nonzero $p \in F(a)[x]$. Since $\log\delta x = a$ is order 1, this has a solution by the axioms of DCF_0. Now the type over F of any solution to $\log\delta x = a$ that is not in $\operatorname{acl}(Fa)$ will satisfy the definition of $\log\delta^{-1}(p)$.

We now prove uniqueness. For $i = 1, 2$, suppose a_i realizes of p, $\log\delta u_i = a_i$, and $u_i \notin \operatorname{acl}(Fa_i)$. We need to prove that $\operatorname{tp}(u_1/F) = \operatorname{tp}(u_2/F)$. Since a_1, a_2 are realizations of p, there is an automorphism $\alpha \in \operatorname{Aut}_F(\mathcal{U})$ such that $\alpha(a_2) = a_1$. Note that $\log\delta(\alpha(u_2)) = \alpha(a_2) = a_1 = \log\delta u_1$, so u_1 and $\alpha(u_2)$ are both in the set $B = \{x : \log\delta x = a_1\}$, which is Fa_1-definable and strongly minimal. Since $u_1, \alpha(u_2) \notin \operatorname{acl}(Fa_1)$, $\operatorname{tp}(u_1/Fa_1) = \operatorname{tp}(\alpha(u_2)/Fa_1)$, so $\operatorname{tp}(u_1/F) = \operatorname{tp}(\alpha(u_2)/F) = \operatorname{tp}(u_2/F)$.

For the U-rank of $\log\delta^{-1}(p)$, let u be a realization of $\log\delta^{-1}(p)$ and $a := \log\delta u$. Then $U(\log\delta^{-1}(p)) = U(u/F) = U(u/aF) + U(a/F) = U(u/aF) + U(p)$. Since $\log\delta u = a$ and $u \notin \operatorname{acl}(Fa)$, $U(u/aF) = 1$, so $U(\log\delta^{-1}(p)) = U(p) + 1$. $\square$

We recall the definition of tensor product of types. Let $p_1, p_2 \in S(A)$. Then the *tensor product of* p_1, p_2, denoted $p_1 \otimes p_2$, is the type that satisfies that $(a_1, a_2) \models p_1 \otimes p_2$ iff $a_1 \models p_1$ and $a_2 \models p_2|Aa_1$. In particular, we have $a_1 \mathop{\smile\!\!\!\!\!\!\;|}_A a_2$. The type $p^{(n)}$ is defined as $\underbrace{p \otimes \cdots \otimes p}_{n}$, and the realizations are exactly Morley sequences of p of length n.

We now state our conjecture.

Conjecture 5.4. Suppose p is an almost $\mathcal{C}$-internal minimal type in $S_1(F)$, where F is an algebraically closed differential field. Let $q = \log\delta^{-1}(p)$. Then the following are equivalent:

(1) q is almost $\mathcal{C}$-internal;

(2) there exist almost $\mathcal{C}$-internal minimal types q_1 and q_2 over F, and an F-definable function f whose domain contains $q(\mathcal{U})$, such that $f|_{q(\mathcal{U})}$ is a finite-to-one map onto $(q_1 \otimes q_2)(\mathcal{U})$;

(3) there is an integer $\ell \neq 0$ such that for some (equivalently any) $u \models q(x)$, $u^\ell = u_1 u_2$ where $u_1 \in F\langle \log\delta u\rangle$ and $\log\delta(u_2) \in F$.

Remark 5.5. It is not hard to see that (3)⇒(2)⇒(1).

For (2)⇒(1), let u be a realization of q. Suppose $f(u) = (u_1, u_2)$. Then $u \in \mathrm{acl}(Fu_1u_2)$, where u_1 and u_2 are realizations of q_1 and q_2, respectively. Since q_1, q_2 are both almost $\mathcal{C}$-internal, we have that q is almost $\mathcal{C}$-internal.

For (3)⇒(2), let u be a realization of q, and u_1, u_2, ℓ as in the statement of (3). Since $u_1 \in F\langle \log\delta u\rangle \subseteq F\langle u\rangle$, there is an F-definable map f_1 such that $f_1(u) = u_1$. Let $f_2(x) = \dfrac{x^\ell}{f_1(x)}$. We claim that $q_1 = \mathrm{tp}(u_1/F), q_2 = \mathrm{tp}(u_2/F)$ and $f = (f_1, f_2)$ satisfy the statement of (2). Note first that since $U(u/F) = 2$, $U(u_2/F) \leq 1$ as $\log\delta u_2 \in F$, and $U(u_1/F) \leq 1$ as $u_1 \in \mathrm{dcl}(F, \log\delta u)$, we must have that $U(u_1/F) = U(u_2/F) = 1$ and $u_1 \mathop{\smile\!\!\!\!\!\!\;|}_F u_2$, so q_1, q_2 are minimal and $f(u) = (u_1, u_2) \models q_1 \otimes q_2$. That f is finite-to-one because there are at most ℓ many elements in $\mathcal{U}$ that satisfy $x^\ell = u_1 u_2$.

Hence, the conjecture is really that (1)⇒(3).

We categorize the almost $\mathcal{C}$-internal minimal types p into two main cases: those that are weakly orthogonal to $\mathcal{C}$, and those that are not. In Section 5.1 we will prove the the conjecture when p is not weakly orthogonal to $\mathcal{C}$. In Section 5.2, we will consider the other case, where p is $\mathcal{C}$-internal but weakly orthogonal to $\mathcal{C}$. We prove the conjecture in that case under some additional differential algebraic assumptions. In particular, when $F \subseteq \mathcal{C}$ the conjecture is true — see Corollary 5.13 below. We provide several examples in Section 5.3, which illustrates cases that are dealt with in Sections 5.1 and 5.2. Section 5.4 gives an example which remains open. In Section 5.5, we discuss the specific condition which is used in Section 5.2. Finally, in Section 5.6, we discuss pullbacks under the derivative map instead of the logarithmic derivative map.

5.1 The non-weakly-orthogonal case

In this section, we prove Conjecture 5.4 when our minimal type p is not weakly orthogonal to $\mathcal{C}$.

Theorem 5.6. *Conjecture 5.4 is true under the additional assumption that p is not weakly orthogonal to $\mathcal{C}$.*

From the discussion in Remark 5.5, we know that we only need to prove (1)⇒(3) of the conjecture. We prove this theorem in several steps, the first few of which do not assume that p is not weakly orthogonal to $\mathcal{C}$, and will be used again later.

We assume the following for the rest of this section:

1. F is an algebraically closed differential field,
2. p is an almost $\mathcal{C}$-internal minimal type over F,
3. $q = \log\delta^{-1}(p)$ is almost $\mathcal{C}$-internal,

4. $(u_1, u_2, \ldots)$ is a Morley sequence in q, and $a_i := \log\delta u_i$,

5. K is the field generated by $F \cup \mathcal{C}$, and

6. $L := K\langle a_1, a_2, \ldots\rangle$.

Lemma 5.7. *For some positive integer n, $\{u_1, \ldots, u_n\}$ is algebraically dependent over L.*

Proof. Since q is almost $\mathcal{C}$-internal, there exists some n such that $u_1 \in \mathrm{acl}(F\mathcal{C}u_2, \ldots, u_n)$. By quantifier elimination of DCF_0, there is a formula $\varphi(x_1) = (\varphi_1(x_1) \wedge \cdots \wedge \varphi_m(x_1))$ with parameters in $K\langle u_2, \ldots, u_n\rangle$ such that u_1 realizes $\varphi(x_1)$, each $\varphi_i(x_1)$ is a literal (an atomic formula or its negation), and $\varphi(\mathcal{U})$ is finite. Since u_1 satisfies the equation $\delta x_1 = a_1 x_1$, which we denote by $\zeta(x_1)$, we have that u_1 realizes $\zeta(x_1) \wedge \varphi(x_1)$. Let $\psi(x_1)$ be the formula obtained by replacing δx_1 with $a_1 x_1$ in $\varphi(x_1)$, and similarly for $\psi_i(x_1)$'s. Then $\zeta(\mathcal{U}) \wedge \varphi(\mathcal{U}) = \zeta(\mathcal{U}) \wedge \psi(\mathcal{U})$. Note that $\psi(x_1)$ is a formula with parameters in $K\langle a_1, u_2, \ldots, u_n\rangle)$. Since $\psi_i(x_1)$ is a literal in the language of rings, each $\psi_i(\mathcal{U})$ is either finite or cofinite in $\mathcal{U}$. But $\zeta(\mathcal{U}) \cap \bigcap_{i=1}^{m} \psi_i(\mathcal{U}) = \zeta(\mathcal{U}) \wedge \psi(\mathcal{U}) = \zeta(\mathcal{U}) \wedge \varphi(\mathcal{U})$ is finite. Since $\zeta(\mathcal{U})$ is infinite, there must be some $t \leq m$ such that $\psi_t(\mathcal{U})$ is finite. Let $\xi(x_1) = \psi_t(x_1)$.

Since $\xi(\mathcal{U})$ is finite but nonempty (as it contains u_1), $\xi(x_1)$ is an atomic formula in the language of rings (rather than a negated atomic formula). Without loss of generality, suppose $\xi(x_1)$ is of the form $f_0(x_1) = 0$ where $f_0(x_1)$ is a nonzero polynomial over $K\langle a_1, u_2, \ldots, u_n\rangle$. We have $f_0(u_1) = 0$.

Since each $\delta u_i = u_i a_i$, we have that

$$K\langle a_1, u_2, \ldots, u_n\rangle = K\langle a_1, a_2, \ldots, a_n\rangle(u_2, \ldots, u_n) \subseteq L(u_2, \ldots, u_n).$$

We may therefore rewrite $f_0(x_1) = 0$ as $f(x_1, u_2, \ldots, u_n) = 0$ where $f \in L[x_1, x_2, \ldots, x_n]$. Note that $f \neq 0$ and $f(u_1, \ldots, u_n) = 0$. We thus have that $\{u_1, \ldots, u_n\}$ is not algebraically independent over L. $\square$

Lemma 5.8. *For some $g \in L$ and some nonzero integer k, $gu_1^k = u_2^k$.*

Proof. Suppose $f(x_1, \ldots, x_n)$ is a polynomial over L with minimal number of terms such that $f(u_1, \ldots, u_n) = 0$ and $f \neq 0$. Such an f exists because of Lemma 5.7.

Let

$$f(\bar{x}) = \sum_{\bar{k} \in I} g_{\bar{k}} \bar{x}^{\bar{k}}$$

where I is a finite set of non-negative integer n-tuples, $g_{\bar{k}} \in L$ nonzero for $\bar{k} \in I$, and $\bar{x}^{\bar{k}} := x_1^{k_1} \cdots x_n^{k_n}$. As $u_i \neq 0$ for all i, f has at least two terms.

Since $f(u_1, \ldots, u_n) = 0$, we have

$$\sum_{\bar{k} \in I} g_{\bar{k}} \bar{u}^{\bar{k}} = 0.$$

Since $\delta u_i = a_i u_i$, we have $\log\delta\left(\bar{u}^{\bar{k}}\right) = \bar{k} \cdot \bar{a}$ (where $\bar{k} \cdot \bar{a} = k_1 a_1 + \cdots + k_n a_n$), so

$$\begin{aligned} 0 &= \delta\left(\sum_{\bar{k} \in I} g_{\bar{k}} \bar{u}^{\bar{k}}\right) \\ &= \sum_{\bar{k} \in I} \left(\delta(g_{\bar{k}}) \bar{u}^{\bar{k}} + \log\delta(\bar{u}^{\bar{k}}) g_{\bar{k}} \bar{u}^{\bar{k}}\right) \\ &= \sum_{\bar{k} \in I} \left(\delta(g_{\bar{k}}) \bar{u}^{\bar{k}} + (\bar{k} \cdot \bar{a}) g_{\bar{k}} \bar{u}^{\bar{k}}\right) \\ &= \sum_{\bar{k} \in I} (\log\delta g_{\bar{k}} + \bar{k} \cdot \bar{a}) g_{\bar{k}} \bar{u}^{\bar{k}}. \end{aligned}$$

Fix some $\bar{k}^* \in I$. Define

$$f^*(\bar{x}) = \sum_{\bar{k} \in I \setminus \{\bar{k}^*\}} (\log\delta g_{\bar{k}^*} + \bar{k}^* \cdot \bar{a} - \log\delta g_{\bar{k}} - \bar{k} \cdot \bar{a}) g_{\bar{k}} \bar{x}^{\bar{k}}.$$

Note that there are fewer terms in f^* than in f, and also we have

$$\begin{aligned}
f^*(\bar{u}) &= \sum_{\bar{k}\in I\setminus\{\bar{k}^*\}} (\log\delta g_{\bar{k}^*} + \bar{k}^* \cdot \bar{a} - \log\delta g_{\bar{k}} - \bar{k} \cdot \bar{a}) g_{\bar{k}} \bar{u}^{\bar{k}} \\
&= \sum_{\bar{k}\in I} (\log\delta g_{\bar{k}^*} + \bar{k}^* \cdot \bar{a} - \log\delta g_{\bar{k}} - \bar{k} \cdot \bar{a}) g_{\bar{k}} \bar{u}^{\bar{k}} \\
&= (\log\delta g_{\bar{k}^*} + \bar{k}^* \cdot \bar{a}) \sum g_{\bar{k}} \bar{u}^{\bar{k}} - \sum_{\bar{k}\in I} (\log\delta g_{\bar{k}} + \bar{k} \cdot \bar{a}) g_{\bar{k}} \bar{u}^{\bar{k}} \\
&= (\log\delta g_{\bar{k}^*} + \bar{k}^* \cdot \bar{a}) f(\bar{u}) - \delta(f(\bar{u})) = 0.
\end{aligned}$$

So $f^*(\bar{x}) = 0$. This implies that for any $\bar{k} \neq \bar{k}^* \in I$, $\log\delta g_{\bar{k}} + \bar{k} \cdot \bar{a} = \log\delta g_{\bar{k}^*} + \bar{k}^* \cdot \bar{a}$. This yields $\log\delta(g_{\bar{k}} \bar{u}^{\bar{k}}) = \log\delta(g_{\bar{k}^*} \bar{u}^{\bar{k}^*})$. Fix $\bar{k}_{(1)} \neq \bar{k}_{(2)} \in I$, and we have $g_{\bar{k}_{(1)}} \bar{u}^{\bar{k}_{(1)}} = c g_{\bar{k}_{(2)}} \bar{u}^{\bar{k}_{(2)}}$ for some $c \in \mathcal{C}$. So $g_0 = \dfrac{c g_{\bar{k}_{(2)}}}{g_{\bar{k}_{(1)}}} \in L$ and $\bar{k} = \bar{k}_{(2)} - \bar{k}_{(1)}$ satisfies that $g_0 \bar{u}^{\bar{k}} = 1$.

