BINDING GROUPS FOR ALGEBRAIC DYNAMICS

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Abstract. A binding group theorem is proved in the context of quantifier-free internality to the fixed field in the theory ACFA_0. This is articulated as a statement about the birational geometry of isotrivial algebraic dynamical systems, and more generally isotrivial σ-varieties. It asserts that if (V, ϕ) is an isotrivial σ-variety then a certain subgroup of the group of birational transformations of V, namely those that preserve all the relations between (V, ϕ) and the trivial dynamics $(\mathbb{A}^1, \mathrm{id})$, is in fact an algebraic group. Several application are given including new special cases of the Zariski Dense Orbit Conjecture and the Dixmier-Moeglin Equivalence Problem in algebraic dynamics, as well as finiteness results about the existence of nonconstant invariant rational functions on cartesian powers of σ-varieties. These applications give algebraic-dynamical analogues of recent results in differential-algebraic geometry.

Contents

1. Introduction

Fix a field k and an endomorphism $\sigma : k \to k$. This paper is concerned with the birational algebraic geometry, and model theory, of *rational σ-varieties* over (k, σ); that is, irreducible algebraic varieties V over k equipped with dominant rational maps $\phi : V \dashrightarrow V^\sigma$. Note that such a structure on V is determined by, and determines, an endomorphism of the rational function field $k(V)$ that extends σ; namely $f \mapsto f^\sigma \circ \phi$.

The algebraic dynamics literature usually only considers the autonomous situation where σ is the identity on k, and hence $V = V^\sigma$. These are *rational dynamical systems.* While they are an important special case for us too, we work generally in the possibly nonautonomous context.[1]

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[1]One reason to not restrict attention only to the autonomous case is to allow the taking of generic fibres: even if both (V, ϕ) and (W, ψ) are rational dynamical systems, the generic fibre of

The natural subobjects here are the *invariant* subvarieties; namely, irreducible subvarieties $X \subseteq V$ over k on which ϕ restricts to a dominant rational map from X to X^σ. And the morphisms of this category, $g : (V_1, \phi_1) \dashrightarrow (V_2, \phi_2)$, are the dominant rational maps $g : V_1 \to V_2$ that are *equivariant* in the sense that $\phi_2 \circ g = g^\sigma \circ \phi_1$.

Our focus is on *isotrivial* σ-varieties: those that are, after base extension, equivariantly birationally equivalent to a trivial σ-variety, namely a variety equipped with the identity transformation. More precisely, a rational σ-variety (V, ϕ) is isotrivial if there is a commuting diagram of the form

$$\begin{array}{ccc} (V \times Z, \phi \times \psi) & \overset{g}{\dashrightarrow} & (\mathbb{A}^\ell \times Z, \mathrm{id} \times \psi) \\ & \searrow \quad \swarrow & \\ & (Z, \psi) & \end{array}$$

where (Z, ψ) is another rational σ-variety over k and g is birational onto its image. A basic example of a nontrivial but isotrivial rational dynamical system is the map $\phi : \mathbb{A}^1 \to \mathbb{A}^1$ given by $\phi(x) = x + 1$. The trivialisation is obtained by taking (Z, ψ) to be $(\mathbb{A}^1, \phi)$ itself, and g to be given by $g(x, y) = (x - y, y)$. In fact, isotrivial σ-varieties are ubiquitous; see, for example, Corollary 3.9 below, which says that if a σ-variety admits any nonvacuous algebraic relation to trivial dynamics, even after base extension, then already without base extension it admits a positive-dimensional isotrivial image. An even more convincing example of the centrality of isotrivial σ-varieties is the Zilber dichotomy (a deep model-theoretic result established in [7]) which implies, roughly speaking, that as soon as some cartesian power of (V, ϕ) admits a sufficiently rich algebraic family of invariant subvarieties, then (V, ϕ) admits a positive-dimensional isotrivial image.

Isotrivial rational dynamics were studied (by very different means) in [2], where they were shown to always come from the action of an algebraic group, very much like in the above example where the relevant group action is that of the additive group on the affine line. We are partly motivated by the desire to extend that work to the possibly nonautonomous setting, and to give a model-theoretic account. Nevertheless, our approach gives significant new information even for rational dynamical systems.

Our main result is that a certain natural group of birational transformations of an isotrivial σ-variety is in fact an algebraic group. To describe the result we need some notation. First of all, for any field extension $K \supseteq k$, let us denote by $\mathrm{Bir}_K(V)$ the group of birational transformations of V over K. That is, birational maps $\delta : V_K \dashrightarrow V_K$ under composition. We are interested in "algebraic subgroups" of $\mathrm{Bir}_k(V)$ in the following sense:

Definition 1.1. By an *algebraic group of birational transformations of* V we mean an algebraic group G over k equipped with a rational map $\theta : G \times V \dashrightarrow V$ over k that determines an injective group homomorphism $G(K) \to \mathrm{Bir}_K(V)$, for any field extension $K \supseteq k$, given by $w \mapsto \theta_w$.[2]

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an equivariant map $g : (V, \phi) \to (W, \psi)$, namely the induced base extension of V to $k(W)$, should be considered with its natural nonautonomous σ-variety structure coming from the endomorphism of $k(W)$ induced by ψ.

[2]Note that θ makes V into a pre-homogeneous variety for G, in the sense of Weil [23], and that it follows, by Weil's group-chunk theorem, that after replacing V by a birationally equivalent

For example, while the usual action of $\mathbb{G}_a$ on $\mathbb{A}^1$ is an algebraic group of automorphisms, the variant given by $(g,x) \mapsto \frac{x}{gx+1}$ is only an algebraic group of birational transformations.[3] These are well-studied objects in birational algebraic geometry, especially in the case when $V = \mathbb{P}^n$; See, for example, [5] and the references therein.

The following abstract subgroup of $\mathrm{Bir}(V)$ captures the interaction between a given rational σ-variety structure on V and trivial dynamics. It appears to us to be a fundamental object in algebraic dynamics that has not been studied before, even in the autonomous case.

Definition 1.2 (Binding group of a σ-variety)**.** Fix an absolutely irreducible rational σ-variety $\mathbb{V} := (V, \phi)$ over (k, σ), and let $\mathbb{L} := (\mathbb{A}^1, \mathrm{id})$ denote the trivial dynamics on the affine line. Let

$$\mathcal{I}_{\mathbb{V}} := \left\{ \begin{array}{l} \text{irreducible invariant subvarieties of } \mathbb{V}^r \times \mathbb{L}^s \text{ over } k \text{ that project} \\ \text{dominantly onto each copy of } V\text{, for all } r \geq 1 \text{ and } s \geq 0 \end{array} \right\}$$

Fix, now, a field extension $K \supseteq k$. For each $r \geq 1$ and $s \geq 0$, embed $\mathrm{Bir}_K(V)$ into $\mathrm{Bir}_K(V^r \times \mathbb{A}^s)$ by acting diagonally on V^r and trivially on $\mathbb{A}^s$. Let

$$\mathrm{Bir}_K(\mathbb{V}/\mathbb{L}) := \left\{ \delta \in \mathrm{Bir}_K(V) : \begin{array}{l} \text{if } X \in \mathcal{I}_{\mathbb{V}} \text{ then } \delta \text{ restricts to a} \\ \text{birational transformation of } X_K \end{array} \right\}$$

That is, $\mathrm{Bir}_K(\mathbb{V}/\mathbb{L})$ is the group of birational transformations of V over K that preserve all invariant k-definable algebraic relations between cartesian powers of $\mathbb{V}$ and $\mathbb{L}$.

We show that the *a priori* abstract binding group of an isotrivial σ-variety is in fact an algebraic group:

Theorem 1.3. *Suppose k is an algebraically closed field of characteristic zero and $\mathbb{V} := (V, \phi)$ is a rational σ-variety over (k, σ). If $\mathbb{V}$ is isotrivial then there exists an algebraic group G of birational transformations of V such that $\mathrm{Bir}_K(\mathbb{V}/\mathbb{L}) = G(K)$, for any field extension $K \supseteq k$.*

The proof of this theorem appears in Section 4 below.

