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The degree of nonminimality is at most 2

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In this paper, it is shown that if $p \in S(A)$ is a complete type of Lascar rank at least 2, in the theory of differentially closed fields of characteristic zero, then there exists a pair of realisations a_1, a_2 such that p has a nonalgebraic forking extension over Aa_1a_2. Moreover, if A is contained in the field of constants then p already has a nonalgebraic forking extension over Aa_1. The results are also formulated in a more general setting.

Keywords: Geometric stability theory; differentially closed fields; compact complex manifolds; degree of nonminimality.

Mathematics Subject Classification: 03C45, 12H05

1. Introduction

In [4], motivated by the search for general techniques that might aid in proving strong minimality for certain algebraic differential equations, the first and third authors introduced *degree of nonminimality* as a measure of how many parameters

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are needed to witness that a type is *not* minimal. Working in a sufficiently saturated model of a stable theory eliminating imaginaries, here is a precise formulation.

Definition 1.1. Suppose $p \in S(A)$ is a stationary type with $U(p) > 1$. The *degree of nonminimality of* p, denoted by $\mathrm{nmdeg}(p)$, is the least positive integer d such that there exist realisations $a_1, \ldots, a_d$ of p and a nonalgebraic forking extension of p over $Aa_1, \ldots, a_d$. If $U(p) \leq 1$ then we set $\mathrm{nmdeg}(p) = 0$ by convention.

Using an analysis of the multiple transitivity of binding group actions, it was shown in [4] that $\mathrm{nmdeg}(p) \leq U(p) + 1$ in the theory of differentially closed fields of characteristic zero (DCF_0). Bounds on the degree of nonminimality have played a significant role in recent proofs of strong minimality; of the generic differential equation in [2] and of the differential equations satisfied by the Schwarz triangle functions in [1]. Based on a maturing of the techniques used in [4], and informed by the approach taken in [3] to a related problem, we give in this note a short proof of a dramatic improvement to that bound.

Theorem. *Suppose* $T = \mathrm{DCF}_0$ *and* p *is a complete stationary type of finite rank. Then* $\mathrm{nmdeg}(p) \leq 2$. *Moreover, if* p *is over constant parameters then* $\mathrm{nmdeg}(p) \leq 1$.

The bound is sharp; see [4, Example 4.2] for types of nonminimality degree 2.

The argument we give for the main clause, namely that $\mathrm{nmdeg}(p) \leq 2$, works equally well in $\mathrm{DCF}_{0,m}$, the theory of differentially closed fields in m commuting derivations, and in CCM, the theory of compact complex manifolds. All one needs is that T be totally transcendental, eliminate imaginaries, eliminate the "there exists infinitely many" quantifier, and admit a 0-definable pure algebraically closed field to which every non-locally-modular minimal type is nonorthogonal. In $\mathrm{DCF}_{0,m}$ that pure algebraically closed field is the field of absolute constants and in CCM it is the (interpretation in $\mathcal{U}$ of the) complex field living on the projective line.

The "moreover" clause of the theorem, however, does make use of the fact that, in DCF_0, the binding group of a type over the constants and internal to the constants cannot be centerless.

The most general setting for the results is articulated, for the record, in Sec. 3.

Remark 1.2. A corollary of our theorem is a significant improvement to the main result of [2], where it was shown that generic algebraic differential equations of order $h \geq 2$ and degree at least $2(h+2)$ are strongly minimal. The proof in [2] used that $\mathrm{nmdeg}(p) \leq U(p) + 1$. The same proof, but using the improved bound of $\mathrm{nmdeg}(p) \leq 2$ obtained here, allows one to replace $2(h+2)$ by 6 in that result.

2. The Proof

We work in a fixed sufficiently saturated model $\mathcal{U}$ of a complete totally transcendental theory T eliminating imaginaries and the "there exists infinitely many" quantifier, with $\mathcal{C}$ a 0-definable pure algebraically closed field such that every non-locally-modular minimal type is nonorthogonal to $\mathcal{C}$.

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Maybe the first thing to observe is that the degree of nonminimality is invariant under interalgebraicity. Here we use the following, possibly nonstandard but unambigious, terminology.

Definition 2.1. Complete types $p, q \in S(A)$ are said to be *interalgebraic* if for each (equivalently some) $a \models p$ there exists $b \models q$ such that $\mathrm{acl}(Aa) = \mathrm{acl}(Ab)$.

That $\mathrm{nmdeg}(p) = \mathrm{nmdeg}(q)$ when p and q are interalgebraic is more or less immediate from the definitions; see for example [4, Lemma 3.1(c)].