Since $\bar{k} \neq 0$, without loss of generality, assume $k_1 \neq 0$. Let $\alpha, \beta \in \mathrm{Aut}_F(\mathcal{U})$ be such that $\alpha(u_1, u_2, u_3, \ldots, u_n) = (u_1, u_3, u_4, \ldots, u_{n+1})$ and $\beta(u_1, u_2, u_3, \ldots, u_n) = (u_2, u_3, u_4, \ldots, u_{n+1})$. We have

$$\begin{aligned}
1 &= \alpha(g_0 \bar{u}^{\bar{k}}) \\
&= \alpha(g_0 u_1^{k_1} \prod_{i=2}^{n} u_{i+1}^{k_i}) \\
&= \alpha(g_0) \alpha(u_1)^{k_1} \prod_{i=2}^{n} \alpha(u_i)^{k_i} \\
&= \alpha(g_0) u_1^{k_1} \prod_{i=2}^{n} u_{i+1}^{k_i}
\end{aligned}$$

and

$$\begin{aligned}
1 &= \beta(g_0 \bar{u}^{\bar{k}}) \\
&= \beta(g_0 u_1^{k_1} \prod_{i=2}^{n} u_{i+1}^{k_i})
\end{aligned}$$

$$= \beta(g_0)\beta(u_1)^{k_1} \prod_{i=2}^{n} \beta(u_i)^{k_i}$$
$$= \beta(g_0)u_2^{k_1} \prod_{i=2}^{n} u_{i+1}^{k_i}.$$

We therefore get $gu_1^{k_1} = u_2^{k_1}$ for nonzero integer k_1 and $g = \dfrac{\alpha(g_0)}{\beta(g_0)} \in L$. □

Lemma 5.9. *There exists a nonzero polynomial $f \in F[y, z]$ such that any realization a of p satisfies $f(a, \delta a) = 0$. In particular, $F\langle a\rangle = F(a, \delta a)$.*

Proof. Since p is almost $\mathcal{C}$-internal, there exists an algebraically closed differential field $\hat{F} \supset F$ such that any realization a of $p|\hat{F}$ satisfies $a \in \operatorname{acl}(\hat{F}\mathcal{C})$. This implies that for some tuple $\bar{c} \in \mathcal{C}$, a and $\bar{c}$ are interalgebraic over $\hat{F}$. Since $\operatorname{tp}(a/\hat{F})$ is minimal, $\operatorname{tp}(c/\hat{F})$ is also minimal, and is completely determined by $\operatorname{tp}(c/\mathcal{C}_{\hat{F}})$ by stable embeddedness of $\mathcal{C}$ in $\mathcal{U}$. Therefore, $1 = \text{Tr.Deg}(C_{\hat{F}}(c)/\mathcal{C}_{\hat{F}}) = \text{Tr.Deg}(\hat{F}(c)/\hat{F}) = \text{Tr.Deg}(\hat{F}\left\langle a\right\rangle/\hat{F})$. Moreover, since $a \downarrow_F \hat{F}$, $\text{Tr.Deg}(F\left\langle a\right\rangle/F) = 1$. Therefore, δa is algebraic over $F(a)$, which yields the existence of $f \in F[y, z]$ such that $f(a, \delta a) = 0$.

The “in particular” clause follows from differentiating $f(a, \delta a) = 0$ to get that
$$\frac{\partial f}{\partial y}(a, \delta a)\delta a + \frac{\partial f}{\partial z}(a, \delta a)\delta^2 a + f^{\delta}(a, \delta a) = 0.$$
Hence $\delta^2 a \in F(a, \delta a)$. Iterating yields $F\langle a\rangle \in F(a, \delta a)$. □

We now invoke the condition of not weakly orthogonal to $\mathcal{C}$.

Proposition 5.10. *Assume p is not weakly orthogonal to $\mathcal{C}$. Then there exist $g \in F\left\langle a_1, a_2\right\rangle$ and a nonzero integer k such that* $\log\delta g = ka_2 - ka_1$.

Proof. Apply Lemma 5.8, we assume $gu_1^k = u_2^k$ for some $g \in L$ and some nonzero integer k.

Since p is not weakly orthogonal to $\mathcal{C}$, $p(\mathcal{U}) \subseteq \operatorname{acl}(F\mathcal{C})$ by Lemma 2.16. It follows that $L \subseteq \operatorname{acl}(KF\mathcal{C}) = K^{\text{alg}}$, so $g \in K^{\text{alg}}$. Let $h_1 = g, h_2, \ldots, h_m$

be conjugates of g over $K\langle a_1, a_2\rangle$, and $h := \prod_{i=1}^{m} h_i$. Note that $h \in K\langle a_1, a_2\rangle$. Since $\log\delta g = ka_2 - ka_1$, $\log\delta h_i = ka_2 - ka_1$ for $i = 1, 2, \ldots, m$, and we have $\log\delta h = kma_2 - kma_1$. Let $g_0 \in F\langle a_1, a_2\rangle(\bar{x})$ be such that $g_0(\bar{c}) = h$. Note that $\bar{c}$ is a solution in $\mathcal{C}$ to $\log\delta g_0(\bar{x}) = kma_2 - kma_1$, and that $\log\delta g_0(\bar{x})$ (when the domain is restricted to $\mathcal{C}$) can be viewed as a rational function of $\bar{x}$ over $F\langle a_1, a_2\rangle$ (see Lemma 2.26). By Lemma 2.19 there exists $\bar{e} \in \mathcal{C}_{F\langle a_1, a_2\rangle}$ which is also a solution. Then $g_0(\bar{e}) \in F\langle a_1, a_2\rangle$ and km witness the claim. □

Proposition 5.11. *Assume p is not weakly orthogonal to $\mathcal{C}$. Then there exist $g(y, z) \in F(y, z)$ and a nonzero integer k such that $\log\delta g(a, \delta a) - ka \in F$ for any realization a of p.*

Proof. By Lemma 5.9, $F\langle a_1, a_2\rangle = F(a_1, \delta a_1, a_2, \delta a_2)$. By Proposition 5.10, there exist $g \in F(y, z, u, v)$ such that

$$\log\delta g(a_1, \delta a_1, a_2, \delta a_2) = ka_2 - ka_1.$$

Since (a_1, a_2, a_3) is a Morley sequence over F, we have

$$\log\delta g(a_1, \delta a_1, a_3, \delta a_3) = ka_3 - ka_1$$

and

$$\log\delta g(a_2, \delta a_2, a_3, \delta a_3) = ka_3 - ka_2,$$

so

$$\log\delta g(a_1, \delta a_1, a_3, \delta a_3) + ka_1 = \log\delta g(a_2, \delta a_2, a_3, \delta a_3) + ka_2.$$

We claim that there exists in F a realization of the formula

$$\log\delta g(a_1, \delta a_1, x, \delta x) + ka_1 = \log\delta g(a_2, \delta a_2, x, \delta x) + ka_2,$$

which we denote by $\varphi(x)$. By Lemma 5.9, p is the generic type of an order 1 definable set D over F. So D has Morley rank 1 (see Lemma 5.8 of [18]), and as we are working over an acl-closed set we may take D to be strongly minimal. On the other hand, since p is not weakly orthogonal to $\mathcal{C}$, it cannot be isolated. Indeed, by Lemma 2.20, a is interalgebraic with a constant c over F, and $\mathrm{tp}(c/F)$ is not isolated because in ACF the only isolated types are the algebraic ones. We now have that D is strongly minimal and $p(\mathcal{U}) \subseteq D$ is not isolated, so $D \cap F$ is infinite. Since φ is realized by some generic element a_3 of D, it is realized by all but finitely many elements in D, so in particular we can find some $f \in D \cap F$ that realizes φ, i.e.,

$$\mathrm{log}\delta g(a_1, \delta a_1, f, \delta f) + ka_1 = \mathrm{log}\delta g(a_2, \delta a_2, f, \delta f) + ka_2.$$

Letting $g_0(y, z) = g(y, z, f, \delta f) \in F(y, z)$, we have shown that the F-definable function $\mathrm{log}\delta g_0(y, \delta y) + ky$ has the same value at a_1 as at a_2. As a_1, a_2 are independent realizations of p, this implies that $\mathrm{log}\delta g_0(y, \delta y) + ky$ is constant on all of $p(\mathcal{U})$. Hence it must be that $\mathrm{log}\delta g_0(a, \delta a) + ka \in F$ for $a \models p$. So g_0 witnesses the truth of the proposition. □

Proof of Theorem 5.6. We now assume (1) in Conjecture 5.4 and we need to prove (3). Let $u \models q$ and $a = \mathrm{log}\delta u$. By Proposition 5.11, we have $\mathrm{log}\delta g(a, \delta a) - ka \in F$ for some $g \in F(y, z)$ and some nonzero integer k. Let $w_1 = g(a, \delta a)$ and $w_2 = \dfrac{u^k}{w_1}$. Then $u^k = w_1 w_2$, $w_1 \in F\langle a\rangle = F\langle \mathrm{log}\delta u\rangle$, and

$$\begin{aligned}
\mathrm{log}\delta w_2 &= \mathrm{log}\delta \frac{u^k}{w_1} \\
&= k\mathrm{log}\delta u - \mathrm{log}\delta w \\
&= ka - g(a, \delta a) \in F.
\end{aligned}$$

That is, condition (3) of Conjecture 5.4 holds. This proves the conjecture when p is not weakly orthogonal to $\mathcal{C}$. □

5.2 The weakly-orthogonal case

We now explore the case when the minimal type $p \in S_1(F)$ is weakly orthogonal to $\mathcal{C}$. We first introduce the following additional assumption:
$(*)$ For every $a \models p$, there exists $v \in F\langle a\rangle \backslash F$ such that $\delta v \in F$ or $\log\delta(v) \in F$.

Theorem 5.12. *Conjecture 5.4 is true for p satisfying $(*)$.*

Before we prove this theorem, let us point out that this implies the truth of the conjecture when working with order 1 degree 1 differential equations over a field of constants.

Corollary 5.13. *Conjecture 5.4 is true when $F \subseteq \mathcal{C}$ and p is the generic type of a differential equation of the form $\delta x = f(x)$ where $f \in F(x)$.*

Proof. By Theorem 5.12, we only need to prove that p satisfies $(*)$.

Let $(a_1, a_2, \ldots)$ be a Morley sequence in p. Notice that $F\langle a_i\rangle = F(a_i)$ as $\delta a_i \in F(a_i)$. Since p is almost $\mathcal{C}$-internal, there exists an integer k such that $\mathrm{tp}(a_k/Fa_1 \cdots a_{k-1})$ is not weakly orthogonal to $\mathcal{C}$. This implies that $\mathcal{C}_{F(a_1,\ldots,a_k)} \neq \mathcal{C}_{F(a_1,\ldots,a_{k-1})}$ by Lemma 2.22. Let i be the least such k. Then by Rosenlicht's Theorem (see Theorem 6.12 of [18]), as $f(x)$ is defined over $\mathcal{C}_{F(a_1,\ldots,a_{i-1})} = \mathcal{C}_F$ and $\mathcal{C}_{F(a_1,\ldots,a_i)} \neq \mathcal{C}_{F(a_1,\ldots,a_{i-1})}$, we have that $\dfrac{1}{f(x)}$ is of the form $c\dfrac{\partial u}{\partial x}/u$ or $c\dfrac{\partial u}{\partial x}$ for some $u \in F(x)$, $c \in F$.

If $\dfrac{1}{f(x)} = c\dfrac{\partial u}{\partial x}/u$, then for any $a \models p$, $u(a) \in F(a)$ satisfies

$$\begin{aligned}
\log\delta u(a) &= \frac{\delta u(a)}{u(a)} \\
&= \frac{\frac{\partial u}{\partial x}(a)\delta a}{u(a)} \\
&= \frac{\frac{\partial u}{\partial x}(a)}{u(a)} f(a)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{cf(a)} f(a) \\
&= \frac{1}{c} \in \mathcal{C}_F.
\end{aligned}
$$

If $\dfrac{1}{f(x)} = c\dfrac{\partial u}{\partial x}$, then for any $a \models p$, $u(a) \in F(a)$ satisfies

$$
\begin{aligned}
\delta u(a) &= \frac{\partial u}{\partial x}(a)\delta a \\
&= \frac{\partial u}{\partial x}(a) f(a) \\
&= \frac{1}{cf(a)} f(a) \\
&= \frac{1}{c} \in \mathcal{C}_F.
\end{aligned}
$$

Therefore $(*)$ holds. □

The goal of the rest of this section is to prove Theorem 5.12. We may assume p is weakly orthogonal to $\mathcal{C}$ by Theorem 5.6. Similar to the previous section, we only need to prove (1)⇒(3) of the conjecture.

We assume the following for the rest of this section:

1. F is an algebraically closed differential field;

2. p is an almost $\mathcal{C}$-internal minimal type over F that is weakly orthogonal to $\mathcal{C}$;

3. p satisfies $(*)$; note then that there is a non-constant δ-rational function $\alpha \in F\langle x \rangle$ such that either $\delta\alpha(a) \in F$ for all $a \models p$ or $\log\delta\alpha(a) \in F$ for all $a \models p$;

4. $q = \log\delta^{-1}(p)$ is almost $\mathcal{C}$-internal;

5. $(u_1, u_2, \ldots)$ is a Morley sequence in q, $a_i := \log\delta u_i$, and $v_i := \alpha(a_i)$;

6. K is the field generated by $F \cup \mathcal{C}$; and

7. $L := K\langle a_1, a_2, \ldots\rangle$.

Note that the above assumptions includes all those that appeared in the previous section, so we may use the results of Lemmas 5.7 through 5.9.

Proposition 5.14. *We have $gu_1^k = u_2^k$ for some $g \in K\langle a_1, a_2\rangle$ and some nonzero integer k.*

Proof. For any integer i, recall that $v_i \in F\langle a_i\rangle \backslash F$, and either $\delta v_i \in F$ or $\log\delta v_i \in F$. Since $\mathrm{tp}(v_i/F) = \mathrm{tp}(v_j/F)$, this means that either $v_i - v_j \in C$ or $\dfrac{v_i}{v_j} \in \mathcal{C}$ for any i, j. In either case, $v_i \in \mathcal{C}(v_j)$. On the other hand, by minimality of p, $a_i \in F\langle v_i\rangle^{\mathrm{alg}} = F(v_i)^{\mathrm{alg}}$ for all i (as we have $v_i \in F\langle a_i\rangle \backslash F$). Therefore, $a_i \in F(v_i)^{\mathrm{alg}} \subseteq K(v_j)^{\mathrm{alg}} \subseteq K\langle a_j\rangle^{\mathrm{alg}}$ for any i, j, which means that $L = K\langle a_1, \ldots, a_n\rangle \subseteq K\langle a_1\rangle^{\mathrm{alg}}$.