We delay discussion of applications of this theorem to later in the Introduction, addressing first its model-theoretic formulation. The model-theorist will by now have realised that Theorem 1.3 has something to do with what is often called "the binding group theorem". Indeed, what we prove is a quantifier-free binding group theorem for the theory of difference-closed fields (ACFA), introduced by Chatzidakis and Hrushovski in [7]. Recall that a *difference field* is a field equipped with an endomorphism, and it is *difference-closed* if every system of algebraic *difference equations* (that is, polynomial equations in variables $x, \sigma(x), \sigma^2(x), \ldots$) that is consistent – in the sense that it has a solution in some difference field extension – has a solution. The connection to algebraic dynamics is that a σ-variety (V, ϕ) can be seen as encoding the first-order difference equation $\sigma(x) = \phi(x)$. That is, given any difference field extension $(K, \sigma) \supseteq (k, \sigma)$, we can consider those K-points of V on which σ and ϕ agree. This is a quantifier-free definable set in (K, σ) that we denote by $(V, \phi)^\sharp(K)$. The fact that the class of difference-closed fields is axiomatisable (by

copy, we get an honest regular (rather than rational) algebraic group action. See [25] for a modern treatment that does not assume, as we do not, the connectedness of G.

[3]This example is slightly artificial since it comes from a regular action on $\mathbb{P}^1 \supset \mathbb{A}^1$, but in higher dimensions it is possible to give examples with V projective.

ACFA) means that, instead of considering all possible difference field extensions of (k, σ), we can work in a fixed large difference-closed extension, $\mathcal{U}$, that serves as a universal domain for difference-algebraic geometry.

We associate to each rational σ-variety (V, ϕ) over (k, σ), the quantifier-free type $q(x)$ over k which asserts that x is a Zariski generic point of V over k and that $\sigma(x) = \phi(x)$. This turns out to be a complete quantifier-free type which we call the *generic type* of (V, ϕ). We call those complete quantifier-free types that arise in this way, *rational types.* The model theory of q in $\mathcal{U}$ will control, and is controlled by, the birational geometry of (V, ϕ). In particular, in characteristic zero, (V, ϕ) being isotrivial corresponds precisely to q being *quantifier-free internal* to the fixed field. Quantifier-free internality in ACFA was introduced in [7] and is discussed at length in §2.4 below. The model-theoretic content of this paper has to do, therefore, with the structure of rational types in ACFA that are quantifier-free internal to the fixed field. In particular, we introduce a binding group:

Definition 1.4 (Binding group of a rational type)**.** Given a rational quantifier-free type q over (k, σ), we denote by $\operatorname{Aut}_{\mathrm{qf}}(q/\operatorname{Fix}(\sigma))$ the (abstract) subgroup of permutations δ of $q(\mathcal{U})$ satisfying:

$$\theta(a, c) \text{ holds } \iff \theta(\delta(a), c) \text{ holds}$$

for any (quantifier-free) formula $\theta(x, y)$ over k in the language of rings, any tuple a of realisations of q, and any tuple c of elements of the fixed field.

And we prove a binding group theorem:

Theorem 1.5. *Working in a sufficiently saturated difference-closed field $\mathcal{U}$ of characteristic zero, suppose q is a rational quantifier-free type over k that is quantifier-free internal to the fixed field. There exists a quantifier-free definable group $\mathcal{G}$ over k, with a relatively quantifier-free definable action on $q(\mathcal{U})$ over k, such that $\mathcal{G}$ and* $\operatorname{Aut}_{\mathrm{qf}}(q/\operatorname{Fix}(\sigma))$ *are isomorphic as groups acting on $q(\mathcal{U})$.*

In fact, if q is the generic type of an isotrivial rational σ-variety (V, ϕ) over (k, σ), and $\theta : G \times V \dashrightarrow V$ is the algebraic group of birational transformations of V given by Theorem 1.3, then there is an isomorphism $\rho : G \to G^{\sigma}$ of algebraic groups over k such that $\mathcal{G} = (G, \rho)^{\sharp}(\mathcal{U})$ and θ restricts to the action of $\mathcal{G}$ on $q(\mathcal{U})$.

This is also proved in Section 4.

Let us briefly recall the model-theoretic precedents to this theorem. A crucial aspect of usual internality in totally transcendental theories, of a complete type p, say, to a definable set X, is that the witness to internality may involve more parameters than those over which p and X are defined. This dependence on additional parameters is controlled by the binding group (or *liaison* group), a definable group acting definably on the realisations of p and agreeing with the action of the group of automorphisms of the universe that fix X pointwise. The existence and importance of the binding group was already recognised by Zilber [26] in the late nineteen-seventies. Poizat [21] realised that when applied to differentially closed fields, binding groups recover Kolchin's differential Galois theory. Hrushovski developed the subject in its current form, first working with stable theories but eventually in complete generality: in [14] the binding group is constructed (as a type-definable group) from internality assuming only that the set X is stably embedded. Based on Hrushovski's construction, the first author, in [16], extended the theory of binding groups to the quantifier-free fragment (or indeed arbitrary fragments) of a theory, very much with ACFA in mind. The focus of [16] is linear difference equations and the development of a binding group theory that recovers the difference Galois theory of Van der Put and Singer. Moreover, it is concerned with the internality of one definable set in another, and not of generic types. In particular, the results there do not immediately apply to the birational geometry of rational σ-varieties. Nevertheless, while we do not directly rely on [16], that work should be considered the immediate predecessor of this one, and it very much influences our construction.

1.1. **Applications.** We now describe several applications. The proofs, and more detailed statements, of the following theorems appear in Section 5 below. Each of these applications has both a formulation in terms of the birational geometry of rational σ-varieties, as well as in terms of the model theory of rational types in ACFA_0. Here, in the Introduction, we focus on the geometric formulations.

First of all, if we restrict attention to rational dynamics $\phi : V \dashrightarrow V$, then we recover some of the main results of [2]. In particular, we are able to show that if (V, ϕ) is isotrivial then ϕ comes from an algebraic group action; this is [2, Corollary A] and appears as Theorem 5.1 below. The proof in [2] is somewhat involved and computational, using mostly elementary methods from algebraic dynamics. We deduce it here by observing that in the autonomous isotrivial case, $\phi \in \mathrm{Bir}(\mathbb{V}/\mathbb{L})$, and so the algebraic group is the one given by Theorem 1.3. We are, similarly, able to give a model-theoretic account of [2, Corollary B] in Corollary 5.2 below, but we leave the formulation of that result for later in the Introduction. Explaining the results of [2] from the point of view of model-theoretic binding groups was one of the motivations for this work.

Our first new application is about the number of *maximal* proper invariant subvarieties. A necessary condition for a rational σ-variety, (V, ϕ) over (k, σ), to admit only finitely many maximal proper invariant subvarieties over k is that (V, ϕ) admit no nonconstant invariant rational functions. Here, an *invariant rational function* on (V, ϕ) is a rational function on V over k that is fixed by the endomorphism of $k(V)$ that ϕ induces. Such a rational function would, by taking level sets, give rise to infinitely many distinct codimension 1, and hence maximal proper, invariant subvarieties. The question of whether this condition is sufficient – that is, whether having no nonconstant invariant rational functions implies having only finitely many maximal proper invariant subvarieties – is sometimes called the *Dixmier-Moeglin equivalence problem* in algebraic dynamics, at least in the case when ϕ is an automorphism of a projective variety (see [3, Conjecture 8.5] and also [19] for a survey of Dixmier-Moeglin-type problems). Using binding groups, we resolve the problem for isotrivial σ-varieties:

Theorem 1.6 (Appearing as Theorem 5.4 below)**.** *Suppose k is an algebraically closed field of characteristic zero and (V, ϕ) is an isotrivial rational σ-variety over (k, σ). If (V, ϕ) has no nonconstant invariant rational functions then it has only finitely many maximal proper invariant subvarieties.*

Model-theoretically, as we show in Proposition 3.4, (V, ϕ) having no nonconstant invariant rational functions over k says that the generic type is weakly orthogonal to the fixed field. Hence, the model-theoretic content of Theorem 1.6 is that a rational type that is both quantifier-free internal and weakly orthogonal to the fixed field is isolated. It should not be surprising, at least to the model-theorist, that this follows rather easily from the existence of a quantifier-free definable binding group.