The following consequences of nmdeg > 1 were observed in [4], but we include some details here for the sake of completeness:

Fact 2.2. Suppose $p \in S(A)$ is stationary of finite rank with $\mathrm{nmdeg}(p) > 1$. Then p is interalgebraic with a stationary type $q \in S(A)$ such that q is $\mathcal{C}$-internal and $q^{(2)}$ is weakly $\mathcal{C}$-orthogonal.

Proof. Note, first of all, that

$(*)$ if $a \models p$ and $b \in \mathrm{acl}(Aa) \setminus \mathrm{acl}(A)$ then $a \in \mathrm{acl}(Ab)$.

Indeed, if a' realises a nonforking extension of $\mathrm{tp}(a/Ab)$ to Aab then $\mathrm{tp}(a'/Aa)$ is a forking extension of p. Since $\mathrm{nmdeg}(p) > 1$ we must have that $a' \in \mathrm{acl}(Aa)$, from which it follows that $a' \in \mathrm{acl}(Ab)$, and hence $a \in \mathrm{acl}(Ab)$.

In the finite rank setting, condition $(*)$, which is a weak form of exchange, implies that either p is interalgebraic with a locally modular minimal type, or p is almost internal to a non-locally-modular minimal type — see [6, Proposition 2.3]. The former is impossible as $U(p) > 1$, and by assumption on T the latter implies p is almost $\mathcal{C}$-internal. We thus find a stationary $\mathcal{C}$-internal $q \in S(A)$ that is interalgebraic with p. Note that $\mathrm{nmdeg}(q) > 1$ as well.

Suppose that q is not weakly $\mathcal{C}$-orthogonal. Since the induced structure on $\mathcal{C}$, namely that of a pure algebraically closed field, eliminates imaginaries, this failure of weak $\mathcal{C}$-orthogonality will be witnessed by some $b \models q$ and $c \in \mathcal{C}$ such that $c \in \mathrm{dcl}(Ab) \setminus \mathrm{acl}(A)$. By $(*)$ applied to q this would force $b \in \mathrm{acl}(Ac)$, contradicting $U(q) > 1$. So q is weakly $\mathcal{C}$-orthogonal. In particular, as it is $\mathcal{C}$-internal, q is isolated. We let Ω be the definable set $q(\mathcal{U})$.

Now suppose that $q^{(2)}$ is not weakly $\mathcal{C}$-orthogonal. Then there are independent b_1, b_2 realizing q and $c \in \mathcal{C}$ such that $c \in \mathrm{dcl}(Ab_1b_2) \setminus \mathrm{acl}(A)$. Note that $b_2 \notin \mathrm{acl}(Ab_1c)$ as $U(b_2/Ab_1) = U(q) > 1$. So there is a partial Ab_1-definable function $f : \Omega \to \mathcal{C}$ with infinite image and infinite generic fiber. It follows, by elimination of the "there exists infinitely many" quantifier, that all but finitely many of the fibers are infinite. As $\mathcal{C} \cap \mathrm{acl}(A)$ is infinite (it is an algebraically closed subfield of $\mathcal{C}$), there exists $b \in \Omega \setminus \mathrm{acl}(Ab_1)$ such that $f(b) \in \mathrm{acl}(A)$. If $b \mathop{\smile\!\!\!\!\!\!\mid}_A b_1$ then $\mathrm{tp}(b/Ab_1) = \mathrm{tp}(b_2/Ab_1)$ contradicting the fact that $f(b_2) = c \notin \mathrm{acl}(A)$. So $b \not\mathop{\smile\!\!\!\!\!\!\mid}_A b_1$. That is, $\mathrm{tp}(b/Ab_1)$ is a nonalgebraic forking extension of q. But this contradicts $\mathrm{nmdeg}(q) > 1$. Hence $q^{(2)}$ is weakly $\mathcal{C}$-orthogonal. $\square$

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The following improvement to Fact 2.2 was *not* remarked in [4].

Lemma 2.3. *Suppose $p \in S(A)$ is stationary of finite rank with* $\mathrm{nmdeg}(p) > 1$. *Then p is interalgebraic with some stationary $q \in S(A)$ such that*

(a) *q is $\mathcal{C}$-internal,*
(b) *$q^{(2)}$ is weakly $\mathcal{C}$-orthogonal, and,*
(c) *any two distinct realisations of q are independent over A.*

Proof. Suppose a, b are realizations of p such that $a \not\mathrel{\perp\!\!\!\perp}_A b$. If $a \notin \mathrm{acl}(Ab)$ then $\mathrm{tp}(a/Ab)$ is a nonalgebraic forking extension of p, contradicting $\mathrm{nmdeg}(p) > 1$. Similarly, we must have $b \in \mathrm{acl}(Aa)$. In other words, $a \not\mathrel{\perp\!\!\!\perp}_A b$ if and only if $\mathrm{acl}(Aa) = \mathrm{acl}(Ab)$. In particular, $a \not\mathrel{\perp\!\!\!\perp}_A b$ is an equivalence relation on $p(\mathcal{U})$, which we now denote by E.