By Lemma 5.8 there is $g \in L = K\langle a_1, a_2, \ldots\rangle$ and a nonzero integer k such that $gu_1^k = u_2^k$. Let $g = g_1, g_2, \ldots, g_m$ be conjugates of g over $K\langle a_1, a_2\rangle$. Taking logarithmic derivative on both sides of $gu_1^k = u_2^k$, we get that $\log\delta g + ka_1 = ka_2$. Since $g_1, \ldots, g_m$ are conjugates of g over $K\langle a_1, a_2\rangle$, $\log\delta g_i + ka_1 = ka_2$ for all $i = 1, \ldots, m$. We therefore have $\log\delta\left(\prod_{i=1}^{m} g_i\right) + kma_1 = kma_2$. Let $g_0 = \prod g_i \in K\langle a_1, a_2\rangle$. Then as $\log\delta g_0 + kma_1 = kma_2$, there exists $c \in \mathcal{C}$ such that $cg_0u_1^{km} = u_2^{km}$. $\square$

Lemma 5.15. *There exist $g \in F(a_1, \delta a_1, a_2, \delta a_2)$ and k a nonzero integer such that $cgu_1^k = u_2^k$ for some $c \in \mathcal{C}$.*

Proof. By Proposition 5.14 and Lemma 5.9, we have $g_0u_1^k = u_2^k$ for $g_0 \in K\langle a_1, a_2\rangle \subseteq K(a_1, \delta a_1, a_2, \delta a_2)$, so $\log\delta g_0 = ka_2 - ka_1$. Since K is the field generated by F and $\mathcal{C}$, we can rewrite $g_0 = g(\bar{c})$ for $\bar{c} \in \mathcal{C}^n$, and $g \in F(a_1, \delta a_1, a_2, \delta a_2)(\bar{x})$. Then $\bar{c}$ is a solution to $\log\delta g(x) = ka_2 - ka_1$, and notice that $\log\delta g(\bar{x})$, restricted to $\mathcal{C}$, is a rational function over $F(a_1, \delta a_1, a_2, \delta a_2)$ (see Lemma 2.26). By Lemma 2.19, let $\bar{c}_2 \in \mathcal{C}^{\mathrm{alg}}_{F(a_1,\delta a_1,a_2,\delta a_2)}$ be a solution of $\log\delta g(\bar{x}) = ka_2 - ka_1$. Let $\bar{e}_1 = \bar{c}_2, \bar{e}_2, \ldots, \bar{e}_m$ be conjugates of

$\bar{c}_2$ over $\mathcal{C}_{F(a_1,\delta a_1,a_2,\delta a_2)}$. Then $\log\delta g(\bar{e}_i) = ka_2 - ka_1$. Let $h = \prod_{i=1}^{m} g(\bar{e}_i) \in F(a_1, \delta a_1, a_2, \delta a_2)$. Then $\log\delta h = kma_2 - kma_1$, which means for some $c \in \mathcal{C}$, $chu_1^{km} = u_2^{km}$, as desired. $\square$

Proposition 5.16. *Suppose that for $a \models p$, $a \in F(\alpha(a))$. Then there exist some $g(y) \in F(y)$ and some nonzero integer k such that $\log\delta g(v_i) - ka_i \in F$ for all i.*

Proof. By Lemma 5.15, we have $cg_0u_1^k = u_2^k$ for some $c \in \mathcal{C}$, k nonzero integer, and $g_0 \in F(a_1, \delta a_1, a_2, \delta a_2)$. By assumption, $a_i \in F(v_i)$ for $i = 1, 2$. Moreover, either $\delta v_i \in F$ or $\log\delta v_i \in F$, so that $F\langle v_i\rangle = F(v_i)$. Hence, we also have that $\delta a_i \in F(v_i)$. Since $a_i, \delta a_i \in F(v_i) = F\langle v_i\rangle$ for $i = 1, 2$, So $g_0 \in F(v_1, v_2)$, and we set $g(y_1, y_2) \in F(y_1, y_2)$ to be such that $g(v_1, v_2) = g_0$.

In the following proof, we will use these facts about v_i.

(1) *$(v_1a_1, v_2a_2, \ldots)$ is a Morley sequence over F.* This is because $(a_1, a_2, \ldots)$ is a Morley sequence and $v_i = \alpha(a_i)$.

(2) *$\log\delta g(v_i, v_j) = ka_j - ka_i$ for all $i \neq j$.* Indeed, this is true for $(i, j) = (1, 2)$ because $g(v_1, v_2) = g_0$ and $cg_0u_1^k = u_2^k$. Now use (1) to see that the statement is true for all $i \neq j$.

(3) *$v_i \notin K^{alg}$.* Since p is weakly orthogonal to the constants, $a_i \notin K^{\mathrm{alg}}$. Since $a_i \in F(v_i)$, this implies $v_i \notin K^{\mathrm{alg}}$.

(4) *$\mathrm{tp}(a_i, v_i/F)$ is minimal.* This is because p is minimal and $v_i \in F\langle a_i\rangle$.

(5) *$\delta v_i \in F$ for all i, or $\log\delta v_i \in F$ for all i.* This is by assumption $(*)$.

To deal with the two cases ($\delta v_i \in F$ and $\log\delta v_i \in F$) uniformly, we define $[x : y] = x - y$ and $x * y = x + y$ if $\delta v_i \in F$ for all i, and $[x : y] = \frac{x}{y}$ and $x * y = xy$ otherwise (i.e., $\log\delta v_i \in F$ for all i). Since $\mathrm{tp}(v_i/F) = \mathrm{tp}(v_j/F)$, $\delta v_i = \delta v_j$ in the first case, and $\log\delta v_i = \log\delta v_j$ in the second. So either way, we have $[v_i : v_j] \in \mathcal{C}$ for all i, j.

Let $g_1(x,y) = g(x, x*y) \in F(x,y)$. Then $g_1(v_1, [v_2 : v_1]) = g(v_1, v_2)$. Note that

$$\begin{aligned}
&\log\delta(g_1(v_1,[v_2:v_1])g_1(v_2,[v_3:v_2])g_1(v_3,[v_1:v_3])) \\
&= ka_2 - ka_1 + ka_3 - ka_2 + ka_1 - ka_3 \\
&= 0,
\end{aligned}$$

so

$$g_1(v_1,[v_2:v_1])g_1(v_2,[v_3:v_2])g_1(v_3,[v_1:v_3]) = e_1$$

for some $e_1 \in \mathcal{C}$. Note also that an automorphism in $\mathrm{Aut}_F(\mathcal{U})$ that takes (v_1, v_2, v_3) to (v_2, v_3, v_1) or (v_3, v_1, v_2) fixes e_1.

Suppose $g_1(x,y) = \dfrac{\sum_{i=0}^{m} p_{1i}(y)x^i}{\sum_{i=0}^{n} p_{2i}(y)x^i}$ where each $p_{1i}(y), p_{2i}(y) \in F(y)$.

If $\delta v_i \in F$ for all i, then

$$\begin{aligned}
e_1 &= g_1(v_1,[v_2:v_1])g_1(v_2,[v_3:v_2])g_1(v_3,[v_1:v_3]) \\
&= g_1(v_1,[v_2:v_1])g_1(v_1+[v_2:v_1],[v_3:v_2])g_1(v_1+[v_3:v_1],[v_1:v_3]) \\
&= \left(\frac{p_{1m}([v_2:v_1])}{p_{2n}([v_2:v_1])}\right)\left(\frac{v_1^m+\sum_{i=0}^{m-1} v_1^i q_{1i}}{v_1^n+\sum_{i=0}^{n-1} v_1^i q_{2i}}\right)\left(\frac{p_{1m}([v_3:v_2])}{p_{2n}([v_3:v_2])}\right) \\
&\quad \left(\frac{v_1^m+\sum_{i=0}^{m-1} v_1^i q_{3i}}{v_1^n+\sum_{i=0}^{n-1} v_1^i q_{4i}}\right)\left(\frac{p_{1m}([v_1:v_3])}{p_{2n}([v_1:v_3])}\right)\left(\frac{v_1^m+\sum_{i=0}^{m-1} v_1^i q_{5i}}{v_1^n+\sum_{i=0}^{n-1} v_1^i q_{6i}}\right) \\
&= \left(\frac{p_{1m}([v_2:v_1])}{p_{2n}([v_2:v_1])}\right)\left(\frac{p_{1m}([v_3:v_2])}{p_{2n}([v_3:v_2])}\right)\left(\frac{p_{1m}([v_1:v_3])}{p_{2n}([v_1:v_3])}\right) \\
&\quad \left(\frac{v_1^{3m}+\sum_{i=0}^{3m-1} v_1^i q_{7i}}{v_1^{3n}+\sum_{i=0}^{3m-1} v_1^i q_{8i}}\right)
\end{aligned}$$

where $q_{ji} \in F([v_2:v_1],[v_3:v_2],[v_1:v_3]) \subseteq K$ for all i, j.

If $\log\delta(v_i) \in F$ for all i, then

$$\begin{aligned}
e_1 &= g_1(v_1,[v_2:v_1])g_1(v_2,[v_3:v_2])g_1(v_3,[v_1:v_3]) \\
&= g_1(v_1,[v_2:v_1])g_1([v_2:v_1]v_1,[v_3:v_2])g_1([v_3:v_1]v_1,[v_1:v_3])
\end{aligned}$$

$$= \left(\frac{p_{1m}([v_2 : v_1])}{p_{2n}([v_2 : v_1])}\right)\left(\frac{v_1^m + \sum_{i=1}^{m-1} v_1^i q_{1i}}{v_1^n + \sum_{i=1}^{n-1} v_1^i q_{2i}}\right)\left(\frac{p_{1m}([v_3 : v_2])}{p_{2n}([v_3 : v_2])}\right)$$
$$\left(\frac{[v_2 : v_1]^m v_1^m + \sum_{i=1}^{m-1} v_1^i q_{3i}}{[v_2 : v_1]^n v_1^n + \sum_{i=1}^{n-1} v_1^i q_{4i}}\right)\left(\frac{p_{1m}([v_1 : v_3])}{p_{2n}([v_1 : v_3])}\right)$$
$$\left(\frac{[v_3 : v_1]^m v_1^m + \sum_{i=1}^{m-1} v_1^i q_{5i}}{[v_3 : v_1]^n v_1^n + \sum_{i=1}^{n-1} v_1^i q_{6i}}\right)$$
$$= \left(\frac{p_{1m}([v_2 : v_1])}{p_{2n}([v_2 : v_1])}\right)\left(\frac{p_{1m}([v_3 : v_2])}{p_{2n}([v_3 : v_2])}\right)\left(\frac{p_{1m}([v_1 : v_3])}{p_{2n}([v_1 : v_3])}\right)$$
$$\left([v_2 : v_1]^{m-n}[v_3 : v_1]^{m-n}\right)\left(\frac{v_1^{3m} + \sum_{i=1}^{3m-1} v_1^i q_{7i}}{v_1^{3n} + \sum_{i=1}^{3m-1} v_1^i q_{8i}}\right)$$

where $q_{ji} \in F([v_2 : v_1], [v_3 : v_2], [v_1 : v_3]) \subseteq K$ for all i, j.

In either case, since $e_1 \in K$, $\dfrac{p_{1m}([v_2 : v_1])p_{1m}([v_3 : v_2])p_{1m}([v_1 : v_3])}{p_{2n}([v_2 : v_1])p_{2n}([v_3 : v_2])p_{2n}([v_1 : v_3])} \in K$, and $[v_2 : v_1]^{m-n}[v_3 : v_1]^{m-n} \in K$, we must have that $\dfrac{v_1^{3m} + \sum_{i=1}^{3m-1} v_1^i q_{7i}}{v_1^{3n} + \sum_{i=1}^{3m-1} v_1^i q_{8i}} \in K$.

However, since $v_1 \notin K^{\text{alg}} = \operatorname{acl}(F\mathcal{C})$. we must also have that $m = n$ and each $q_{7i} = q_{8i}$, which yields $\dfrac{v_1^{3m} + \sum_{i=1}^{3m-1} v_1^i q_{7i}}{v_1^{3n} + \sum_{i=1}^{3m-1} v_1^i q_{8i}} = 1$. Therefore

$$e_1 = \frac{p_{1m}([v_2 : v_1])p_{1m}([v_3 : v_2])p_{1m}([v_1 : v_3])}{p_{2n}([v_2 : v_1])p_{2n}([v_3 : v_2])p_{2n}([v_1 : v_3])} \tag{5.1}$$

Let $\mathcal{C}_1 = \mathcal{C}_F(v_1, v_2, v_3)^{\text{alg}}$. Since $\{v_1, v_2, v_3\}$ is algebraically independent over F, $\mathcal{C}_1$ and F are independent field extensions of $\mathcal{C}_F$. Write $\dfrac{p_{1m}(y)}{p_{2n}(y)}$ as $f_1(\bar{\alpha}, y)$ with $\bar{\alpha}$ a tuple from F and $f_1 \in \mathcal{C}_F(\bar{x}, y)$. We have $f_1(\bar{\alpha}, [v_2 : v_1])f_1(\bar{\alpha}, [v_3 : v_2])f_1(\bar{\alpha}, [v_1 : v_3]) = e_1$.

We now show that $e_1 \in \mathcal{C}_1$. By construction, $e_1 \in F(v_1, v_2, v_3) \cap \mathcal{C} = \mathcal{C}_{F(v_1,v_2,v_3)}$. Since $\text{Tr.Deg}(F(v_1, v_2, v_3)/F(v_1)) = 2$, $\text{Tr.Deg}(\mathcal{C}_{F(v_1,v_2,v_3)}/\mathcal{C}_{F(v_1)})$ is at most 2. In fact, $\text{Tr.Deg}(\mathcal{C}_{F(v_1,v_2,v_3)}/\mathcal{C}_{F(v_1)}) = 2$, since $[v_2 : v_1], [v_3 : v_1]$ are two algebraically independent elements in $\mathcal{C}_{F(v_1,v_2,v_3)}$ over $\mathcal{C}_{F(v_1)}$. As p is weakly orthogonal to $\mathcal{C}$, $\mathcal{C}_F = \mathcal{C}_{F(v_1)}$ by Lemma 2.22. Therefore, $\mathcal{C}^{\mathrm{alg}}_{F(v_1,v_2,v_3)} = \mathcal{C}_F([v_2 : v_1],[v_3 : v_1])^{\mathrm{alg}} \subseteq \mathcal{C}_1$, so $e_1 \in \mathcal{C}_1$. Now by Lemma 5.30 of the appendix, there are $d_1(y), d_2(y), d_3(y) \in C_F(y)$ such that

$$e_1^n d_1([v_2 : v_1]) d_2([v_3 : v_2]) d_3([v_1 : v_3]) = 1$$

for some $n > 0$.

Since e_1 is fixed under any automorphism that fixes F and takes (v_1, v_2, v_3) to (v_2, v_3, v_1) or (v_3, v_1, v_2), we have

$$e_1^{3n} \prod_{i=1}^{3} d_i([v_2 : v_1]) d_i([v_3 : v_2]) d_i([v_1 : v_3]) = 1.$$

Let $g_2(x_1, x_2) := (g_1(x_1, [x_2 : x_1]))^n \prod_{i=1}^{3} d_i([x_2 : x_1])$, which is over F. We have $\log\delta g_2(v_1, v_2) = ka_2 - ka_1$ and

$$g_2(v_1, v_2) g_2(v_2, v_3) g_2(v_3, v_1) = 1.$$

Note that $g_2(v_2, v_1) g_2(v_3, v_2) g_2(v_1, v_3) = 1$ as well because v_1, v_2, v_3 is indiscernible over F. Let $g_3(x_1, x_2) = \dfrac{g_2(x_1, x_2)}{g_2(x_2, x_1)}$. Then we have $\log\delta g_2(v_1, v_2) = 2ka_2 - 2ka_1$, $g_3(v_1, v_2) g_3(v_2, v_3) g_3(v_3, v_1) = 1$, and $g_3(v_2, v_3) g_3(v_3, v_2) = 1$.