Theorem 1.6 can be seen as the difference-algebraic analogue of a theorem in differential-algebraic geometry, appearing in [4], about isotrivial D-varieties.[4]

As a more or less immediate corollary, we resolve a special case of the *Zariski dense orbit conjecture* (from [18]) that we don't think has been observed before, namely the isotrivial case:

Corollary 1.7 (Appearing as Corollary 5.5 below)**.** *Suppose k is an algebraically closed field of charactetistic zero, and $\phi : V \to V$ is an automorphism of an algebraic variety over k such that (V, ϕ) is isotrivial. If (V, ϕ) admits no nonconstant invariant rational functions then there is $a \in V(k)$ such that the orbit of a under ϕ is Zariski dense in V.*

Indeed, if we take $a \in V(k)$ outside of the finitely many maximal proper invariant subvarieties then the Zariski closure of its orbit, being invariant, will be all of V.

Our final application has to do with rational σ-varieties with the property that some cartesian power admits a nonconstant invariant rational function. We show that there is a bound on how high a cartesian power one must look at:

Theorem 1.8 (Appearing as Theorem 5.6 and Corollary 5.2 below)**.** *Suppose (V, ϕ) is a rational σ-variety over an algebraically closed difference field of characteristic 0. If some cartesian power of (V, ϕ) admits a nonconstant invariant rational function then already (V^n, ϕ) does, where $n = \dim V + 3$. In the autonomous case, when $\phi : V \dashrightarrow V$ is a rational dynamical system, we can take $n = 2$, regardless of $\dim V$.*

Actually, we can weaken the antecedent of this implication somewhat, to the existence of an invariant rational function on $(V \times W, \phi \times \psi)$, for some (W, ψ), that is not the pullback of a rational function on (W, ψ). See the geometric formulation of Theorem 5.6, below. As we show in Corollary 3.9, this condition on (V, ϕ) turns out to be equivalent to the existence of a positive-dimensional isotrivial image.

For rational dynamics (with the bound of 2) this theorem appears already as [2, Corollary B], using very different methods. But the general case is new. An additional ingredient in its proof is the truth of the Borovik-Cherlin Conjecture in the theory of algebraically closed fields of characteristic zero, established in [12] using the work of Popov [22] as proposed by Borovik and Cherlin in [6]. This statement bounds the degree of generic multiple transitivity of an algebraic group action; and we apply that bound to the binding group action of a positive-dimensional isotrivial image of (V, ϕ). The differential-algebraic analogue of Theorem 1.8 (which also uses the Borovik-Cherlin Conjecture in ACF_0) comes out of work in [12, 10, 15]. This too was part of our original motivation for developing binding groups in the difference-algebraic context. Let us also mention that, while the bound of $\dim V + 3$ seems quite weak compared to the absolute bound of 2 for rational dynamics, it turns out to be sharp for the analogous result in differential-algebraic geometry, and we expect it to be sharp here too. But that remains as yet unverified.

[4] In fact, Proposition 2.3 of [4] proves the analogous result for the more general "compound isotrivial" D-varieties – so for types *analysable*, rather than internal, in the constants. It is likely that compound isotriviality can be made sense of for rational dynamics also, and the extension of Theorem 1.6 to that case would be a desirable objective of future work.

Theorem 1.8 also has a model-theoretic articulation: a rational quantifier-free type p of dimension d is nonorthogonal to the fixed field if and only if the Morley power $p^{(d+3)}$ is not weakly orthogonal to the fixed field.

Finally, let us mention that one of the primary motivations for this work is the extension, to the setting of algebraic dynamics, of the results in [11] on the structure of algebraic differential equations having the property that any three solutions are independent. As that work uses model-theoretic binding groups in a crucial way, it is our hope that the theory developed here will lead to such an extension.

1.2. **A word about characteristic.** While we have assumed characteristic zero in the statements of our theorems above, much of what we do in this paper goes through for arbitrary characteristics, and in what follows we will make clear where and why characteristic zero is required. While positive characteristic analogues of the theorems presented here can be articulated, we have decided not to do so, partly because in positive characteristic one should not only consider isotriviality and internality with respect to the fixed field, but rather to the various fixed fields of σ composed with powers of the Frobenius automorphism. Working out a general theory of binding groups in that setting is desirable, but is deferred to future work.

1.3. **Plan of the paper.** We conclude the Introduction by fixing our algebraic geometric conventions. Then, in Section 2, we discuss/review in some detail the various elements of the quantifier-free fragment of ACFA that concern us. In particular, we discuss rational types, canonical bases, nonorthogonality to the fixed field, and internality to the fixed field, all in the quantifier-free setting. In Section 3 we develop algebraic dynamics in the general nonautonomous context, and produce a dictionary translating between algebraic dynamics and model theory. In particular, invariant rational functions and isotriviality are discussed at length here. Section 4 is dedicated to the proofs of our main binding group theorems, namely Theorems 1.3 and 1.5. Finally, in Section 5, we state and prove the applications we have discussed above.

1.4. **Algebraic geometric conventions.** Here we make explicit some more or less standard notational conventions.

We drop the assumption of characteristic zero, asserting it explicitly when needed from now on. We will tend to work in a sufficiently saturated algebraically closed field $\mathcal{U}$ that serves as a universal domain for algebraic geometry, in the sense of Weil. In particular, all tuples and fields are assumed to live in $\mathcal{U}$, and all varieties are identified with their $\mathcal{U}$-points.

Varieties are reduced and of finite type over a field, but not necessarily irreducible. Nothing will be lost by restricting to quasi-projective varieties.

If the characteristic is $p > 0$ then we denote by Fr the Frobenius automorphism of $\mathcal{U}$ given by $x \mapsto x^p$. In characteristic 0, we take Fr to be the identity. We denote by k^{perf} the perfect closure of a field k, and by k^{alg} the algebraic closure.

A subfield k is a *field of definition* for an affine variety V (equivalently V is *over* k) if the ideal $I(V) \subseteq \mathcal{U}[x]$ of polynomials vanishing on V has a set of generators with coefficients in k. In characteristic 0 this coincides with being $\mathcal{L}_{\mathrm{ring}}$-definable over k, but in positive characteristic it is a stronger notion: being $\mathcal{L}_{\mathrm{ring}}$-definable over k only ensures that k^{perf} is a field of definition.

Given a variety V over k we will tend not to distinguish notationally between V and its base change to a field extension $K \supseteq k$, except when confusion could arise, in which case we use V_K for the base extension.

Given a variety V over k we will say that a property holds of *general* $a \in V$ to mean that it holds on a Zariski dense open subset over k.

Given an n-tuple a, and a perfect field k, we denote by $\mathrm{loc}(a/k)$ the Zariski locus of a over k, the smallest closed subvariety of $\mathbb{A}^n$ over k that contains a as a $\mathcal{U}$-point.

If $\phi : V \dashrightarrow W$ is a rational map of varieties, we denote by $\mathrm{dom}(\phi) \subseteq V$ the largest (open) subset on which ϕ is defined.

We will often consider algebraic families of varieties. These will usually be presented as follows: we have irreducible varieties V and Z over a field k, as well as an irreducible subvariety $X \subseteq V \times Z$ over k. For $e \in Z$ we denote the set-theoretic fibre by

$$X_e := \{v \in V : (v, e) \in X\}.$$

That is, X_e denotes the underlying reduced variety, over $k(e)$, of the subscheme of $V_{k(e)}$ given by the scheme-theoretic fibre. In particular, $X_e = X_{e'}$ if and only if $X_e(L) = X_{e'}(L)$ for some (equivalently any) algebraically closed field extending $k(e)$. These fibres form a family of subvarieties of V parameterised by Z. We will tend to use $\pi_1 : X \to V$ and $\pi_2 : X \to Z$ to denote the co-ordinate projections. Given another such family, say $Y \subseteq W \times Z$, and a rational map

$$\begin{array}{ccc} X & \overset{g}{\dashrightarrow} & Y \\ & \searrow \quad \swarrow & \\ & Z & \end{array}$$

for any $e \in \pi_2(\mathrm{dom}(g))$, we denote by $g_e : X_e \dashrightarrow Y_e$ the $k(e)$-rational map given by $v \mapsto \pi_1(g(v, e))$.

It is well known that any algebraic family of subvarieties admits a rational quotient family where the parameters are canonical – this is essentially a Hilbert scheme argument with projective varieties, but we present it here as a consequence of elimination of imaginaries. We restrict attention to characteristic zero as we will only use this fact in that case, and the statement in positive characteristic is slightly more involved (requiring precomposition with a purely inseparable map).