Applying Fact 2.2, we may assume that p is $\mathcal{C}$-internal and $p^{(2)}$ is weakly $\mathcal{C}$-orthogonal. In particular, both p and $p^{(2)}$ are isolated, say by the L_A-formulae $\phi(x)$ and $\psi(x, y)$, respectively. Note then, that $\phi(x) \wedge \phi(y) \wedge \neg\psi(x, y)$ defines the forking relation E. So E is an A-definable equivalence relation.

Each class of E is finite. Indeed, if $a \models p$ has an infinite E-class then there is $b \in p(\mathcal{U}) \setminus \mathrm{acl}(Aa)$ with aEb. But that means that $\mathrm{tp}(b/Aa)$ is a nonalgebraic forking extension of p, contradicting $\mathrm{nmdeg}(p) > 1$.

Fixing $a \models p$, let $e := a/E$ and $q := \mathrm{tp}(e/A)$. Note that $e \in \mathrm{dcl}(Aa)$, and so we still have that q is $\mathcal{C}$-internal and $q^{(2)}$ is weakly $\mathcal{C}$-orthogonal. Also, as the E-classes are finite, p and q are interalgebraic. So it remains to show that any two distinct realisations of q are independent. Suppose $e' \models q$ with $e' \neq e$. Then $e' = a'/E$ for some $a' \models p$ such that $\neg(aEa')$. That is $a \mathrel{\perp\!\!\!\perp}_A a'$. As $\mathrm{acl}(Aa) = \mathrm{acl}(Ae)$ and $\mathrm{acl}(Aa') = \mathrm{acl}(Ae')$, we have that $e \mathrel{\perp\!\!\!\perp}_A e'$, as desired. $\square$

We now work toward a proof of the main clause of the Theorem. That is, fixing a finite rank stationary type $p \in S(A)$, we wish to show that $\mathrm{nmdeg}(p) \leq 2$. Let $\overline{p}$ denote the unique extension of p to $\mathrm{acl}(A)$. It is immediate from the definition that $\mathrm{nmdeg}(\overline{p}) = \mathrm{nmdeg}(p)$. We may therefore assume that $A = \mathrm{acl}(A)$. Let $k := A \cap \mathcal{C}$, it is an algebraically closed subfield of $\mathcal{C}$.

In order to prove that $\mathrm{nmdeg}(p) \leq 2$ we may of course assume that $\mathrm{nmdeg}(p) > 1$. Hence, by Lemma 2.3, we can further reduce to the case that p is $\mathcal{C}$-internal, $p^{(2)}$ is weakly $\mathcal{C}$-orthogonal, and any two distinct realisations of p are independent over A.

Let $\Omega := p(\mathcal{U})$ and let $G := \mathrm{Aut}(p/\mathcal{C})$ be the binding group of p relative to $\mathcal{C}$. So (G, Ω) is an A-definable faithful group action. The action is transitive because p is weakly $\mathcal{C}$-orthogonal. Weak $\mathcal{C}$-orthogonality of p also implies, along with $A = \mathrm{acl}(A)$, that G is connected. The fact that $p^{(2)}$ is weakly $\mathcal{C}$-orthogonal implies that G acts transitively on $p^{(2)}(\mathcal{U})$. But $p^{(2)}(\mathcal{U}) = \Omega^2 \setminus \Delta$ where Δ is the diagonal, because any two distinct realisations of p are independent over A. So (G, Ω) is a 2-transitive connected A-definable homogeneous space.

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Now, the binding group action of any $\mathcal{C}$-internal type is isomorphic to the $\mathcal{C}$-points of an algebraic group action, though possibly over additional parameters. More precisely, let $M \preceq \mathcal{U}$ be a prime model over A. Note that $M \cap \mathcal{C} = k$. There exists an algebraic homogeneous space $(\overline{G}, \overline{\Omega})$ defined over k, and an M-definable isomorphism $\alpha : (G, \Omega) \to (\overline{G}(\mathcal{C}), \overline{\Omega}(\mathcal{C}))$.

In particular, $(\overline{G}, \overline{\Omega})$ is a 2-transitive connected algebraic homogeneous space. This is a very restrictive condition; a theorem of Knop [5] tells us that $(\overline{G}, \overline{\Omega})$ is either isomorphic to the action of PGL_{n+1} on $\mathbb{P}^n$, or is isomorphic to the action of an algebraic subgroup of the group of affine transformations on $\mathbb{A}^n$, for some $n > 1$. In either case we have a notion of *collinearity* which is preserved by the group action. That is, given distinct $u, v \in \overline{\Omega}(\mathcal{C})$ we can talk about the line $\ell_{u,v} \subseteq \overline{\Omega}(\mathcal{C})$ connecting u and v, and for all $g \in \overline{G}(\mathcal{C})$ we have that $g\ell_{u,v} = \ell_{gu,gv}$.