Now v_3 satisfies

$$g_3(v_1, v_2) = \frac{g_3(y, v_2)}{g_3(y, v_1)} \tag{5.2}$$

and is independent from v_1, v_2 over F. Since (5.2) is a field-theoretic equation, we get that it has infinite (therefore co-finite) many solutions. Let $c_3 \in F$ be a solution, and let $g_4(x) = g_3(c_3, x) \in F(x)$. Then $\log\delta \dfrac{g_4(v_2)}{g_4(v_1)} = 2ka_2 - 2ka_1$. Therefore $\log\delta g_4(v_2) - 2ka_2 = \log\delta g_4(v_1) - 2ka_1$. Since $\mathrm{tp}(v_i/Fv_1)$ are the same for all $i = 2, 3, \ldots$, we have that $\log\delta g_4(v_i) - 2ka_i$ does not depend on i. Since $(a_1 v_1, a_2 v_2, \ldots)$ is a Morley sequence over F, this implies that the

F-definable function $\log\delta g_4(y) - 2kx$ is constant on the set of realizations of $\mathrm{tp}(a_i, v_i/F)$. Hence it is F-valued, so $\log\delta g_4(v_i) - 2ka_i \in F$ for all i, as desired. $\square$

Proposition 5.17. *There exist some $g \in F(x, z)$ and some nonzero integer k such that $\log\delta g(a_i, \delta a_i) - ka_i \in F$ for all i.*

Proof. If $a \in F(\alpha(a))$ for $a \models p$, then by Proposition 5.16, there exist $g(y) \in F(y)$ and nonzero integer k such that $\log\delta g(v_i) - ka_i \in F$ for all i. Since $v_i \in F(a_i)$, we can write $g(v_i) = f(a_i)$, and setting $g^*(x, z) = f(x)$ witnesses the proposition.

Now assume $a \notin F(\alpha(a))$ for $a \models p$.

By Lemma 5.15, we have $cg_0(a_1, \delta a_1, a_2, \delta a_2)u_1^k = u_2^k$ for some $c \in \mathcal{C}$, k nonzero integer, and $g_0(x_1, z_1, x_2, z_2) \in F(x_1, z_1, x_2, z_2)$, so $\log\delta g_0(a_1, \delta a_1, a_2, \delta a_2) = ka_2 - ka_1$. Let $a_{i1} = a_i, a_{i2}, \ldots, a_{im}$ be the conjugates of a_i over $F(v_i)$, and let $b_i = \sum_{j=1}^{m} a_{ij}$. Note that since $a_1, v_1 \mathop{\smile\!\!\!\!\!\!\;|}_F a_2, v_2$, $\mathrm{tp}(a_{1\alpha}, a_{2\beta}) = \mathrm{tp}(a_1, a_2)$ for any $\alpha, \beta \in \{1, \ldots, m\}$. Then we have $\prod_{\alpha=1}^{m}\prod_{\beta=1}^{m} g_0(a_{1\alpha}, \delta a_{1\alpha}, a_{2\beta}, \delta a_{2\beta}) \in F(v_1, v_2)$ and

$$\log\delta\left(\prod_{\alpha=1}^{m}\prod_{\beta=1}^{m} g_0(a_{1\alpha}, \delta a_{1\alpha}, a_{2\beta}, \delta a_{2\beta})\right) = mkb_2 - mkb_1.$$

Let $g(y_1, y_2) \in F(y_1, y_2)$ be such that $g(v_1, v_2) = \prod_{\alpha=1}^{m}\prod_{\beta=1}^{m} g_0(a_{1\alpha}, \delta a_{1\alpha}, a_{2\beta}, \delta a_{2\beta})$. Note that the proof of Proposition 5.16 applies here (as all facts used in the proof are satisfied if we replace a_i with mkb_i), so $\log\delta g_1(v_i) - \ell mkb_i \in F$ for some $g_1(y) \in F(y)$ and nonzero integer ℓ.

Since $\prod_{\beta=1}^{m} g(a_1, \delta a_1, a_{2\beta}, \delta a_{2\beta}) \in F(a_1, \delta a_1, v_2)$, let $g_2(x, z, y) \in F(x, z, y)$

be such that $g_2(a_1, \delta a_1, v_2) = \prod_{\beta=1}^{m} g(a_1, \delta a_1, a_{2\beta}, \delta a_{2\beta})$. We have $\mathrm{log}\delta g_2(a_1, \delta a_1, v_2) = \sum_{\beta=1}^{m}(ka_{2\beta} - ka_1) = kb_2 - mka_1$, so

$$
\begin{aligned}
&\mathrm{log}\delta \frac{g_2(a_1, \delta a_1, v_2)^\ell}{g_1(v_2)} - (-\ell mk)a_1 \\
&= \ell \mathrm{log}\delta g_2(a_1, \delta a_1, v_2) - \mathrm{log}\delta g_1(v_2) - (-\ell mk)a_1 \\
&= \ell kb_2 - \ell mka_1 - \mathrm{log}\delta g_1(v_2) - (-\ell mk)a_1 \\
&= -(\mathrm{log}\delta g_1(v_2) - \ell kb_2) \in F.
\end{aligned}
$$

Let $f = \mathrm{log}\delta \frac{g_2(a_1, \delta a_1, v_2)^\ell}{g_1(v_2)} - (-\ell mk)a_1$. Note that $\frac{g_2(a_1, \delta a_1, v_2)^\ell}{g_1(v_2)} \in K(a_1, \delta a_1)$ since $v_2 \in \mathcal{C}(v_1) \subseteq K(a_1, \delta a_1)$. Let $g_3(\bar{x}) \in F(a_1, \delta a_1)(\bar{x})$ be such that $g_3(\bar{c}) = \frac{g_2(a_1, \delta a_1, v_2)^\ell}{g_1(v_2)}$. So $\bar{c}$ is a solution to $h(\bar{x}) = 0$ where $h(\bar{x}) = \mathrm{log}\delta g_3(\bar{x}) - (-\ell mk)a_1 - f$. Note that $h\restriction_{\mathcal{C}}$ is a rational function over $F(a_1, \delta a_1)$, so by Lemma 2.19 there exists $\bar{e} \in \mathcal{C}_{F(a_1, \delta a_1)}$ which is a solution of this equation. Since p is weakly orthogonal to $\mathcal{C}$, $\mathcal{C}_{F(a_1, \delta a_1)} = \mathcal{C}_F$. Therefore, $\mathrm{log}\delta g_3(a_1, \delta a_1, \bar{e}) - (-\ell mk)a_i \in F$. And since $\mathrm{tp}(a_i/F)$ does not depend on i, $\mathrm{log}\delta g_3(a_i, \delta a_i, \bar{e}) - (-\ell mk)a_i \in F$ for $(-\ell mk)$ a nonzero integer and $g_3(x, z, \bar{e}) \in F(x, z)$. □

Proof of Theorem 5.12. We need to show that condition (3) of Conjecture 5.4 holds. Let $u \models q$ and $a = \mathrm{log}\delta u$. By Proposition 5.17, $\mathrm{log}\delta g(a, \delta a) - ka \in F$ for some $g(x, z) \in F(x, z)$ and some nonzero integer k. Let $w_1 = g(a, \delta a)$ and $w_2 = \frac{u^k}{w_1}$, so we have $u^k = w_1 w_2$ where $w_1 \in F\langle a \rangle = F\langle \mathrm{log}\delta u \rangle$ and $\mathrm{log}\delta w_2 = k\mathrm{log}\delta u - \mathrm{log}\delta w_1 = ka - \mathrm{log}\delta g(a, \delta a) \in F$, as desired. □

5.3 Examples

In this section we describe some examples to which our theorems apply. We first show two examples where Theorem 5.6 applies.

Example 5.18. Let $p(x)$ be the generic type of the constants. We know by Fact 2.27 that $q = \log\delta^{-1}(p)$ is not almost $\mathcal{C}$-internal. But this can also be seen to follow from Theorem 5.6, which applies as p is not weakly orthogonal to $\mathcal{C}$. Suppose q is almost $\mathcal{C}$-internal. Then by the truth of Conjecture 5.4 in this case, we have, for $u \models q$, that $u^{\ell} = u_1 u_2$ where $u_1 \in \mathbb{Q}^{\mathrm{alg}} \langle \log\delta u \rangle$ and $\log\delta u_2 \in \mathbb{Q}^{\mathrm{alg}}$. Since $\log\delta u \in \mathcal{C}$, $u_1 \in \mathcal{C}$, and so $\log\delta u_1 = 0$. Hence $\log\delta u^{\ell} = \log\delta u_2 \in \mathbb{Q}^{\mathrm{alg}}$, contradicting the fact that q is of U-rank 2.

Example 5.19. Fix t such that $\delta t = 1$, and set $F = \mathbb{Q}(t)^{\mathrm{alg}}$. Suppose $p \in S_1(F)$ is the generic type of the strongly minimal set D defined by the equation $\delta(\frac{1}{x}) = 1$. This is really Example 5.1, except that we work over $\mathbb{Q}(t)^{\mathrm{alg}}$ rather than $\mathbb{Q}^{\mathrm{alg}}$. A consequence of working over F is that we can express D as $D = \{x : x = \dfrac{1}{t+c}, c \in \mathcal{C}\}$, so that p is not weakly orthogonal to $\mathcal{C}$. Hence Theorem 5.6 applies. But in this case we already know that $\log\delta^{-1}(D)$, and hence $q := \log\delta^{-1}(p)$, is $\mathcal{C}$-internal. Moreover, our proof of this in Example 5.1 goes by decomposing $u \models q$ as $u = \left(\dfrac{u}{\delta u}\right)(u)$, witnessing condition (3) of Conjecture 5.4.

Here are examples where Theorem 5.12 applies.

Example 5.20. Consider again the minimal set D of Example 5.1 given by $\delta(\frac{1}{x}) = 1$, but this time let p be the generic type of D over $F := \mathbb{Q}^{\mathrm{alg}}$. Note that p is weakly orthogonal to $\mathcal{C}$ because otherwise, by Lemma 2.16, any realization a of p would be in $\mathrm{acl}(F\mathcal{C}) = \mathcal{C}$, but no constant satisfies $\delta(\frac{1}{x}) = 1$. So Theorem 5.6 does not apply but Theorem 5.12 does, since if $a \models p$ then $v = \dfrac{1}{a}$ satisfies $\delta v = 1 \in F$. In any case, we already know that $\log\delta^{-1}(p)$ is $\mathcal{C}$-internal, by Example 5.1.

Example 5.21. Let $F := \mathbb{Q}^{\text{alg}}$ and $p(x) \in S_1(F)$ be the generic type of the equation $\delta x = 1$. It is $\mathcal{C}$-internal but weakly orthogonal to $\mathcal{C}$. Theorem 5.12 applies and $q := \log\delta^{-1}(p)$ is not almost $\mathcal{C}$-internal.

Proof. To see that Theorem 5.12 applies, note that if $a \models p$ then $v := a$ satisfies $\delta v = 1 \in F$. Now, let u be a realization of q, and $a = \log\delta u$ a realization of p. Suppose $u^\ell = u_1 u_2$ with $u_1 \in F\langle a\rangle$ and $\log\delta u_2 \in F$. Let t be an element in the universe such that $\delta t = 1$, so that $a = t + c_1$ for some $c_1 \in \mathcal{C}$. Hence $u_1 \in \mathcal{C}(t)$. Suppose $u_1 = e_0 \prod_{i=1}^{n}(t - e_i)^{k_i}$ where $e_i \in \mathcal{C}$ and $k_i \in \mathbb{Z}$. Then $\log\delta u_1 = \sum_{i=1}^{n} \frac{k_i(t-e_i)^{k_i-1}}{(t-e_i)^{k_i}} = \sum_{i=1}^{n} \frac{k_i}{t-e_i}$. But $\log\delta u_1 = \ell\log\delta u - \log\delta u_2 = \ell t + \ell c_1 - \log\delta u_2 = \ell t + c_2$ for some $c_2 \in \mathcal{C}$, which means $\sum_{i=1}^{n} \frac{k_i}{t-e_i} = \ell t + c_2$. This implies that $t \in \mathcal{C}^{\text{alg}} = \mathcal{C}$, contradicting the fact that $\delta t = 1$. Hence there can be no such decomposition of u^ℓ. By Theorem 5.12, p is not almost $\mathcal{C}$-internal. $\square$

Example 5.22. This is a generalization of Example 5.20. Let $F = \mathbb{Q}^{\text{alg}}$. Fix $n \in \mathbb{Z}\backslash\{0\}$ and suppose $p \in S_1(F)$ is the generic type of the strongly minimal set D defined by the equation $(\delta x)^n = n^n x^{n-1}$. Note that when $n = -1$, this equation becomes $\frac{1}{\delta x} = -x^{-2}$, which is equivalent to the equation in Examples 5.19 and 5.20. In any case, p is a $\mathcal{C}$-internal minimal type weakly orthogonal to $\mathcal{C}$. Moreover, Theorem 5.12 applies and $\log\delta^{-1}(p)$ is not almost $\mathcal{C}$-internal unless $n = -1$.

Proof. Taking derivative on both sides of $(\delta x)^n = n^n x^{n-1}$, and we get that $n(\delta x)^{n-1}\delta^2 x = n^n(n-1)x^{n-2}\delta x$, so $\delta^2 x = \frac{n^{n-1}(n-1)x^{n-2}}{(\delta x)^{n-2}}$. Note that

$$
\begin{aligned}
\delta\left(\frac{x}{\delta x}\right) &= \frac{(\delta x)^2 - x\delta^2 x}{(\delta x)^2} \\
&= \frac{(\delta x)^n - x(\delta x)^{n-2}\delta^2 x}{(\delta x)^n}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{(\delta x)^n - n^{n-1}(n-1)x^{n-1}}{(\delta x)^n} \\
&= \frac{n^n x^{n-1} - n^{n-1}(n-1)x^{n-1}}{n^n x^{n-1}} \\
&= \frac{1}{n}.
\end{aligned}
$$

If we work over $\mathbb{Q}(t)^{\text{alg}}$ where t is such that $\delta t = 1$, then $\dfrac{x}{\delta x} = \dfrac{t+c}{n}$ for some $c \in \mathcal{C}$, so $x = n^n \dfrac{x^n}{(\delta x)^n} = (t+c)^n$. On the other hand, for every $c \in \mathcal{C}$, $(t+c)^n$ is a solution to $(\delta x)^n = n^n x^{n-1}$. So we can express D as $D = \{x : x = (t+c)^n, c \in \mathcal{C}\}$, and there exists a t-definable map from $\mathcal{C}$ to D that maps c to $(t+c)^n$. This shows us that D is strongly minimal and $\mathcal{C}$-internal. Moreover, since $D(F) = \varnothing$, p is weakly orthogonal to $\mathcal{C}$. Theorem 5.12 applies because for any $a \models p$, $v = \dfrac{a}{\delta a}$ satisfies $\delta v = \dfrac{1}{n}$, so $(*)$ holds.