Fact 1.9. *Let k be a field of characteristic zero, V, Z irreducible varieties over k, and $X \subseteq V \times Z$ an irreducible subvariety projecting dominantly to Z. Then there is a variety Z_0 over k and a dominant rational map $\mu = \mu_X : Z \dashrightarrow Z_0$ such that*

(a) *For general $a, b \in Z$, if $X_a = X_b$ then $\mu(a) = \mu(b)$*

(b) *Universality: For any dominant rational map $t : Z \dashrightarrow W$ such that for general $a, b \in Z$, $X_a = X_b$ implies $t(a) = t(b)$, there is a unique dominant rational map $\bar{t} : Z_0 \dashrightarrow W$ with $t = \bar{t} \circ \mu$*

Furthermore, for general $a, b \in Z$, if $\mu(a) = \mu(b)$, then $X_a = X_b$.

Proof. By elimination of imaginaries in ACF, the analogous statement holds in the definable category: there is an $\mathcal{L}_{\mathrm{ring}}$-definable map $\tilde{\mu}$ on Z such that, for all $a, b \in Z$, $X_a = X_b$ if and only if $\tilde{\mu}(a) = \tilde{\mu}(b)$. By quantifier elimination, and the fact that we are in characteristic zero, there is a nonempty Zariski open subset of Z on which $\tilde{\mu}$ agrees with a dominant rational map, $\mu : Z \dashrightarrow Z_0$. In particular, μ satisfies (a) and the “furthermore” clause.

We claim it also satisfies (b). If $t : Z \dashrightarrow W$ is dominant rational, and satisfies the condition on the fibres, then there is an $\mathcal{L}_{\text{ring}}$-*definable* $\tilde{t} : Z_0 \to W$ such that t agrees with $\tilde{t} \circ \tilde{\mu}$ on a nonempty Zariski open set. Once again, we have $\tilde{t}$ agrees on a nonempty Zariski open set with a (necessarily dominant) rational map $\bar{t} : Z_0 \to W$. Hence $t = \bar{t} \circ \mu$ as rational maps on Z. Uniqueness is by dominance of μ. □

2. The quantifier-free model theory of ACFA

Everything we do in this section is known to the experts, and much of it can be found in, or easily deduced from, the literature on the model theory of difference fields, in particular [7] and[16]. Our purpose here is to give a self-contained and complete account.

Let $\mathcal{L}_{\text{ring}} = \{0, 1, +, -, \times\}$ be the language of rings and $\mathcal{L}_\sigma = \{0, 1, +, -, \times, \sigma\}$ the language of difference rings. Fix a sufficiently saturated model $(\mathcal{U}, \sigma) \models$ ACFA. When working in this universal domain we follow the usual conventions that sets of parameters are small in cardinality compared to the degree of saturation – unless explicitly stated otherwise.

We set $\text{Fix}(\sigma) = \{a \in \mathcal{U} : \sigma(a) = a\}$ to be the fixed field of $(\mathcal{U}, \sigma)$.

We do not assume that our difference fields are inversive – that is they are simply fields k equipped with an endomorphism σ, viewed as $\mathcal{L}_\sigma$-structures. We use $\langle A \rangle$ to denote the difference field generated by the set A. If k is a difference subfield of $\mathcal{U}$, and a is a tuple, then by $k\langle a \rangle$ we mean the difference subfield of $\mathcal{U}$ generated by a over k, namely $k(a, \sigma(a), \sigma^2(a), \dots)$. For natural m, we denote by $\nabla_m(a)$ the tuple $(a, \sigma(a), \dots, \sigma^m(a))$.

We use nonforking independence in ACFA freely: $A \mathop{\smile\!\!\!\!\!\!\;|}_C B$ means that $\langle A \cup C \rangle^{\text{alg}}$ is algebraically disjoint from $\langle B \cup C \rangle^{\text{alg}}$ over $\langle C \rangle^{\text{alg}}$. In particular, dependence is always witnessed by quantifier-free formulas.

We are concerned with the quantifier-free fragment of $(\mathcal{U}, \sigma)$. This means that, given a parameter set A, we will be primarily interested in $S_{\text{qf}}(A)$, the set of complete quantifier-free types over A.

Definition 2.1. Suppose $p \in S_{\text{qf}}(A)$. We say that p is *stationary* if for any extension of parameters $B \supseteq A$ there is a unique extension of p to a complete quantifier-free type over B whose realisations are independent of B over A. This extension is the *nonforking extension of* p *to* B.

Complete quantifier-free types over algebraically closed difference fields are always stationary. In fact, let us record for future use the following well known strengthening of quantifier-free stationarity over algebraically closed sets:

Lemma 2.2. *Suppose k is an algebraically closed difference field, with four difference field extensions K_1, K_2, L_1, L_2. Assume that*

(i) $\text{qftp}(K_1/k) = \text{qftp}(K_2/k)$,
(ii) $\text{qftp}(L_1/k) = \text{qftp}(L_2/k)$, *and*
(iii) $L_i \mathop{\smile\!\!\!\!\!\!\;|}_k K_i$, *for* $i = 1, 2$.

Then $\text{qftp}(K_1L_1/k) = \text{qftp}(K_2L_2/k)$.

Proof. Because K_i is linearly disjoint from L_i over k, we have a canonical identification of the field K_iL_i with the fraction field of $K_i \otimes_k L_i$. Moreover, this is an identification of difference fields where σ acts on $K_i \otimes_k L_i$ by $\sigma(a \otimes b) = \sigma(a) \otimes \sigma(b)$.

Now $\mathrm{qftp}(K_1/k) = \mathrm{qftp}(K_2/k)$ is witnessed by an isomorphism $\alpha : K_1 \to K_2$ of difference fields over k, and similarly we have $\beta : L_1 \to L_2$. We obtain an isomorphism $\alpha \otimes \beta : K_1 \otimes_k L_1 \to K_2 \otimes_k L_2$ of difference rings over k, which will extend to the fraction fields. That isomorphism witnesses $\mathrm{qftp}(K_1L_1/k) = \mathrm{qftp}(K_2L_2/k)$. □

In particular, if $p \in S_{\mathrm{qf}}(A)$ is stationary then, for all $n \geq 1$, all n-tuples of independent realisations of p will have the same complete quantifier-free type over A, which we denote by $p^{(n)}$, and call the *n-th Morley power of p*.

2.1. **Rational types.** Fix a difference field k.

Among the complete quantifier-free types over k we will be primarily interested in what we will call *rational types*, namely those $p(x) \in S_{\mathrm{qf}}(k)$ that imply the formula $\sigma(x) = f(x)$ for some rational function $f \in k(x)$. In that case, p is determined by this formula along with the $\mathcal{L}_{\mathrm{ring}}$-formulas in p.

Note that if p is rational and $a \models p$ then $k\langle a\rangle = k(a)$ is a finitely generated field extension of k. Conversely, if $k\langle a\rangle$ is finitely generated over k as a field then $\mathrm{qftp}(\nabla_m(a)/k)$ is rational for some $m \geq 0$. So, to study rational types is to study difference field extensions that are finitely generated as field extensions.

Lemma 2.3. *Suppose k is a perfect difference field and $\mathrm{qftp}(a/k)$ is rational. If $e \in k(a)^{\mathrm{perf}}$ then $\mathrm{qftp}(\nabla_m(e)/k)$ is rational for some $m \geq 0$.*

Proof. Let $\ell > 0$ be such that $\mathrm{Fr}^{\ell}(e) \in k(a) = k\langle a\rangle$. It follows that $k\langle \mathrm{Fr}^{\ell}(e)\rangle$ is a finitely generated field extension of k, and hence, for some $m \geq 0$, if

$$b := \nabla_m \mathrm{Fr}^{\ell}(e) = \mathrm{Fr}^{\ell} \nabla_m(e)$$

then $\mathrm{qftp}(b/k)$ is rational. Let $f \in k(x)$ be such that $\sigma(b) = f(b)$. Then,

$$\begin{aligned}
\sigma(\nabla_m(e)) &= \sigma(\mathrm{Fr}^{-\ell}(b)) \\
&= \mathrm{Fr}^{-\ell}(\sigma(b)) \\
&= \mathrm{Fr}^{-\ell}(f(b)) \\
&= f^{\mathrm{Fr}^{-\ell}}(\mathrm{Fr}^{-\ell}(b)) \\
&= f^{\mathrm{Fr}^{-\ell}}(\nabla_m(e)).
\end{aligned}$$

Here $f^{\mathrm{Fr}^{-\ell}}$ denotes the transform of $f \in k(x)$ obtained by applying $\mathrm{Fr}^{-\ell}$ to k. As k is perfect, $f^{\mathrm{Fr}^{-\ell}}$ is again a rational function over k, witnessing the rationality of $\mathrm{qftp}(\nabla_m(e)/k)$. □

We define the *dimension* of a rational type $p \in S_{\mathrm{qf}}(k)$ to be the transcendence degree of $k(a)$ over k for any $a \models p$.