Fix distinct $a, b \in \Omega$, and consider the set $X := \alpha^{-1}(\ell_{\alpha(a),\alpha(b)})$. Then X is a rank 1 Mab-definable subset of Ω.

Claim 2.4. *There is a finite tuple c from $\mathcal{C}$ such that X is $Aabc$-definable.*

Proof. It suffices to show that if $\sigma \in \mathrm{Aut}_{Aab}(\mathcal{U}/\mathcal{C})$, that is, if σ is an automorphism of $\mathcal{U}$ that fixes $A \cup \{a, b\} \cup \mathcal{C}$ point-wise, then $\sigma(X) = X$. Now, the restriction of σ to Ω is an element of the binding group, say $g_\sigma \in G$, which fixes a and b. Hence $\alpha(g_\sigma) \in \overline{G}(\mathcal{C})$ fixes $\alpha(a)$ and $\alpha(b)$, and hence preserves the line $\ell_{\alpha(a),\alpha(b)}$. It follows that

$$\begin{aligned}\alpha(\sigma(X)) &= \alpha(g_\sigma(\alpha^{-1}(\ell_{\alpha(a),\alpha(b)}))) \\ &= \alpha(g_\sigma)(\ell_{\alpha(a),\alpha(b)}) \\ &= \ell_{\alpha(a),\alpha(b)}.\end{aligned}$$

Applying α^{-1} to both sides we obtain that $\sigma(X) = X$, as desired. □

Let $\theta(x, y)$ be an L_{Aab}-formula such that $X = \theta(\mathcal{U}, c)$. If, in addition, we chose $a, b \in \Omega(M)$, then X and $\theta(x, y)$ are over M, and it follows that there is $c' \in M \cap \mathcal{C}$ such that $X = \theta(\mathcal{U}, c')$. But $M \cap \mathcal{C} = k \subseteq A$, so that this witnesses the definability of X over Aab.

We have thus found $a, b \in \Omega$ and an Aab-definable subset $X \subseteq \Omega$ of rank 1. Since $U(p) > 1$, the generic type of X over Aab is a nonalgebraic forking extension of p. Since a and b realize p, this witnesses that $\mathrm{nmdeg}(p) = 2$.

This completes the proof of the main clause of the Theorem.

For the "moreover" clause, we return to the particular setting of $T = \mathrm{DCF}_0$ and $\mathcal{C}$ the field of constants. We make the additional assumption that $A \subseteq \mathcal{C}$ and show that $\mathrm{nmdeg}(p) > 1$ leads to a contradiction. Indeed, that (G, Ω) is 2-transitive forces G to be centerless; see for example the elementary argument at the beginning of the proof of Satz 2 in [5]. But, in DCF_0, the binding group of a type that is $\mathcal{C}$-internal and over constant parameters cannot be centerless; see for example the proof of [3, Theorem 3.9]. This contradiction proves that $\mathrm{nmdeg}(p) \leq 1$.

3. Some Remarks on the Assumptions

We carried out the above proof under assumptions on T that were suitable for generalization to both $\mathrm{DCF}_{0,m}$ and CCM. But it may be worth recording the minimal hypotheses on T required for the proofs to go through. We leave it to the reader to inspect those proofs and verify that what is actually proved are the following two statements.

Theorem 3.1. *Suppose T is a complete totally transcendental theory eliminating imaginaries and the "there exists infinitely many" quantifier. Let $\mathcal{U} \models T$ be a sufficiently saturated model and $A \subseteq \mathcal{U}$ a parameter set.*

(a) *Suppose each non-locally-modular minimal type in T is nonorthogonal to some A-definable pure algebraically closed field. Then* $\mathrm{nmdeg}(p) \leq 2$ *for all stationary $p \in S(A)$ of finite rank.*

(b) *Suppose there exists a collection $\{\mathcal{C}_i : i \in I\}$ of A-definable non-locally-modular strongly minimal sets such that each non-locally-modular minimal type in T is nonorthogonal to $\mathcal{C}_i$ for some $i \in I$, and such that for all $i \in I$,*

  (i) *$\mathcal{C}_i \cap \mathrm{acl}(A)$ is infinite, and,*

  (ii) *for all weakly $\mathcal{C}_i$-orthogonal $\mathcal{C}_i$-internal $q \in S(\mathrm{acl}(A))$, the binding group $\mathrm{Aut}(q/\mathcal{C}_i)$ has a nontrivial center.*

  Then $\mathrm{nmdeg}(p) \leq 1$ *for all stationary $p \in S(A)$ of finite rank.*

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