Let $q = \log\delta^{-1}(p)$ and let u be a realization of q. If q is almost $\mathcal{C}$-internal, then by Theorem 5.12 we can write $u = u_1 u_2$ where $u_1 \in F\langle \log\delta u \rangle$ and $d := \log\delta u_2 \in F$. Writing $\log\delta u = (t+c)^n$ with $c \in \mathcal{C}$, we have $u_1 \in F(c,t)$. Suppose $u_1 = s_0 \prod_i (t - s_i)^{\ell_i}$ where s_0 and all s_i are in $F(c)$. Then $\log\delta u_1 = \sum_i \dfrac{\ell_i}{t - s_i}$. As $\log\delta u = \log\delta u_1 + \log\delta u_2$, we have

$$
(t+c)^n = \sum_i \frac{\ell_i}{t - s_i} + d.
$$

The only way the above equality will hold is when $n = -1$, in which case $d = 0$, there is only one summand, ℓ_1=1, and $s_1 = -c$. $\square$

Equivalent condition (3) of Conjecture 5.4 states that there exists a nonzero ℓ such that $u^\ell = u_1 u_2$ where u_1, u_2 satisfy some conditions. Note that ℓ is equal to 1 in all of the above examples. The following example shows that this is not always the case. That is, we cannot replace "finite-to-one" with "bijective" in equivalent condition (2) of Conjecture 5.4.

Example 5.23. Let $F = \mathbb{Q}^{\text{alg}}$, and suppose p is the generic type of the definable set D defined by $\{x : \delta\left(\frac{1}{x}\right) = 2\}$. Let $q := \text{log}\delta^{-1}(p)$. Theorem 5.12 applies, since for any $a \models p$ we have that $v := \frac{1}{a}$ satisfies $\delta v = 2 \in F$. For any $u \models q$, $a := \text{log}\delta u$ satisfies

$$\delta a = \delta\left(\frac{1}{a^{-1}}\right) = \frac{-\delta(a^{-1})}{a^{-2}} = -2a^2.$$

So

$$\begin{aligned}\delta(u^2a) &= a\delta(u^2) + u^2\delta a \\ &= au^2\text{log}\delta(u^2) + u^2\delta a \\ &= 2a^2u^2 - 2u^2a^2 \\ &= 0.\end{aligned}$$

This means that there exists $c \in \mathcal{C}$ such that $u^2 = \frac{c}{a}$. As $a \in F\langle \text{log}\delta u\rangle$ and $\text{log}\delta c = 0 \in F$, this shows that p satisfies condition (3) of Conjecture 5.4. In particular, by Theorem 5.12, q is almost $\mathcal{C}$-internal.

We now show that $u \models q$ cannot be expressed as the product of u_1 and u_2 where $u_1 \in F\langle \text{log}\delta u\rangle$ and $c := \text{log}\delta u_2 \in F$. Suppose for a contradiction that such u_1, u_2 do exist. Let $a := \text{log}\delta u$ and $b := \frac{1}{2a}$. Note that $\delta b = \frac{1}{2}\delta\left(\frac{1}{a}\right) = 1$. Then $\text{log}\delta u_1 = \text{log}\delta u - \text{log}\delta u_2 = a - c = \frac{1}{2b} - c$ for some $c \in F$. Since $u_1 \in F\langle a\rangle = F(a) = F(b)$ as $\delta a = 2a^2$, we can write $u_1 = c_0\prod_i(b - c_i)^{\ell_i}$, where the c_i's are in F. Then $\text{log}\delta u_1 = \sum_i \frac{\ell_i\delta(b - c_i)}{b - c_i} = \sum_i \frac{\ell_i}{b - c_i}$. We thus have

$$\sum_i \frac{\ell_i}{b - c_i} = \frac{1}{2b} - c.$$

As all ℓ_i's are integers, the above equation is a nontrivial equation over F satisfied by b. However, this is impossible as b is transcendental over F. $\square$

In examples 5.20 through 5.23, condition $(*)$ was realized by finding $v \in F\langle a\rangle$ with $\delta v \in F$. The following is an example where the other alternative in $(*)$ is realized; i.e., we find $v \in F\langle a\rangle$ such that $\log\delta v \in F$.

Example 5.24. Let $D = \{x : \log\delta\left(-\frac{x-1}{x-2}\right) = 1\}$, which is defined over $F := \mathbb{Q}^{\text{alg}}$. For any $x \in D$, let $v = -\frac{x-1}{x-2}$. Since $\log\delta v = 1$, and x is interdefinable with v, we have that $p := \operatorname{tp}(x/F)$ is $\mathcal{C}$-internal. In addition, v witnesses that condition $(*)$ is satisfied. Hence Theorem 5.12 applies. In this case, $q = \log\delta^{-1}(p)$ is $\mathcal{C}$-internal.

Proof. Suppose u realizes q and $\log\delta u = a$. Let $v = -\frac{a-1}{a-2}$. We have

$$\begin{aligned}\log\delta\left(v^2+v\right) &= \frac{\delta(v^2+v)}{v^2+v}\\ &= \frac{2v\delta v+\delta v}{v^2+v}\\ &= \frac{2v^2+v}{v^2+v}\\ &= a,\end{aligned}$$

so that $u = c(v^2+v)$ for some $c \in \mathcal{C}$. Note that $v^2+v \in F\langle a\rangle = F\langle\log\delta u\rangle$ and $\log\delta c = 0 \in F$, so that this decomposition witnesses condition (3) of Conjecture 5.4. In particular, q is $\mathcal{C}$-internal. $\square$

5.4 A counterexample to $(*)$

We now give an example to which our theorems do not apply; namely, where $p \in S_1(F)$ is minimal, $\mathcal{C}$-internal, weakly orthogonal to $\mathcal{C}$, but $(*)$ fails. This serves also as a counterexample to the extension of Rosenlicht's theorem (see Theorem 6.12 of [18]; also see Corollary 5.13) to nonconstant parameters, which had been claimed in a preprint of James Freitag but withdrawn upon my communicating to him the following counterexample.

We will make use of the following consequence of $(*)$.

Lemma 5.25. *Suppose $p \in S_1(F)$ is the generic type of a $\mathcal{C}$-internal strongly minimal F-definable set D that is weakly orthogonal to $\mathcal{C}$. If p satisfies $(*)$, then the binding group $\mathrm{Aut}_F(D/\mathcal{C})$ is of Morley rank 1.*

Proof. See Section 2.1 for a review of the concept of binding group. If p satisfies $(*)$, then fixing some a realizing p, there exists some $v \in F(a)\backslash F$ such that $\delta v \in F$ or $\log\delta v \in F$. Let $r := \mathrm{tp}(v/F)$, and we have that r is $\mathcal{C}$-internal and weakly orthogonal to $\mathcal{C}$. Moreover, the binding group $\mathrm{Aut}_F(r/\mathcal{C})$ is strongly minimal (either $\mathbb{G}_a(\mathcal{C})$ in the case $\delta v \in F$ or $\mathbb{G}_m(\mathcal{C})$ if $\log\delta v \in F$).

Since $v \in F(a)$, there is a natural surjective group homomorphism

$$\pi : \mathrm{Aut}_F(p/\mathcal{C}) \to \mathrm{Aut}_F(r/\mathcal{C}),$$

given as follows: if $\sigma \in \mathrm{Aut}_F(p/\mathcal{C})$ extends to $\hat{\sigma} \in \mathrm{Aut}(\mathcal{U})$, then set $\pi(\sigma) = \hat{\sigma}\restriction_{r(\mathcal{U})}$. That this is well-defined uses the fact that $v \in F(a)$: if $\tau_1, \tau_2 \in \mathrm{Aut}(\mathcal{U})$ both extend σ, then $\tau_1\tau_2^{-1}(a) = a$ so that $\tau_1\tau_2^{-1}(v) = v$ since $v \in \mathrm{dcl}(Fa)$, and hence $\tau_1\restriction_{r(\mathcal{U})} = \tau_2\restriction_{r(\mathcal{U})}$.

Note that π is definable. Indeed, since $v \in F(a)$, there is an F-definable function f such that $f(a) = v$. Fix $\sigma \in \mathrm{Aut}_F(p/\mathcal{C})$. If $v' \models r$, then $f(a') = v'$ for some $a' \models p$, so $\hat{\sigma}(v') = f(\hat{\sigma}(a'))$ for any extension $\hat{\sigma} \in \mathrm{Aut}(\mathcal{U})$ of σ. Hence $\pi(\sigma)(v') = f(\sigma(a'))$. Since this is true for any a' satisfying $f(a') = v'$, and since the actions of $\mathrm{Aut}_F(p/\mathcal{C})$ on $p(\mathcal{U})$ and $\mathrm{Aut}_F(r/\mathcal{C})$ on $r(\mathcal{U})$ are both F-definable, this proves that the homomorphism π defined above is an F-definable homomorphism.

We now look for the kernel of π. By Lemma 2.10, an element of the binding group $\mathrm{Aut}_F(p(\mathcal{U})/\mathcal{C})$ is determined by its action on a finite set of elements in $p(\mathcal{U})$, say $\{a_1, \ldots, a_k\}$. Let $v_1, \ldots, v_k$ be such that $\mathrm{tp}(v_i a_i/F) = \mathrm{tp}(va/F)$. Suppose $\alpha \in \ker(\pi)$, which means any extension $\hat{\alpha}$ of α to $\mathcal{U}$

fixes $r(\mathcal{U})$ pointwise. As $v_i \in F(a_i)\backslash F$, $a_i \in \mathrm{acl}(Fv_i)$, and since α fixes v_i for each i, $\alpha(a_i)$ can only be one of the finitely many conjugates of a_i over $F(v_i)$. This means that the action of α on $a_1, \ldots, a_k$ has only finitely many possibilities, i.e., the kernel of π is finite, so the Morley rank of $\mathrm{Aut}_F(p/\mathcal{C})$ is equal to that of $\mathrm{Aut}_F(r/\mathcal{C})$, which is 1. □

Here is a general context in which the binding group is *not* of Morley rank 1.

Lemma 5.26. *Let*

$$\delta x = ax + b$$

be an inhomogeneous differential equation with D as its set of solutions. Note that D is strongly minimal and $\mathcal{C}$-internal. Let W be defined by

$$\delta x = ax,$$

the corresponding homogeneous differential equation. Let $F = \mathbb{Q}\langle a, b\rangle^{\mathrm{alg}}$ be an algebraically closed δ-field of parameters, and p be the generic type of D over F. If $W(F) = \{0\}$ and $D(F) = \varnothing$, then p is weakly orthogonal to $\mathcal{C}$ and $\mathrm{Aut}_F(p/\mathcal{C})$ is of Morley rank > 1.

Proof. The following is clear: for any $v_1, v_2 \in D$, $v_1 - v_2 \in W$; for any nonzero $w_1, w_2 \in W$, $\dfrac{w_1}{w_2} \in \mathcal{C}$.

First note that W is $\mathcal{C}$-internal and as $W(F) = \{0\}$, Example 2.25 tells us that $\mathrm{Aut}_F(W/\mathcal{C}) = \mathbb{G}_m(\mathcal{C})$ acting by multiplication on W. Also note that since $D(F) = \varnothing$, p is an isolated type and is weakly orthogonal to $\mathcal{C}$.

Claim 1. There is a surjective definable group homomorphism

$$\pi : \mathrm{Aut}_F(p/\mathcal{C}) \longrightarrow \mathbb{G}_m(\mathcal{C}) = \mathrm{Aut}_F(W/\mathcal{C})$$

given by

$$\pi(\beta) = \hat{\beta}\restriction_W,$$

where $\hat{\beta}$ is some extension of β to the universe. This does not depend on the choice of $\hat{\beta}$. Moreover, for any $v_1 \neq v_2 \in D$,

$$\pi(\beta) = \frac{\beta(v_1) - \beta(v_2)}{v_1 - v_2}.$$

Proof of Claim 1. We first prove that π is well-defined, i.e., it does not depend on the choice of $\hat{\beta}$. Suppose β_1, β_2 are two extensions of β to the universe. We fix $v_1 \in D$, and note that for any $w \in W$, $v_1 + w \in D$. Note that

$$\beta_1(w) = \beta_1((v_1 + w) - v_1) = \beta(v_1 + w) - \beta(v_1) = \beta_2((v_1 + w) - v_1) = \beta_2(w),$$

for any $w \in W$, so $\beta_1 \restriction_W = \beta_2 \restriction_W$, which means that π is well-defined.

For any $\gamma \in \mathrm{Aut}_F(W/\mathcal{C})$, $\pi(\hat{\gamma} \restriction_D) = \gamma$, where $\hat{\gamma}$ is any extension of γ to $\mathrm{Aut}_F(\mathcal{U})$. So π is surjective.

Let $\beta_1, \beta_2 \in \mathrm{Aut}_F(p/\mathcal{C})$, and let $\hat{\beta}_1, \hat{\beta}_2$ be any extensions of β_1, β_2 to the universe, respectively. Note that $\hat{\beta}_1 \hat{\beta}_2^{-1}$ is an extension of $\beta_1 \beta_2^{-1}$ to the universe. Then for any $w \in W$,

$$\pi(\beta_1 \beta_2^{-1})(w) = (\hat{\beta}_1 \hat{\beta}_2^{-1})(w) = \hat{\beta}_1(\hat{\beta}_2^{-1}(w)) = \pi(\beta_1)\pi(\beta_2^{-1})(w),$$

so π is a group homomorphism.

Finally, for any $\beta \in \mathrm{Aut}_F(p/\mathcal{C})$, let $\hat{\beta}$ be any extension of β to the universe. Note that

$$\pi(\beta)(v_1 - v_2) = \hat{\beta}(v_1 - v_2) = \beta(v_1) - \beta(v_2),$$

so $\pi(\beta) = \dfrac{\beta(v_1) - \beta(v_2)}{v_1 - v_2}$.

This proves Claim 1.

It remains to prove:

Claim 2. If $\mathrm{Aut}_F(p/\mathcal{C})$ is of Morley rank 1, then $D(F) \neq \varnothing$.

Proof of Claim 2. Let H be the connected component of $\mathrm{Aut}_F(p/\mathcal{C})$. Then H is strongly minimal and F-definable. Since $\pi(\mathrm{Aut}_F(p/\mathcal{C})) = \mathbb{G}_m(\mathcal{C})$ and $\mathbb{G}_m(\mathcal{C})$ is connected, $\pi(H) = \mathbb{G}_m(\mathcal{C})$. Note that for all $\beta \in H$, if $\pi(\beta) \neq 1$, then β fixes a unique $v_\beta \in D$. Indeed, fix $v_1 \in D$. Since $\beta(v) - \beta(v_1) = \pi(\beta)(v - v_1)$, we have that $\beta(v) = v$ has the unique solution $v := \dfrac{\beta(v_1) - \pi(\beta)v_1}{1 - \pi(\beta)}$. Since $\pi\restriction_H$ is surjective, there are infinitely many $\beta \in H$ that satisfy $\pi(\beta) \neq 1$, so there are infinitely many $\beta \in H$ that fixes a unique v_β. Moreover, since H is strongly minimal, all but finitely many $\beta \in H_0$ fix a unique v_β.