By a *rational map* $\gamma : p \to q$, between rational types $p, q \in S_{\mathrm{qf}}(k)$, we mean that γ is a rational map over k and that for every (equivalently some) $a \models p$, $\gamma(a) \models q$. This is equivalent to asking that there are $a \models p$ and $b \models q$ with $b \in k(a)$. We say that p and q are *birationally equivalent* if there exist rational maps $\gamma : p \to q$ and $\delta : q \to p$ such that $\delta\gamma(a) = a$ and $\gamma\delta(b) = b$ for all (equivalently some) $a \models p$ and $b \models q$.

The following fact about the interaction between rational types and the fixed field is essentially a special case of [16, Prop. 26], but we recall the proof for convenience.

Proposition 2.4. *Suppose k is a difference field, $p \in S_{\mathrm{qf}}(k)$ is rational, and $\mathcal{C} = \mathrm{Fix}(\sigma)$. Then $\mathrm{qftp}(a/k, k(a) \cap \mathcal{C})$ isolates $\mathrm{qftp}(a/k, \mathcal{C})$, for any $a \models p$.*

Proof. Let us add constants for k to the language, for notational convenience. Also, in what follows we use tp^- to denote the $\mathcal{L}_{\mathrm{ring}}$-type.

Fix $a_1, a_2 \models p$ such that $D := k(a_1) \cap \mathcal{C} = k(a_2) \cap \mathcal{C}$, and $\mathrm{qftp}(a_1/D) = \mathrm{qftp}(a_2/D)$. We wish to show that $\mathrm{qftp}(a_1/\mathcal{C}) = \mathrm{qftp}(a_2/\mathcal{C})$.

Note that the σ-transforms of a_i are all given by k-rational functions as p is a rational type. It follows that $(\mathrm{tp}^-(a_i/\mathcal{C}) \cup p) \vdash \mathrm{qftp}(a_i/\mathcal{C})$. So it suffices to prove that $\mathrm{tp}^-(a_1/\mathcal{C}) = \mathrm{tp}^-(a_2/\mathcal{C})$. Let C_i be the canonical base for $\mathrm{tp}^-(a_i/\mathcal{C})$ in the sense of ACF. That is, C_i is the minimal field of definition of the Zariski locus of a_i over $\mathcal{C}$. Since $\mathrm{tp}^-(a_i/\mathcal{C})$ is the unique non-forking extension of its restriction to C_i, it suffices to show that $\mathrm{tp}^-(a_1/C_1, C_2) = \mathrm{tp}^-(a_2/C_1, C_2)$.

Note that $C_i \subseteq \mathrm{dcl}(a_i)$. Indeed, $\mathcal{C}$ is invariant under any automorphism of $(\mathcal{U}, \sigma)$, and hence if $\alpha \in \mathrm{Aut}(\mathcal{U}, \sigma)$ is such that $\alpha(a_i) = a_i$ then α preserves the Zariski locus of a_i over $\mathcal{C}$, and hence is the identity on C_i. We thus have

$$C_i \subseteq \mathrm{dcl}(a_i) \cap \mathcal{C} \subseteq \mathrm{acl}(a_i) \cap \mathcal{C} = k(a_i)^{\mathrm{alg}} \cap \mathcal{C} =: E_i.$$

It suffices to show, therefore, that $\mathrm{tp}^-(a_1/E_1, E_2) = \mathrm{tp}^-(a_2/E_1, E_2)$. In fact, we will show that $E_1 = E_2$ and that $\mathrm{tp}^-(a_1/E_1) = \mathrm{tp}^-(a_2/E_2)$.

Choose $c \in E_i$, and let $P(x)$ be its minimal polynomial over $k(a_i)$. Applying σ, we see that c is also a root of P^σ, hence $P = P^\sigma$ (noting that $k(a_i)$ is a σ-field by rationality of p). So P is over $\mathcal{C}$, and hence over D. This shows that $E_i = D^{\mathrm{alg}} \cap \mathcal{C} =: E$ is independent of i.

The above argument gives a bit more: whenever c is a finite tuple from E, $\mathrm{tp}^-(c/D) \vdash \mathrm{tp}^-(c/k(a_i))$ for $i = 1, 2$. Indeed, let Σ be the (finite) set of realisations of $\mathrm{tp}^-(c/k(a_1))$. Since $\sigma(c) = c$ and $\sigma(k(a_1)) \subseteq k(a_1)$, we have that $\mathrm{tp}^-(c/k(a_1))^\sigma \subseteq \mathrm{tp}^-(c/k(a_1))$, and hence $\Sigma \subseteq \sigma(\Sigma)$, which by finiteness forces $\Sigma = \sigma(\Sigma)$. This means that Σ is $\mathcal{L}_{\mathrm{ring}}$-definable over $k(a_1) \cap \mathcal{C} = D$. So $\mathrm{tp}^-(c/D) \vdash \mathrm{tp}^-(c/k(a_1))$, and similarly $\mathrm{tp}^-(c/D) \vdash \mathrm{tp}^-(c/k(a_2))$.

Finally, let us show that $\mathrm{tp}^-(a_1/E) = \mathrm{tp}^-(a_2/E)$. Given a finite tuple c from E, we show that there is a field-automorphism taking (a_1, c) to (a_2, c). This will suffice. Let τ be a field-automorphism of $\mathcal{U}$ fixing D pointwise and such that $\tau(a_1) = a_2$. Then, by the previous paragraph, $\mathrm{tp}^-(\tau(c)/k(a_2)) = \mathrm{tp}^-(c/k(a_2))$, witnessed, say, by a field-automorphism ι. Hence, $\iota\tau$ is a field-automorphism that takes (a_1, c) to (a_2, c), as desired. □

2.2. **Canonical bases.** Given a quantifier-free type $p = \mathrm{qftp}(a/k)$ over a perfect difference field k, the *canonical base of* p is the difference subfield of k generated by the minimal fields of definition of the Zariski loci $\mathrm{loc}(\nabla_n(a)/k)$, as $n \geq 0$ varies.[5] Note that this does not depend on the realisation of p chosen. We will denote the canonical base by $\mathrm{Cb}(a/k)$ or by $\mathrm{Cb}(p)$. When k is not necessarily perfect, we will still write $\mathrm{Cb}(a/k)$ to mean $\mathrm{Cb}(a/k^{\mathrm{perf}})$

Lemma 2.5. *Suppose k is a perfect difference field and $p = \mathrm{qftp}(a/k)$.*

(a) *For any difference subfield $L \subseteq k$, $a \mathop{\smile\!\!\!\!\!\!\raisebox{1ex}{$|$}}_L k$ if and only if $\mathrm{Cb}(a/k) \subseteq L^{\mathrm{alg}} \cap k$.*

(b) *If p is rational then $\mathrm{Cb}(p)$ is the difference field generated by the minimal field of definition of $\mathrm{loc}(\nabla(a)/k)$. That is, for rational types, one need only consider $n = 1$ in the definition of canonical base.*

[5]This disagrees mildly with the terminology of Chatzidakis and Hrushovski in [7, §2.13]; they take as the canonical base the perfect closure of what we are calling the canonical base.