Fix $\alpha \in H$ such that $\pi(\alpha) = 2$. Then, as $\alpha^n(v_\alpha) = v_\alpha$, we have $v_{\alpha^n} = v_\alpha$ for all $n > 0$. Note that the α^n's are distinct since $\pi(\alpha^n) = 2^n$. Hence $\{\beta \in H : v_\beta = v_\alpha\}$ is infinite. By strong minimality, there is an $N > 0$ such that

$$|\{\beta \in H : \pi(\beta) = 1 \text{ or } v_\beta \neq v_\alpha\}| \leq N.$$

Let $\varphi(v)$ be the formula

$$(\delta v = av + b) \wedge \exists^{\geq N+1}\beta(\beta \in H \wedge \beta(v) = v),$$

which is over F. Then $\varphi(\mathcal{U}) = \{v_\alpha\}$. So $v_\alpha \in F$. Therefore $D(F) \neq \varnothing$.

This proves Claim 2, and hence the lemma. $\square$

We can now describe our counterexample to $(*)$. Let t be such that $\delta t = 1$. We claim that the generic type of $\delta x = (1 - \dfrac{\sqrt{2}}{t})x + 1$ over $F := \mathbb{Q}(t)^{\mathrm{alg}}$ fails $(*)$. Note that if we set $a := 1 - \dfrac{\sqrt{2}}{t}$ and $b := 1$, then this equation becomes $\delta x = ax + b$ and $F = \mathbb{Q}\langle a, b\rangle^{\mathrm{alg}}$. Hence, by Lemmas 5.25 and 5.26, it suffices to verify that $W(F) = \{0\}$ and $D(F) = \varnothing$, where D is defined by $\delta x = ax + b$ and W is defined by $\delta x = ax$.

We first prove that $W(F) = \{0\}$. Suppose $\alpha \in F$ is a non-zero solution to $\delta x = ax$. Let $\alpha_1 = \alpha, \ldots, \alpha_k$ be conjugates of α over $\mathbb{Q}^{\mathrm{alg}}(t)$, and $\beta = \prod_i \alpha_i$.

Then $\beta \in \mathbb{Q}^{\mathrm{alg}}(t)$ and $\log\delta\beta = \sum_i \log\delta\alpha_i = ka$ (we are using $\alpha \neq 0$ here, so that $\beta \neq 0$). Suppose $\beta = e_0 \prod_j (t - e_j)^{k_j}$ where $e_j \in \mathbb{Q}^{\mathrm{alg}}$ and $k_j \in \mathbb{Z}$. Then $\log\delta\beta = \sum_j \frac{k_j}{t - e_j}$. So $\sum_j \frac{k_j}{t - e_j} = ka = k(1 - \frac{\sqrt{2}}{t})$. That is, t is a solution to the $\mathbb{Q}^{\mathrm{alg}}$-definable equation $\sum_j \frac{k_j}{y - e_j} = k - \frac{k\sqrt{2}}{y}$. Note that $k\sqrt{2}$ is the only parameter that is not rational in the equation, so the equation is nontrivial, but t is transcendental over $\mathbb{Q}^{\mathrm{alg}}$ as $\delta t = 1$. This contradiction proves $W(F) = \{0\}$.

Finally, we prove that $D(F) = \varnothing$. Suppose $\gamma \in F$ is a solution to $\delta x = ax + b$. Let $\gamma_1 = \gamma, \ldots, \gamma_\ell$ be conjugates of γ over $\mathbb{Q}^{\mathrm{alg}}(t)$, and $\epsilon = \sum_i \gamma_i$. Then $\delta\epsilon = a\epsilon + \ell b$, and $\epsilon \in \mathbb{Q}^{\mathrm{alg}}(t)$. Clearly $\epsilon \neq 0$. Suppose $\epsilon = s_0 \prod_j (t - s_j)^{\ell_j}$ where $s_j \in \mathbb{Q}^{\mathrm{alg}}$ and $\ell_j \in \mathbb{Z}$. Then $\log\delta\epsilon = \sum_j \frac{\ell_j}{t - s_j}$. Hence $\sum_j \frac{\ell_j}{t - s_j} = a + \frac{\ell b}{\epsilon} = 1 - \frac{\sqrt{2}}{t} + \frac{\ell}{\epsilon}$. That is, t satisfies

$$\sum_j \frac{\ell_j}{y - s_j} = 1 - \frac{\sqrt{2}}{y} + \ell s_0^{-1} \prod_j (y - s_j)^{-\ell_j}.$$

Note that $\sqrt{2}$ is the only parameter in the equation which is not rational, so the equation is nontrivial, which contradicts the fact that t is transcendental over $\mathbb{Q}^{\mathrm{alg}}$.

5.5 A binding group analysis of condition $(*)$

We wish to analyse the assumption $(*)$ further so as to make precise what remains to be done to prove the conjecture.

We are given an algebraically closed δ-field F and a minimal type $p \in$

$S_1(F)$ that is almost $\mathcal{C}$-internal. By Lemma 2.7, p is algebraic over another minimal type over F that is $\mathcal{C}$-internal. For the sake of simplifying some of the technical and notational complications, let us assume that p itself is $\mathcal{C}$-internal. In this case, we can consider the (F-definable) binding group $G := \mathrm{Aut}_F(p/\mathcal{C})$ together with its F-definable action on the type-definable set $S := p(\mathcal{U})$. Moreover, by Remark 4.9 in Chapter 7 of [27], G is definably isomorphic to a group living in the constants. By the structure of definable group in ACF (see Theorem 5.7 of [17]), we have that G is definably isomorphic to $H(\mathcal{C})$ for some algebraic group H over $\mathcal{C}$.

Note that when p is not weakly orthogonal to $\mathcal{C}$, G is the trivial group. In this case, we do not require condition $(*)$ as the conjecture follows from Theorem 5.6. We assume therefore that p is weakly orthogonal to $\mathcal{C}$ and hence G acts transitively on S (by Lemma 2.15). In particular, S is a definable set, and hence strongly minimal. So (G, S) is a definable homogeneous space.

Fact 5.27 (See Fact 6.25 of Chapter 1 of [27])**.** *Working in a model of any stable theory, suppose (G, S) is an F-definable homogeneous space, where S is strongly minimal. Then one of the following holds:*

1. *G is strongly minimal and the action of G on S is regular;*
2. *The U-rank of the generic type of G over F is 2, and there is an F-definable field structure $(K, +, \cdot)$ on S such that G is precisely the group of transformations $\{x \mapsto ax + b : a, b \in K\}$; or*
3. *The U-rank of the generic type of G over F is 3, S has the structure of $P^1(K)$ for some F-definable field $(K, +, \cdot)$, and G is the group* $\mathrm{PSL}_2(K)$ *of linear fractional transformations $\{x \mapsto \dfrac{ax+b}{cx+d} : a, b, c, d \in K\}$.*

By Lemma 5.25, if p satisfies $(*)$ then G is of Morley rank 1, so that we are in case (1). In fact, the proof of Lemma 5.25, together with Fact 5.27, implies that G is isomorphic to either $\mathbb{G}_m(\mathcal{C})$ or $\mathbb{G}_a(\mathcal{C})$. This actually characterizes condition $(*)$:

Proposition 5.28. *Suppose F is an algebraically closed δ-field, and $p \in S_1(F)$ is $\mathcal{C}$-internal and weakly orthogonal to $\mathcal{C}$. Then p satisfies $(*)$ if and only if $G = \mathrm{Aut}_F(p/\mathcal{C})$ is F-definably isomorphic to either $\mathbb{G}_m(\mathcal{C})$ or $\mathbb{G}_a(\mathcal{C})$.*

Proof. In the proof of Lemma 5.25, assuming $(*)$, we exhibited a surjective F-definable group homomorphism $\pi : G \to \mathbb{G}_0(\mathcal{C})$ with finite kernel, where $\mathbb{G}_0$ is either $\mathbb{G}_m$ or $\mathbb{G}_a$. Since G is definably isomorphic to $H(\mathcal{C})$ for some algebraic group H over $\mathcal{C}$, we get a (field)-definable surjective homomorphism $\beta : H(\mathcal{C}) \to \mathbb{G}_0(\mathcal{C})$ with finite kernel. Note that as a consequence of Fact 5.27, G, and therefore $H(\mathcal{C})$, is connected. This forces β to be an isomorphism if $\mathbb{G}_0 = \mathbb{G}_a$, and the raising to the n-th power map on $\mathbb{G}_m$ when $\mathbb{G}_0 = \mathbb{G}_m$. In the former case we get that π is an F-definable isomorphism between G and $\mathbb{G}_a(\mathcal{C})$. In the latter case we have the commuting diagram of definable group homomorphisms.

$$\begin{tikzcd} G \arrow[r, "\alpha"] \arrow[dr, "\pi"'] & \mathbb{G}_m(\mathcal{C}) \arrow[d, "\beta"] \\ & \mathbb{G}_m(\mathcal{C}) \end{tikzcd}$$

It remains to show that α is F-definable. But if α' is an F-conjugate of α, then

$$\begin{aligned} \alpha'/\alpha : G &\to \ker(\beta) \\ x &\mapsto \frac{\alpha'(x)}{\alpha(x)} \end{aligned}$$

is a definable group homomorphism by the commutative diagram and the F-definability of β and π. By connectedness of G and finiteness of $\ker(\beta)$, we must have $\alpha = \alpha'$. That is, α is F-definable.

For the converse let us fix $a \models p$. We claim first of all that the differential field $F\langle a\rangle$ admits infinitely many automorphisms fixing F pointwise. First, note that as $S := p(\mathcal{U})$ is acted upon transitively and F-definably by a definable group in the constants, and since the induced structure on

$\mathcal{C}$ eliminates imaginaries, we have an Fa-definable embedding $f : S \to \mathcal{C}^n$ for some n. In particular, $S \cap \mathrm{acl}(Fa)$ is infinite — this is because in $\mathcal{C}^n$ every infinite definable set has infinitely many points in an algebraically closed parameter set. But if $b \in S \cap \mathrm{acl}(Fa)$ then each coordinate of $f(b)$ is in $\mathrm{acl}(Fa) \cap \mathcal{C} = (F\langle a \rangle \cap \mathcal{C})^{\mathrm{alg}} = (F \cap \mathcal{C})^{\mathrm{alg}}$ by Lemma 2.22. Note that $(F \cap \mathcal{C})^{\mathrm{alg}} = F \cap \mathcal{C}$ as F is algebraically closed. So each coordinate of $f(b)$ is in F, and hence $b \in F\langle a \rangle$. That is, $S \cap F\langle a \rangle$ is infinite. Now, every element $b \in S \cap F\langle a \rangle$ induces an automorphism $\alpha_b \in \mathrm{Aut}_F(\mathcal{U})$ such that $\alpha_b(a) = b$. Since $b \in F\langle a \rangle = F(a, \delta a) \subseteq F(a)^{\mathrm{alg}}$, we get $a \in F(b)^{\mathrm{alg}}$, so $a \in \mathrm{acl}(Fb)$, and hence (by symmetry, and following the same proof as above) $a \in F\langle b \rangle$. Therefore $F\langle a \rangle = F\langle b \rangle$. The restrictions $\alpha_b \restriction_{F\langle a \rangle}$ thus give us infinitely many differential automorphisms of $F\langle a \rangle$ fixing F.

Now, because $F\langle a \rangle$ has infinitely many differential automorphisms fixing F pointwise, a theorem of Matsuda (see the main theorem of [21]) tells us that there exist v such that $F\langle v \rangle = F\langle a \rangle$ and one of the following folds:

(i) $\delta v \in F$,

(ii) $\log \delta v \in F$, or

(iii) $(\delta v)^2 = cv(v^2 - 1)(v - e)$, where $c \in F$, $e \in \mathcal{C}_F$, $c \neq 0$, and $e \neq -1, 0, 1$.

If case (i) or (ii) holds then p satisfies $(*)$. It remains, therefore, to rule out case (iii). Indeed, if case (iii) holds, then $r = \dfrac{1}{v} - \dfrac{1}{3e}$ satisfies $(\delta r)^2 = \dfrac{ce}{4}(4r^3 - g_1 r - g_2)$ for some $g_1, g_2 \in \mathcal{C}_F$, and $4r^3 - g_1 r - g_2$ has 3 distinct roots in $\mathcal{C}_F$. We know from Section 6 of [13], where equations of this form are studied, that the the group of differential automorphisms of $F\langle r \rangle$ that fixes F is isomorphic to the $\mathcal{C}_F$-points of an elliptic curve over $\mathcal{C}_F$, and in particular, if d is a root of $4r^3 - g_1 r - g_2$, then there exists a differential automorphism α_d of $F\langle r \rangle$ of order 2 that fixes F given by

$$\alpha_d(r) = -d - r + \frac{4r^3 - g_1 r - g_2}{4(r - d)^2}.$$

We extend α_d to $\hat{\alpha}_d \in \mathrm{Aut}_F(\mathcal{U}/\mathcal{C})$. Since $S \subseteq \mathrm{dcl}(F\mathcal{C}a)$, and a and r are interdefinable over F, $\hat{\alpha}_d\restriction_S \in \mathrm{Aut}_F(p/\mathcal{C})$ is uniquely determined by α_d (indeed, by its action on r) and is of order 2. Since $4r^3 - g_1 r - g_2$ has three distinct roots, there are three different elements in $\mathrm{Aut}_F(p/\mathcal{C})$ of order 2. This means that $\mathrm{Aut}_F(p/\mathcal{C})$ is not isomorphic to either $\mathbb{G}_m(\mathcal{C})$ or $\mathbb{G}_a(\mathcal{C})$. $\square$

The counterexample to $(*)$ produced in Section 5.4 yields a binding group of U-rank 2. Anand Pillay has suggested to us ways of producing examples in DCF_0 where the binding group is isomorphic to $\mathrm{GL}_2(\mathcal{C})/\mathbb{G}_m(\mathcal{C})$, which is of U-rank 3.

So, in the wake of Fact 5.27, to complete the proof of Conjecture 5.4, it remains to consider the following cases: G is definably isomorphic to $H(\mathcal{C})$ where H is an elliptic curve over the constants, or G is of U-rank 2 or 3. This is something I am actively pursuing but at the time of the writing of this thesis I have not yet obtained a complete proof.

5.6 Pullbacks under the derivative map

Instead of considering $\log\delta^{-1}(D)$, it is natural to ask when $\delta^{-1}(D)$ is almost $\mathcal{C}$-internal for $D \subseteq \mathcal{U}$ strongly minimal and almost $\mathcal{C}$-internal. Note that as in the $\log\delta^{-1}(D)$ case, $\delta^{-1}(D)$ is $\mathcal{C}$-analysable in at most 2 steps. As the following example shows, however, $\delta^{-1}(D)$ can be $\mathcal{C}$-internal without decomposing into a product of $\mathcal{C}$-internal sets. That is, the analogue of Conjecture 5.4 fails.