(c) *Suppose p is rational and $q = \mathrm{qftp}(a/K)$, where $K \supseteq k$ is a perfect difference field extension. Then there is an $\ell \geq 0$ such that $\mathrm{Cb}(q)$ is contained in the perfect closure of the field generated over k by any ℓ independent realisations of q.*

Proof. For part (a) we note that

$$\begin{aligned} a \mathop{\smile\!\!\!\!\!\!\,|}_L k &\iff \mathrm{trdeg}(\nabla_n(a)/L) = \mathrm{trdeg}(\nabla_n(a)/k), \text{ for all } n \\ &\iff \mathrm{loc}(\nabla_n(a)/k) \text{ is over } L^{\mathrm{alg}} \cap k, \text{ for all } n \\ &\iff \mathrm{Cb}(a/k) \subseteq L^{\mathrm{alg}} \cap k. \end{aligned}$$

For part (b), by rationality, we have $\sigma(a) = f(a)$ for some rational function f over k. Let $V = \mathrm{loc}(a/k)$. Note that $\Gamma := \mathrm{loc}(\nabla_1(a)/k) \subseteq V \times V^\sigma$ is the graph of f viewed as a rational map on V. Let F be the minimal field of definition of Γ. One shows, inductively, that $\mathrm{loc}(\nabla_n(a)/k)$ is over $\langle F\rangle$. Consider $n = 2$. Then

$$\mathrm{loc}(\nabla_2(a)/k) = \mathrm{loc}(a, f(a), f^\sigma(f(a))/k) = \Gamma \times_{V^\sigma} \Gamma^\sigma.$$

where the fibre product here is taken with respect to the co-ordinate projections $\pi_2 : \Gamma \to V^\sigma$ and $\pi_1 : \Gamma^\sigma \to V^\sigma$. Since V, V^σ, and Γ are over F, and Γ^σ is over $\sigma(F)$, we get that $\mathrm{loc}(\nabla_2(a)/k)$ is over $\langle F\rangle$. This argument can be iterated.

Finally, for part (c), let F be the minimal field of definition of $\Gamma := \mathrm{loc}(\nabla(a)/K)$. By a general property of canonical bases in stable theories (see [20, Lemma 1.2.28]), here applied to ACF, there exists $\ell \geq 0$ such that F is contained in the perfect closure of any ℓ independent Zariski generic points of Γ. Let $a_1, \dots, a_\ell$ be independent realisation of q. So F is contained in the perfect closure of the field generated by $\nabla(a_1), \dots, \nabla(a_\ell)$, and hence

$$\begin{aligned} \mathrm{Cb}(q) &\subseteq \langle F\rangle \quad \text{by part (b) applied to } q, \text{ which is rational} \\ &\subseteq \langle \nabla(a_1), \dots, \nabla(a_\ell)\rangle^{\mathrm{perf}} \\ &\subseteq k(a_1, \dots, a_\ell)^{\mathrm{perf}} \end{aligned}$$

where the last containment uses that $q|_k = p$ is rational. □

By part (b) of the above lemma, we have that, in the rational case, $\mathrm{Cb}(p)$ is finitely generated as a difference field. In this case we may abuse notation by writing that $e = \mathrm{Cb}(p)$ to mean that $\langle e\rangle$, the difference field generated by e, is the canonical base of p.

2.3. **Nonorthogonality to the fixed field.**

Recall that a complete type $\mathrm{tp}(a/k)$ is *weakly orthogonal* to a k-definable set $\mathcal{C}$ if $a \mathop{\smile\!\!\!\!\!\!\,|}_k c$ for any finite tuple c from $\mathcal{C}$, and it is *orthogonal*[6] to $\mathcal{C}$ if every nonforking extension is weakly orthogonal to $\mathcal{C}$.

Proposition 2.6. *Suppose k is a difference field, $\mathcal{C} = \mathrm{Fix}(\sigma)$, and $\mathrm{qftp}(a/k)$ is rational. Then $\mathrm{tp}(a/k)$ is weakly orthogonal to $\mathcal{C}$ if and only if $k(a) \cap \mathcal{C} \subseteq k^{\mathrm{alg}}$.*

In particular, weak orthogonality to $\mathcal{C}$ depends only on $\mathrm{qftp}(a/k)$.

[6] Maybe *foreign* and *weakly foreign*, as in [20], are better terms than orthogonal and weakly orthogonal, as the latter are often, and were originally, used for a related but symmetric notion between complete types.

Proof. This is a corollary of Proposition 2.4.

Note that $\mathrm{tp}(a/k)$ is weakly orthogonal to $\mathcal{C}$ if and only if $\mathrm{tp}(a/k^{\mathrm{alg}})$ is. Also, as the fixed field of an algebraic difference-field extension is algebraic over the fixed field of the base, we also have that $k(a)\cap\mathcal{C}\subseteq k^{\mathrm{alg}}$ if and only if $k^{\mathrm{alg}}(a)\cap\mathcal{C}\subseteq k^{\mathrm{alg}}$. So, replacing k by k^{alg}, we may assume that k is algebraically closed.

The left-to-right implication is clear. For the converse, suppose $k(a)\cap\mathcal{C}\subseteq k$ and let c be a finite tuple from $\mathcal{C}$. Applying Proposition 2.4 to $p:=\mathrm{qftp}(a/k)$, we deduce that $\mathrm{qftp}(a/k)\vdash\mathrm{qftp}(a/kc)$. It follows by the existence of nonforking extensions, and the quantifier-free nature of nonforking, that $a\mathop{\smile\!\!\!\!\!\mid}_k c$. □

It therefore makes sense to talk about orthogonality to the fixed field for rational types $p\in S_{\mathrm{qf}}(k)$. Namely, p is *weakly orthogonal to* $\mathrm{Fix}(\sigma)$ if $a\mathop{\smile\!\!\!\!\!\mid}_k c$ for some (equivalently any) $a\models p$ and any finite tuple c from $\mathrm{Fix}(\sigma)$; and p is *orthogonal to* $\mathrm{Fix}(\sigma)$ if every nonforking extension is weakly orthogonal to $\mathrm{Fix}(\sigma)$. It turns out that to verify nonorthogonality one need not consider all nonforking extensions:

Proposition 2.7. *Suppose $p\in S_{\mathrm{qf}}(k)$ is rational, with k an algebraically closed difference field. Then p is nonorthogonal to* $\mathrm{Fix}(\sigma)$ *if and only if $p^{(\ell)}$ is not weakly orthogonal to* $\mathrm{Fix}(\sigma)$*, for some $\ell\geq 1$.*

Proof. This is a standard argument using canonical bases and forking calculus.

The right-to-left direction is clear. For the converse, suppose p is nonorthogonal to $\mathrm{Fix}(\sigma)$, and let this be witnessed by a difference field extension $K\supseteq k$, $a\models p$ with $a\mathop{\smile\!\!\!\!\!\mid}_k K$, and c from $\mathrm{Fix}(\sigma)$ with $a\not\mathop{\smile\!\!\!\!\!\mid}_K c$. Extending K, if necessary, we may assume that K is perfect. Let $e=\mathrm{Cb}(ac/K)$. Since $\mathrm{qftp}(ac/K)$ is rational, Lemma 2.5(c) gives us that there are independent realisations $a_1c_1, a_2c_2,\dots,a_\ell c_\ell$ of $\mathrm{qftp}(ac/K)$ such that $e\in k(a_1c_1,\dots,a_\ell c_\ell)^{\mathrm{perf}}$. (Here ℓ could be 0.) Moreover, we can choose the a_ic_i such that $ac\mathop{\smile\!\!\!\!\!\mid}_K a_1c_1\dots a_\ell c_\ell$.

We first claim that $a\not\mathop{\smile\!\!\!\!\!\mid}_{ke} c$. Indeed, from $a\not\mathop{\smile\!\!\!\!\!\mid}_K c$ we get that

$$a\not\mathop{\smile\!\!\!\!\!\mid}_{ke} Kc, \tag{1}$$

and from $ac\mathop{\smile\!\!\!\!\!\mid}_{ke} K$ we get that

$$a\mathop{\smile\!\!\!\!\!\mid}_{kec} K. \tag{2}$$

From (1) and (2) we get the desired $a\not\mathop{\smile\!\!\!\!\!\mid}_{ke} c$.

Next, we claim that $a\not\mathop{\smile\!\!\!\!\!\mid}_{ka_1c_1\dots a_\ell c_\ell} c$. Indeed, from $ac\mathop{\smile\!\!\!\!\!\mid}_{ke} K$ and

$$ac\mathop{\smile\!\!\!\!\!\mid}_K a_1c_1\dots a_\ell c_\ell$$

we get that $ac\mathop{\smile\!\!\!\!\!\mid}_{ke} Ka_1c_1\dots a_\ell c_\ell$. In particular, $a\mathop{\smile\!\!\!\!\!\mid}_{ke} a_1c_1\dots a_\ell c_\ell$. So, if it were the case that $a\mathop{\smile\!\!\!\!\!\mid}_{ka_1c_1\dots a_\ell c_\ell} c$ then we would have $a\mathop{\smile\!\!\!\!\!\mid}_{ke} ca_1c_1\dots a_\ell c_\ell$ which contradicts $a\not\mathop{\smile\!\!\!\!\!\mid}_{ke} c$. (Here we are using that $e\in k(a_1c_1,\dots,a_\ell c_\ell)^{\mathrm{alg}}$ in order to apply the transitivity of nonforking.)

Finally, from $a\not\mathop{\smile\!\!\!\!\!\mid}_{ka_1c_1\dots a_\ell c_\ell} c$ it follows that $(a,a_1,\dots,a_\ell)\not\mathop{\smile\!\!\!\!\!\mid}_k (c,c_1,\dots,c_\ell)$. This suffices as $(a,a_1,\dots,a_\ell)\models p^{(\ell+1)}$ and $(c,c_1,\dots,c_\ell)$ is a tuple from the fixed field. □

One of our main applications of binding groups is the existence of a bound on ℓ in the statement of the above proposition – this is Theorem 5.6 below.

We will make use of the following immediate corollary, which says that nonorthogonality to the fixed field is always witnessed by parameters that themselves realise rational types:

Corollary 2.8. *Suppose k is algebraically closed and $p = \operatorname{qftp}(a/k)$ is a rational type that is nonorthogonal to the fixed field. Then there is a tuple b such that $\operatorname{qftp}(b/k)$ is rational, $a \mathop{\smile\!\!\!\!\!\!\mid}_k b$, and $a \not\mathop{\smile\!\!\!\!\!\!\mid}_{kb} c$ for some tuple c from $\operatorname{Fix}(\sigma)$.*

Proof. Let $\ell \geq 1$ be such that $p^{(\ell)}$ is not weakly orthogonal to the fixed field, and let $(a = a_1, \dots, a_\ell) \models p^{(\ell)}$. Then $b = (a_2, \dots, a_\ell)$ has the desired properties. □

2.4. **Quantifier-free internality to the fixed field.** While orthogonality to the fixed field behaves well with the quantifier-free fragment of ACFA, at least for rational types, *internality* is harder to pin down because we do not quite understand dcl in ACFA. Following [7, §5], we therefore take a rather strong condition for our notion of quantifier-free internality:

Definition 2.9 (Quantifier-free internality)**.** Suppose k is a difference field, $\mathcal{C}$ is a quantifier-free k-definable set, and $p \in S_{\mathrm{qf}}(k)$ is stationary. We say that p is *qf-internal to* $\mathcal{C}$ if for all $a \models p$ there is a difference field extension $K \supseteq k$ such that $a \mathop{\smile\!\!\!\!\!\!\mid}_k K$ and $a \in K\langle c\rangle^{\mathrm{perf}}$ for some tuple c from $\mathcal{C}$.

The condition is strong in that we ask for a to be in the perfect closure of $K\langle c\rangle$ rather than simply to be quantifier-free definable from c over K. In fact, as the following proposition shows, this condition is even stronger than it looks when we restrict to rational types and the fixed field:

Proposition 2.10. *Suppose $p \in S_{\mathrm{qf}}(k)$ is rational, where k is an algebraically closed difference field. Then the following are equivalent:*

(i) *p is qf-internal to* $\operatorname{Fix}(\sigma)$.

(ii) *For all (equivalently for some) $a \models p$ there is a difference field extension $K \supseteq k$ such that $a \mathop{\smile\!\!\!\!\!\!\mid}_k K$ and $a \in K(c)$ for some tuple c from* $\operatorname{Fix}(\sigma)$.

(iii) *For all (equivalently for some) $a \models p$ there is a difference field extension $K \supseteq k$ and c from $\operatorname{Fix}(\sigma)$ such that $a \mathop{\smile\!\!\!\!\!\!\mid}_k K$ and $K(a) \subseteq K(c) \subseteq K(a)^{\mathrm{perf}}$.*

(iv) *For all $a \models p$ there exists $K = k(a_1, \dots, a_n, d)$ where*
   - *$a_1, \dots, a_n$ are independent realisations of p over k, and*
   - *d is from* $\operatorname{Fix}(\sigma)$,

   such that $a \mathop{\smile\!\!\!\!\!\!\mid}_k K$ and $K(a)^{\mathrm{perf}} = K(c)^{\mathrm{perf}}$, for some c from $\operatorname{Fix}(\sigma)$.

Proof. Let $\mathcal{C} := \operatorname{Fix}(\sigma)$.

Assuming (i) we prove the "for all $a \models p$" version of (ii). Let $a \models p$. By definition, there is a difference field extension $K \supseteq k$ such that $a \mathop{\smile\!\!\!\!\!\!\mid}_k K$ and $a \in K\langle c\rangle^{\mathrm{perf}}$ where c is a tuple from $\mathcal{C}$. We may assume that K is perfect. As c is in the fixed field, we have that $a \in K(c)^{\mathrm{perf}}$. It follows that for some $\ell \geq 0$ and $c' := \operatorname{Fr}^{-\ell}(c)$, we have $a \in K(c')$. As $\mathcal{C}$ is perfect, c' is also from $\mathcal{C}$.

Assuming the "for some $a \models p$" version of (ii) we prove the "for some $a \models p$" version of (iii). Let $a \models p$ and $K \supseteq k$ be such that $a \mathop{\smile\!\!\!\!\!\!\mid}_k K$ and $a \in K(c)$, where c is a tuple from $\mathcal{C}$. Replacing K by $K(c_0)$ where c_0 is a maximal sub-tuple of c such that $a \mathop{\smile\!\!\!\!\!\!\mid}_k Kc_0$, we may assume that $c \in K(a)^{\mathrm{alg}}$. We may also assume that K is perfect. Let $e = \operatorname{Cb}(c/K(a)) \in K(a)^{\mathrm{perf}}$. In terms of the pure algebraically closed field $\mathcal{U}$, this means that e is a code for the (finite) set E of Ka-conjugates of c. As σ fixes c, it fixes E, and hence e is a tuple from $\mathcal{C}$. On the other hand, as $a \in K(c)$ we have that $a = f(c)$ for some f a rational function over K. So $a = f(c')$ for *every* $c' \in E$. Hence, any field automorphism fixing $K(e)$ will fix a, proving that $a \in K(e)^{\mathrm{perf}}$. Replacing e by some $e' = \mathrm{Fr}^{-\ell}(e')$ we have that $a \in K(e')$, and it is still the case that e' is from $\mathcal{C}$ and that $e' \in K(a)^{\mathrm{perf}}$.

Next, we show that the "for some" version of (iii) implies the "for all" version. Fix $a \models p$ and $K \supseteq k$ satisfying (iii). Let $a' \models p$ be another realisation. Choose $K' \models \mathrm{tp}(K/ka)$ with $K' \mathop{\smile\!\!\!\!\!\!\;|}_{ka} a'$. Then $K' \mathop{\smile\!\!\!\!\!\!\;|}_{k} aa'$. By Lemma 2.2, $\mathrm{qftp}(Ka/k) = \mathrm{qftp}(K'a'/k)$. Hence, $K(a)$ and $K'(a')$ are difference-field isomorphic over k. The fact that there is c from $\mathcal{C}$ such that $K(a) \subseteq K(c) \subseteq K(a)^{\mathrm{perf}}$ implies that there must be some c' from $\mathcal{C}$, namely the image of c under the above isomorphism, such that $K'(a') \subseteq K'(c') \subseteq K'(a')^{\mathrm{perf}}$, as desired.