Example 5.29. Let s be a differentially transcendental element over $\mathbb{Q}^{\mathrm{alg}}$. Let $F = \mathbb{Q}(s)^{\mathrm{alg}}$ and D be the solution set of $\delta x = s$. Then $E := \delta^{-1}(D)$ is $\mathcal{C}$-internal as it is defined by the inhomogeneous linear differential equation $\delta^2 x = s$. However, there do not exist almost $\mathcal{C}$-internal minimal types q_1 and q_2 over F, and an F-definable finite-to-one surjective map from the generic type of E to $q_1 \otimes q_2$.

Proof. Let p be the generic type of D over F and q the generic type of E over F.

We first note that q is isolated by the formula $\delta^2 x = s$. That is, $q(\mathcal{U}) = E$. Note that if $a \in D$, then a is differentially transcendental over $\mathbb{Q}^{\mathrm{alg}}$ since $\delta a = s$ is. So $a \notin F = \mathbb{Q}\langle \delta a \rangle^{\mathrm{alg}}$ and hence $a \models p$ since D is strongly minimal. This means that p is isolated by the formula $\delta x = s$. Now let u be such that $\delta^2 u = s$. We have just seen that δu is generic in D over F. The same argument shows that u is generic in $\delta^{-1}(\delta u)$ over $F\langle \delta u \rangle$, so u is generic in E over F and $u \models q$. Thus q is isolated by the formula $\delta^2 x = s$.

Next, we compute the binding group $G = \mathrm{Aut}_F(E/\mathcal{C})$. Fix $u_0 \in E$ and t such that $\delta t = 1$ Then every $u \in E$ is of the form $u = d_1 + d_2 t + u_0$ for some $d_1, d_2 \in \mathcal{C}$, and vice versa. We can therefore identify E, definably over $F\langle u_0, t \rangle$, with the set of column vectors $\left\{ \begin{bmatrix} d_1 \\ d_2 \\ 1 \end{bmatrix} : d_1, d_2 \in \mathcal{C} \right\}$. Now, let $U \leq \mathrm{GL}_3(\mathcal{C})$ be the unipotent subgroup of upper triangular matrices of the form $\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$. It acts on E in the natural way:

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ 1 \end{bmatrix} = \begin{bmatrix} d_1 + a d_2 + b \\ d_2 + c \\ 1 \end{bmatrix}.$$

We will show that this is isomorphic to the action of G on E.

Let $g \in G$. Since u_0 and $u_0 + t$ are both in E, so is $g(u_0)$ and $g(u_0 + t)$. Write

$$g(u_0) = \alpha + \beta t + u_0$$

$$g(u_0 + t) = \alpha' + \beta' t + u_0,$$

for some $\alpha, \alpha\,\beta, \beta' \in \mathcal{C}$. Letting $\bar{g} \in \mathrm{Aut}_F(\mathcal{U})$ be any extension of g, we have

$$\bar{g}(t) = g(u_0 + t) - g(u_0) = \alpha' - \alpha + (\beta' - \beta)t.$$

On the other hand, $\delta\bar{g}(t) = \bar{g}(\delta t) = \bar{g}(1) = 1$, so $\beta' - \beta = 1$, and $\bar{g}(t) = \alpha' - \alpha + t$. Hence, for an arbitrary $u = d_1 + d_2 t + u_0 \in E$,

$$\begin{aligned} g(u) &= d_1 + d_2\bar{g}(t) + g(u_0) \\ &= d_1 + d_2(\alpha' - \alpha) + d_2 t + \alpha + \beta t + u_0 \\ &= (d_1 + (\alpha' - \alpha)d_2 + \alpha) + (d_2 + \beta)t + u_0 \end{aligned}$$

That is, g acts on E exactly as $\begin{bmatrix} 1 & \alpha' - \alpha & \alpha \\ 0 & 1 & \beta \\ 0 & 0 & 1 \end{bmatrix} \in U$. Since g is determined by its action on E, this gives us an embedding of G in U.

To see that this embedding is surjective, we need only show that for any $(a, b, c) \in \mathcal{C}^3$, the following map

$$\begin{aligned} g : F \cup \mathcal{C} \cup E &\to F \cup \mathcal{C} \cup E \\ u = d_1 + d_2 t + u_0 &\mapsto (d_1 + ad_2 + b) + (d_2 + c)t + u_0 \\ v \in F \cup \mathcal{C} &\mapsto v \end{aligned}$$

is a partial elementary map, as the image of g under the embedding would be $\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$. Indeed, we only need to show that

$$\begin{aligned} h : F \cup \mathcal{C} \cup \{u_0, t\} &\to F \cup \mathcal{C} \cup E \\ u_0 &\mapsto b + ct + u_0 \\ t &\mapsto a + t \\ v \in F \cup \mathcal{C} &\mapsto v \end{aligned}$$

is a partial elementary map, as it is clear that h has a unique partial elementary map extension to $\mathrm{Dom}(h) \cup E$, which we would call $\hat{h}$, and $\hat{h}\restriction_E = g\restriction_E$. In order to show this, we need to prove that $\mathrm{tp}(u_0, t/F\mathcal{C}) = \mathrm{tp}(h(u_0), h(t)/F\mathcal{C})$. Since $u_0, h(u_0) \in E$, u_0 and $h(u_0)$ both realizes $q \in S_1(F)$, and since q is weakly orthogonal to $\mathcal{C}$, there is a unique extension of q to $F\mathcal{C}$, to which u_0 and $h(u_0)$ are both realizations. Therefore $\mathrm{tp}(u_0/F\mathcal{C}) = \mathrm{tp}(h(u_0)/F\mathcal{C})$. To show that $\mathrm{tp}(u_0, t/F\mathcal{C}) = \mathrm{tp}(h(u_0), h(t)/F\mathcal{C})$, it suffices to show that $\mathrm{tp}(t/u_0F\mathcal{C})$ is isolated by the formula $\delta t = 1$. As this formula is strongly minimal, it suffices to show that there are no solutions in $\mathrm{acl}(u_0F\mathcal{C})$. Suppose, towards a contradiction, that there exists some $t_0 \in \mathrm{acl}(u_0F\mathcal{C})$ such that $\delta t_0 = 1$. Note that $\mathrm{acl}(u_0F\mathcal{C}) = \mathcal{C}\langle u_0\rangle^{\mathrm{alg}}$ as $F = \mathbb{Q}\langle s\rangle^{\mathrm{alg}} \subseteq \mathcal{C}\langle u_0\rangle^{\mathrm{alg}}$. Let $t_0 = v_1, v_2, \ldots, v_m$ be conjugates of t_0 over $\mathcal{C}\langle u_0\rangle$. Then $v := \sum_i v_i \in \mathcal{C}\langle u_0\rangle$ satisfies $\delta v = m \in \mathbb{Z}^+$. Suppose $v = \eta(u_0)$ where $\eta \in \mathcal{C}\langle x\rangle$. Then u_0 satisfies $\delta(\eta(x)) = m$. Moreover, since $\eta \in \mathcal{C}\langle x\rangle$, $\eta(0) \in \mathcal{C}$, so $\delta(\eta(0)) \neq m$. Hence $\delta(\eta(x)) = m$ is a nontrivial differential equation. This implies that u_0 is differentially algebraic over $\mathcal{C}$, which is differentially algebraic over $\mathbb{Q}^{\mathrm{alg}}$. By the Corollary in Section II.8 of [14], u_0 is differentially algebraic over $\mathbb{Q}^{\mathrm{alg}}$; however, we already know that u_0 is differentially transcendental over $\mathbb{Q}^{\mathrm{alg}}$, a contradiction. Therefore, the embedding of G in U is surjective. We now identify G with U.

We have now computed the binding group of E (equivalently, the binding group of q) to be $U = U(3, \mathcal{C})$. If the analogue of Conjecture 5.4 holds for E, then there would be an F-definable finite-to-one surjective function $f : E \to E_1 \times E_2$ where E_1, E_2 are strongly minimal $\mathcal{C}$-internal definable sets. This induces surjective F-definable group homomorphisms:

$$\pi_1 : U = \mathrm{Aut}_F(E/\mathcal{C}) \to \mathrm{Aut}_F(E_1/\mathcal{C}),$$

$$\pi_2 : U = \mathrm{Aut}_F(E/\mathcal{C}) \to \mathrm{Aut}_F(E_2/\mathcal{C}).$$

Let $N_1 = \ker(\pi_1)$ and $N_2 = \ker(\pi_2)$. If N_i is trivial, then $\mathrm{Aut}_F(E_i/\mathcal{C}) \cong U$.

But by Fact 5.27, since the rank of U is 3, this would force $U \cong \mathrm{PSL}_2(\mathcal{C})$, which is not the case as there are non-identity torsion elements in $\mathrm{PSL}_2(\mathcal{C})$, but U is torsion-free. Hence N_1 and N_2 are nontrivial normal algebraic subgroups of U. Since U is nilpotent, any nontrivial normal subgroup intersect $Z(U)$ nontrivially (see Proposition 5.2.1 of [30]). In addition, as $Z(U) = \left\{ \begin{bmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} : b \in \mathcal{C} \right\}$ is strongly minimal and has no finite subgroups, we get that $Z(U) \subseteq N_i$ for each $i = 1, 2$. Let $g \in N_1 \cap N_2$ be such that $g \neq \mathrm{id}$. Then g fixes $E_1 \times E_2$ pointwise and hence preserves the fibration induced by $f : E \to E_1 \times E_2$. As the fibres of f are finite and uniformly bounded, there exists $\ell > 0$ such that $g^\ell = \mathrm{id}$. This is a contradiction as U is torsion-free. Hence no such $f : E \to E_1 \times E_2$ exists. $\square$

5.7 Appendix

This section contains a general algebraic lemma that was used in the proof of Proposition 5.16.

Lemma 5.30. *Let F_1, F_2 be two independent field extensions of an algebraically closed field F. Let $f_1(\bar{x}, \bar{y}_1), \ldots, f_k(\bar{x}, \bar{y}_k)$ be rational functions with parameters in F. Suppose $\prod_i f_i(\bar{\alpha}, \bar{\beta}_i) \in F_2 \setminus \{0\}$ for $\bar{\alpha} \in F_1$ and $\bar{\beta}_i \in F_2$. Then there exist a positive integer n and rational functions $g_1(\bar{y}_1), \ldots, g_k(\bar{y}_k)$ over F such that $\prod_i f_i(\bar{\alpha}, \bar{\beta}_i)^n g_i(\bar{\beta}_i) = 1$.*

Proof. First we drop the assumption that F is algebraically closed. We only assume the following (a consequence of F being algebraically closed and F_1, F_2 being independent over F): if L/K is a finite field extension where $F \subseteq K \subseteq L \subseteq F_1$, then $[L : K] = [LF_2 : KF_2]$.

If $\bar{x}$ is a 0-ary tuple (i.e., $f_i \in F$ does not depend on x), then let $n = 1$ and $g_i(y_i) = f_i(y_i)^{-1}$ and we are done.

Suppose $\bar{x}$ is a singleton.

If α is algebraic over F, assume $p(x) = x^\ell + a_{\ell-1}x^{\ell-1} + \cdots + a_0$ is the minimal polynomial of α over F (and over F_2). Then $f_i(\alpha, \bar{\beta}_i)$ is of the form $\sum_{k=0}^{\ell-1} h_{ik}(\bar{\beta}_i)\alpha^k$ where h_{ik} are rational functions over F. Let d_i be the determinant of the linear transformation on $F_2(\alpha)$ over F_2 defined by $x \mapsto f_i(\alpha, \bar{\beta}_i)x$. If we use the basis $\{1, \alpha, \ldots, \alpha^{\ell-1}\}$, it is easy to see that $d_i \in F(\bar{\beta}_i)$. Now let $m = \ell$ and $h_i(y_i)$ be such that $h_i(\bar{\beta}_i) = d_i^{-1}$, and we have that the determinant of the linear transformation $x \mapsto f_i(\alpha, \beta_i)^m h_i(\bar{\beta}_i)x$ is $d_i^m h_i(\bar{\beta}_i)^m = d_i^m d_i^{-m} = 1$. Let $c := \prod_i f_i(\alpha, \bar{\beta}_i)^m h_i(\bar{\beta}_i)$. Note that $c \in F_2$, and since we know the determinant of $x \mapsto cx$ is 1, we get that $c^\ell = 1$. Now let $n = \ell^2$ and $g_i(\bar{\beta}) = h_i(\bar{\beta})^\ell$, and we get that $\prod_i f_i(\alpha, \bar{\beta}_i)^n g_i(\bar{\beta}_i) = c^\ell = 1$.

If α is transcendental over F (therefore over F_2), then

$$f_i(\alpha, \bar{\beta}_i) = \frac{\sum_{k=0}^{\ell_i} s_{ik}(\bar{\beta}_i)\alpha^k}{\sum_{k=0}^{m_i} t_{ik}(\bar{\beta}_i)\alpha^k},$$

where s_{ik} and t_{tk} are rational functions over F. So if we let $n = 1$ and $g_i(\bar{y}_i) = \dfrac{t_{im_i}(\bar{y}_i)}{s_{i\ell_i}(\bar{y}_i)}$, then $\prod_i f_i(\bar{\alpha}, \bar{\beta}_i)^n g_i(\bar{\beta}_i)$ is of the form

$$\frac{\alpha^\ell + s_{\ell-1}\alpha^{\ell-1} + \cdots + s_0}{\alpha^m + t_{m-1}\alpha^{m-1} + \cdots + t_0},$$

where $s_i, t_i \in F_2$ and ℓ, m are nonnegative integers. Since $\prod_i f_i(\bar{\alpha}, \bar{\beta}_i)^n g_i(\bar{\beta}_i)$ is in F_2, we get that in fact $\prod_i f_i(\bar{\alpha}, \bar{\beta}_i)^n g_i(\bar{\beta}_i) = 1$

Now, suppose the result holds for $(k-1)$-ary tuples. Let $\bar{\alpha}$ be a k-ary tuple. Apply the above result to $F(\bar{\alpha})$, $F_2(\alpha_1, \ldots, \alpha_{k-1})$, two field extensions of $F(\alpha_1, \ldots, \alpha_{k-1})$ that satisfy our assumption, and we get that there exists n and $g_i(x_1, \ldots, x_{k-1}, \bar{y}_i)$ such that $\prod_i f(\bar{\alpha}, \bar{\beta}_1)^n g_i(\alpha_1, \ldots, \alpha_{k-1}, \bar{\beta}_i) = 1$. Also, since $\prod_i g_i(\alpha_1, \ldots, \alpha_{k-1}, \bar{y}_i) = \prod_i f(\bar{\alpha}, \bar{\beta}_1)^{-n} \in F_2$, we apply the result again to rational functions $g_i(x_1, \ldots, x_{k-1}, \bar{y}_i)$ over F, and $F(\alpha_1, \ldots, \alpha_{k-1})$ and F_2 which are field extensions of F that satisfy the assumption, and get that there exists m and $h_i(\bar{y}_i)$ such that $\prod_i g_i(\alpha_1, \ldots, \alpha_{k-1}, \bar{y}_i)^m h_i(\bar{\beta}_i) = 1$. Thus $\prod_i f(\bar{\alpha}, \bar{\beta}_i)^{mn} h_i(\bar{\beta}_i)^{-1} = 1$ and we get that the conclusion is true for k-ary tuples. $\square$

6 Two commissioned examples

In this final chapter we work out, using techniques similar to those appearing elsewhere in this thesis, two specific examples of $\mathcal{C}$-internality and $\mathcal{C}$-analysability in DCF_0. The results here were asked for and have been cited already by other authors in published work.

6.1 A twisted D-group

The first example plays a crucial role in the study of so-called "twisted D-groups", see §3 of [3]. In particular, the following is cited in Example 3.4 of that paper.

Fix $c \in \mathcal{C}$ and let $F := \mathbb{Q}(c)^{\mathrm{alg}}$. Consider the following system of differential equations:

$$\begin{cases} \delta x = xy, \\ \delta y = \dfrac{y^2}{2} + c(1 - x^2). \end{cases} \tag{6.1}$$

Let (a, b) be a generic solution over F. We show that $\mathrm{tp}(a, b/F)$ is $\mathcal{C}$-internal.

If $c = 0$, then

$$\begin{aligned} \delta^3\left(\frac{1}{a}\right) &= \delta^2\left(-\frac{\delta a}{a^2}\right) \\ &= \delta^2\left(-\frac{ab}{a^2}\right) \\ &= \delta^2\left(-\frac{b}{a}\right) \\ &= \delta\left(-\frac{a\delta b - b\delta a}{a^2}\right) \\ &= \delta\left(-\frac{\frac{1}{2}ab^2 - ab^2}{a^2}\right) \\ &= \delta\left(\frac{b^2}{2a}\right) \end{aligned}$$

$$
\begin{aligned}
&= \frac{2a\delta(b^2) - b^2\delta(2a)}{2a} \\
&= \frac{4ab\delta b - 2ab^3}{2a} \\
&= \frac{2ab^3 - 2ab^3}{2a} \\
&= 0.
\end{aligned}
$$

So $\mathrm{tp}(a/F)$ is $\mathcal{C}$-internal, and $\mathrm{tp}(a, b/F)$ is $\mathcal{C}$-internal as $b = \dfrac{\delta a}{a} \in \mathrm{dcl}(a)$.

If $c \neq 0$, let e be one of the square roots of $-2c \in F$, $f = ea + b$, and $g = \dfrac{f-e}{f+e}$. We have

$$
\begin{aligned}
\mathrm{log}\delta g &= \mathrm{log}\delta \frac{f-e}{f+e} \\
&= \mathrm{log}\delta \frac{ea+b-e}{ea+b+e} \\
&= \mathrm{log}\delta(ea+b-e) - \mathrm{log}\delta(ea+b+e) \\
&= \frac{\delta(ea+b-e)}{ea+b-e} - \frac{\delta(ea+b+e)}{ea+b+e} \\
&= \frac{e\delta a + \delta b}{ea+b-e} - \frac{e\delta a + \delta b}{ea+b+e} \\
&= \frac{2e}{(ea+b)^2 - e^2}(e\delta a + \delta b) \\
&= \frac{2e}{(ea+b)^2 - e^2}(eab + \frac{1}{2}b^2 + c - ca^2) \\
&= \frac{2e}{(ea+b)^2 - e^2}(eab + \frac{1}{2}b^2 + c + \frac{1}{2}e^2a^2) \\
&= \frac{2e}{(ea+b)^2 - e^2}(\frac{1}{2}(ea+b)^2 - \frac{1}{2}e^2) \\
&= e
\end{aligned}
$$

This shows that $\mathrm{tp}(g/F)$ is $\mathcal{C}$-internal. From $g = \dfrac{f-e}{f+e}$ we get $f = \dfrac{2e}{1-g} - e$, so $\mathrm{tp}(f/F)$ is $\mathcal{C}$-internal.

We now show that $\mathrm{tp}(a/F)$ and $\mathrm{tp}(b/F)$ are $\mathcal{C}$-internal. Note that

$$\begin{aligned}
\delta\left(\frac{g}{a(1-g)^2}\right) &= \frac{\delta g a(1-g)^2 - \delta a(1-g)^2 g - 2a(1-g)(-\delta g)g}{a^2(1-g)^4} \\
&= \frac{ega(1-g)^2 - a(f-ea)(1-g)^2 g + 2a(1-g)eg^2}{a^2(1-g)^4} \\
&= \frac{eg(1-g) - (f-ea)(1-g)g + 2eg^2}{a(1-g)^3} \\
&= \frac{eg(1+g) - (f-ea)(1-g)g}{a(1-g)^3} \\
&= \frac{eg(1+g) - (\frac{2e}{1-g} - e - ea)(1-g)g}{a(1-g)^3} \\
&= \frac{eg(1+g) - (2e - (e+ea)(1-g))g}{a(1-g)^3} \\
&= \frac{e(g + g^2 - 2g + (1+a)(1-g)g)}{a(1-g)^3} \\
&= \frac{e(-g(1-g) + (g+ag)(1-g))}{a(1-g)^3} \\
&= \frac{eag}{a(1-g)^2} \\
&= \frac{eg}{(1-g)^2}.
\end{aligned}$$

So a is a solution to

$$\delta\left(\frac{g}{x(1-g)^2}\right) = \frac{eg}{(1-g)^2}.$$

We also have

$$\begin{aligned}
\delta\left(\frac{1}{1-g}\right) &= \frac{\delta g}{(1-g)^2} \\
&= \frac{eg}{(1-g)^2}
\end{aligned}$$

since $\log\delta g = e$. So $\dfrac{g}{a(1-g)^2} = \dfrac{1}{1-g} + D$ for some $D \in \mathcal{C}$, i.e., $a =$

$\dfrac{g}{(1-g)(1+D-Dg)}$ for some $D \in \mathcal{C}$. Since $\mathrm{tp}(D/F)$ and $\mathrm{tp}(g/F)$ are both $\mathcal{C}$-internal, $\mathrm{tp}(a/F)$ is $\mathcal{C}$-internal. In addition, $b = f - ea$, $\mathrm{tp}(f/F)$ is $\mathcal{C}$-internal, and $e \in F$, so $\mathrm{tp}(b/F)$ is $\mathcal{C}$-internal also.

Hence, $\mathrm{tp}(ab/F)$ is $\mathcal{C}$-internal, as desired.

6.2 A two-step $\mathcal{C}$-analysis with independent fibres

In [9], an example was asked for in DCF_0 of a two-step "analysable cover" of the constants whose fibres were "independent". We can rephrase this more concretely in our language as follows:

Working over an algebraically closed differential field $F \subset \mathcal{U}$, we seek a definable set S and a surjective F-definable function $\pi : S \to A$ such that

(1) $A \subseteq \mathcal{C}^\ell$ for some $\ell > 0$,

(2) each fibre S_a is $\mathcal{C}$-internal, for all $a \in A$,

(3) S in not almost $\mathcal{C}$-internal, and

(4) Given $n > 0$, distinct $a_1, \ldots, a_n \in A$, and $u_i, v_i \in S_{a_i}$ for each $i = 1, 2, \ldots, n$ with $\mathrm{tp}(u_i/F\mathcal{C}) = \mathrm{tp}(v_i/F\mathcal{C})$, we have $\mathrm{tp}(u_1 \cdots u_n/F\mathcal{C}) = \mathrm{tp}(v_1 \cdots v_n/F\mathcal{C})$.

We give such an example.

Fix $t \in \mathcal{U}$ such that $\delta t = 1$ and let $F = \mathbb{Q}(t)^{\mathrm{alg}}$. Consider the F-definable set

$$S := \{x \in \mathcal{U} \backslash \{0\} : \mathrm{log}\delta x = \frac{1}{(t+c)^2} \text{ for some } c \in \mathcal{C}\},$$

and the F-definable function $\pi : S \to \mathcal{C}$ given by $\pi(u) = \delta\left(\frac{u}{2\delta u}\right) - t$. Note that

$$\delta\pi(u) = \delta^2\left(\frac{(t+c)^2}{2}\right) - 1 = 0,$$

so that π does indeed map S to $\mathcal{C}$. For surjectivity, given $c \in \mathcal{C}$, let $u \in \mathrm{log}\delta^{-1}\left(\frac{1}{(t+c)^2}\right)$ and you will see that $\pi(u) = c$.

Condition (1) is satisfied as in this case $A = \mathcal{C}$.

For condition (2), note that for $c \in \mathcal{C}$ the fibre is given by $S_c = \{x \in \mathcal{U}\backslash\{0\} : \text{log}\delta x = \dfrac{1}{(t+c)^2}\}$, which being a translate of $\mathbb{G}_m(\mathcal{C})$ is $\mathcal{C}$-internal.

The fact that S is not almost $\mathcal{C}$-internal is shown in Example 5.22.

Finally, we need to show condition (4), the independence of the fibres. Note that by induction and an automorphism argument it suffices to consider the case when $n > 1$ and $v_i = u_i$ for $i = 2, 3, \ldots, n$. We need to show that $\text{tp}(u_1/F\mathcal{C}u_2 \cdots u_n) = \text{tp}(v_1/F\mathcal{C}u_2 \cdots u_n)$. Since the fibres are strongly minimal, this will follow if $u_1, v_1 \notin \text{acl}(F\mathcal{C}u_2 \cdots u_n)$. That is, we need to prove: *Given $u_1, \ldots, u_n \in S$ with $c_i := \pi(u_i)$ distinct for $i = 1, 2, \ldots, n$, we must have $u_1 \notin \text{acl}(F\mathcal{C}u_2 \ldots u_n)$.* This is what we now prove.

First, notice the fact that $\delta u_i = u_i \text{log}\delta u_i = \dfrac{u_i}{(t+c)^2} \in C(t, u_i)$ for $i = 2, \ldots, n$, so we have $\mathcal{C}(t)\langle u_2, \ldots, u_n\rangle = \mathcal{C}(t, u_2, \ldots, u_n)$. As a result, we have $\text{acl}(F\mathcal{C}u_2 \cdots u_n) = \mathcal{C}(t, u_2, \ldots, u_n)^{\text{alg}}$.

Suppose, for a contradiction, that $u_1 \notin \mathcal{C}(t, u_2, \ldots, u_n)^{\text{alg}}$. That is, there exists a nonzero $f_0 \in C(t, u_2, \ldots, u_n)[x_1]$ such that $f_0(u_1) = 0$. We may rewrite $f_0(x_1) = 0$ as $f_1(x_1, u_2, \ldots, u_n) = 0$ where $f_1 \in C(t)[x_1, x_2, \ldots, x_n]$. Note that $f_1 \neq 0$ and $f_1(u_1, \ldots, u_n) = 0$.

Suppose $f \in C(t)[x_1, x_2, \ldots, x_n]$ is a polynomial over $C(t)$ with minimal number of terms such that $f \neq 0$ and $f(u_1, \ldots, u_n) = 0$. Such f exists because of the existence of f_1. Let

$$f(\bar{x}) = \sum_{\bar{k}\in I} g_{\bar{k}}\bar{x}^{\bar{k}}$$

where I is a finite set of non-negative integer n-tuples, and $g_{\bar{k}} \in C(t)$ nonzero for $\bar{k} \in I$. As $u_i \neq 0$ for all i (since $\text{log}\delta u_i$ is well-defined), f has at least two terms.

Applying the same argument as in the proof of Lemma 5.8, we get that there exists $g \in \mathcal{C}(t)$ and a nonzero n-tuple $\bar{k}$ such that $g\bar{u}^{\bar{k}} = 1$. Thus $\text{log}\delta\bar{u}^{\bar{k}} = -\text{log}\delta g$.

Note that $\log\delta\bar{u}^{\bar{k}} = \sum_i k_i \log\delta u_i = \sum_i \frac{k_i}{(t+c_i)^2}$. Now suppose $g = e_0 \prod_j (t-e_j)^{\ell_j}$ where $e_i \in \mathcal{C}$ and $\ell_j \in \mathbb{Z}$. Then $\log\delta g = \sum \frac{\ell_j}{t-e_j}$. Since $\log\delta\bar{u}^{\bar{k}} = -\log\delta g$, it is clear from the transcendence of t over $\mathcal{C}$ that the only possibility is $\log\delta g = \log\delta\bar{u}^{\bar{k}} = 0$, and specifically, $k_i = 0$ for all i and $\ell_j = 0$ for all j. This contradicts the fact that $\bar{k}$ is nonzero.

We therefore have that $u_1 \notin \operatorname{acl}(\mathcal{C}tu_2 \cdots u_n)$.

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Glossary

algebraic closure (field-theoretic) The field-theoretic algebraic closure of a set is the smallest algebraically closed field that contains the set.

algebraic closure (model-theoretic) The model-theoretic algebraic closure of a set A, denoted $\mathrm{acl}(A)$, is the set of all elements that realize a formula over A with only finitely many realizations. Elements in $\mathrm{acl}(A)$ are said to be algebraic over A.

algebraic type An algebraic type is the type of an algebraic element.

definable closure The definable closure of a set A, denoted $\mathrm{dcl}(A)$, is the set of all element that realize a formula over A with only one realization. Elements in $\mathrm{dcl}(A)$ are said to be definable over A.

elimination of imaginaries A theory admits elimination of imaginaries if, given any definable set X, any equivalence relation E on X, and any equivalence class a/E, there exists a tuple b such that a/E and b are interdefinable.

forking Suppose $B \subseteq C$. Then $\mathrm{tp}(a/C)$ does not fork over B if, intuitively, $\mathrm{tp}(a/C)$ does not have significantly fewer realizations. In ACF_0, for $B \subseteq C$ algebraically closed fields, $\mathrm{tp}(a/C)$ does not fork over B if the Zariski locus of a over C is equal to the Zariski locus of a over B. In DCF_0, for $B \subseteq C$ algebraically closed differential fields, $\mathrm{tp}(a/C)$ does not fork over B if the Kolchin locus of a over C is equal to the Kolchin locus of a over B.

minimal type A type is minimal if it has a unique non-algebraic extension to any set of parameter.

non-forking extension Let q be a type over C and p its restriction to $B \subseteq C$. We say that q is a non-forking extension of p if for some (equivalently any) realization a of q, we have $a \mathop{\smile\!\!\!\!\!\!|}_B C$.

saturated model A model $\mathcal{M}$ is saturated if it realizes all types whose parameter set is of size $< |\mathcal{M}|$.

stable theory A theory is κ-stable for some infinite cardinal κ if for all set A of size κ, the number of types over A is also κ. A theory is stable if it is κ-stable for some κ.

stationary type A type is stationary if it has a unique non-forking extension to any parameter set. In particular, minimal types and types over algebraically closed sets are stationary.

strong type A strong type over a set A is a type over $\mathrm{acl}(A)$.

strongly minimal set A definable set is strongly minimal if its definable subsets are either finite or cofinite.

U**-rank** U-rank is the foundation rank of forking extension. More specifically, algebraic types are of U-rank 0, and a type is of U-rank $\geq \alpha + 1$ for some ordinal $\alpha + 1$ if it has a forking extension whose U-rank is at least α.