(iii) $\Longrightarrow$ (iv). This is similar to the proof of Proposition 2.7. Fix $a \models p$, and let $K \supseteq k$ and c from $\mathrm{Fix}(\sigma)$ satisfying (iii). We may assume that K is perfect. Let $e = \mathrm{Cb}(ac/K)$. The fact that $K(a)^{\mathrm{perf}} = K(c)^{\mathrm{perf}}$ is reflected in $\mathrm{loc}(a, c/K)$, whose minimal field of definition is contained in $k\langle e\rangle$. It follows that $k\langle e\rangle(a)^{\mathrm{perf}} = k\langle e\rangle(c)^{\mathrm{perf}}$. On the other hand, Lemma 2.5(c) gives us that there are independent realisations $a_1c_1, a_2c_2, \dots, a_\ell c_\ell$ of $\mathrm{qftp}(ac/K)$ such that $e \in k(a_1c_1, \dots, a_\ell c_\ell)^{\mathrm{perf}}$. Moreover, we can choose the a_ic_i such that $a \mathop{\smile\!\!\!\!\!\!\;|}_{K} a_1c_1 \dots a_\ell c_\ell$ and hence $a \mathop{\smile\!\!\!\!\!\!\;|}_{k} a_1c_1 \dots a_\ell c_\ell$. Hence, letting $K' := k(a_1c_1, \dots, a_\ell c_\ell)$, we get that $a \mathop{\smile\!\!\!\!\!\!\;|}_{k} K'$ and $K'(a)^{\mathrm{perf}} = K'(c)^{\mathrm{perf}}$. Finally, observe that K' is of the form called for by (iv).

(iv) $\Longrightarrow$ (i) is clear. $\square$

The following proposition shows that qf-internality to the fixed field arises whenever there is nonorthogonality.

Proposition 2.11. *Suppose p is a rational type over an algebraically closed difference field k. The following are equivalent:*

(i) *p is nonorthogonal to* $\mathrm{Fix}(\sigma)$.
(ii) *There is a rational map $p \to q$ where $q \in S_{\mathrm{qf}}(k)$ is positive-dimensional rational and qf-internal to* $\mathrm{Fix}(\sigma)$.

Proof. Let $\mathcal{C} = \mathrm{Fix}(\sigma)$.

(i) $\Longrightarrow$ (ii). Let $a \models p$. By Corollary 2.8, there is b such that $\mathrm{qftp}(b/k)$ is rational, $a \mathop{\smile\!\!\!\!\!\!\;|}_{k} b$, and $a \not\mathop{\smile\!\!\!\!\!\!\;|}_{kb} c$ for some tuple c from $\mathcal{C}$. Consider $r := \mathrm{qftp}(bc/k(a))$ and the canonical base $e := \mathrm{Cb}(r) \in k(a)^{\mathrm{perf}}$. As r is rational and $bc \not\mathop{\smile\!\!\!\!\!\!\;|}_{k} a$, Lemma 2.5(a) implies that $e \notin k$. Moreover, by part (c) of that lemma, there are independent realisations $b_1c_1, \dots, b_nc_n$ of r, such that $e \in k(b_1c_1, \dots, b_nc_n)^{\mathrm{perf}}$.

Since $e \in k(a)^{\mathrm{perf}}$ and p is rational, Lemma 2.3 implies that $\mathrm{qftp}(\nabla_m(e)/k)$ is rational, for some $m \geq 0$. There is also some $\ell \geq 0$ such that $\mathrm{Fr}^{\ell}(\nabla_m(e)) \in k(a)$. Let $e' := \mathrm{Fr}^{\ell}(\nabla_m(e))$ and set $q := \mathrm{qftp}(e'/k)$. Then q is rational and positive-dimensional, and we have a rational map $p \to q$. It remains to show that q is qf-internal to $\mathcal{C}$.

Let $b' := (b_1, \dots, b_n)$. We claim that $e' \mathop{\smile\!\!\!\!\!\!\;|}_{k} b'$. Indeed, as $b_1, \dots, b_n$ are independent realisations of $\mathrm{qftp}(b/k(a))$, and $b \mathop{\smile\!\!\!\!\!\!\;|}_{k} a$, it follows that $a \mathop{\smile\!\!\!\!\!\!\;|}_{k} b'$. Since $e' \in k(a)$, we get $e' \mathop{\smile\!\!\!\!\!\!\;|}_{k} b'$, as desired.

Let $c' := (c_1, \dots, c_n)$. Recall that $e \in k(b'c')^{\mathrm{perf}}$. Increasing ℓ if necessary, we may assume that $e' \in k(b'c')$. As c' is a tuple from $\mathcal{C}$, this witnesses that q is qf-internal to $\mathcal{C}$.

(ii) $\Longrightarrow$ (i). Fix $e \models q$. By qf-internality to $\mathcal{C}$, there is $K \supseteq k$, and c from $\mathcal{C}$, such that $e \mathop{\smile\!\!\!\!\!\mid}_k K$ and $e \in K(c)$. Since $\dim(q) > 0$, we have that $e \notin K^{\mathrm{alg}}$, and hence $e \not\mathop{\smile\!\!\!\!\!\mid}_K c$. Choose $a \models p$ such that $e \in k(a)$ and $a \mathop{\smile\!\!\!\!\!\mid}_{ke} K$. We get that $a \mathop{\smile\!\!\!\!\!\mid}_k K$ and $a \not\mathop{\smile\!\!\!\!\!\mid}_K c$, witnessing that p is nonorthogonal to $\mathcal{C}$. □

3. Algebraic dynamics

As we will see, studying rational types corresponds to a certain general setting for algebraic dynamics. Given a perfect difference field (k, σ), by a *rational σ-variety over k* we mean an irreducible variety V over k equipped with a dominant rational map $\phi : V \dashrightarrow V^\sigma$ over k. Here V^σ denotes the transform of V with respect to the action of σ on the field of definition k. Note that the rational σ-variety structures on V correspond precisely to the extensions of σ from k to the rational function field $k(V)$, given by $f \mapsto f^\sigma \circ \phi$.

Let us emphasise our (somewhat unfortunate) convention that while varieties need not be irreducible in general, the underlying variety of a rational σ-variety is assumed to be irreducible.

If σ is trivial on k then we say that (V, ϕ) is a *rational dynamical system*; in that case $\phi : V \dashrightarrow V$ is a rational transformation of V. This is often the setting that algebraic dynamics is restricted to, but we will work generally.

The study of σ-varieties as a geometric category in its own right was initiated in [17] and applied to algebraic dynamics in [8, 9].

We will sometimes be interested in the cartesian powers of a rational σ-variety, which we will tend to denote either by $(V, \phi)^n$ or by (V^n, ϕ). We mean, of course, the rational σ-variety whose underlying variety is the cartesian power V^n equipped with the dominant rational map $V^n \dashrightarrow (V^n)^\sigma = (V^\sigma)^n$ given co-ordinatewise by ϕ, but which we continue to denote by ϕ. In order to preserve irreducibility on passing to cartesian powers (or products) we will only do so when V is absolutely irreducible.

An *invariant* subvariety of (V, ϕ) is an irreducible subvariety $X \subseteq V$ over k such that $X \cap \mathrm{dom}(\phi)$ is nonempty and $\phi(X)$ is Zariski dense in X^σ. Equivalently, $(X, \phi|_X)$ is itself a rational σ-variety.

By an *equivariant rational map* $g : (V, \phi) \dashrightarrow (W, \psi)$ we mean that $g : V \dashrightarrow W$ is a rational map and that $\psi g = g^\sigma \phi$ as rational maps from V to W^σ. The equivariant map g is said to be *dominant*, *birational*, etc., if it is such as a rational map of algebraic varieties. Note that the inverse of an equivariant birational map of σ-varieties is itself equivariant.

The following straightforward observation is often used:

Lemma 3.1. *Suppose (V, ϕ) and (W, ψ) are absolutely irreducible σ-varieties over (k, σ), and $f : (V, \phi) \dashrightarrow (W, \psi)$ is a dominant equivariant rational map. Then the graph of f is an invariant subvariety of $(V \times W, \phi \times \psi)$.*

Proof. Let $\Gamma(f) \subseteq V \times W$ denote the graph of f. It is an irreducible closed subvariety that projects dominantly onto both V and W, and such that the projection onto V is a birational equivalence.

Since $f : V \dashrightarrow W$ is dominant it takes the nonempty Zariski open subset $\mathrm{dom}(\phi) \cap \mathrm{dom}(f)$ of V to a Zariski dense subset of W. In particular, there exists a point $v \in \mathrm{dom}(\phi) \cap \mathrm{dom}(f)$ such that $f(v) \in \mathrm{dom}(\psi)$. Hence, $(v, f(v))$ witnesses that $\Gamma(f) \cap \mathrm{dom}(\phi \times \psi)$ is nonempty.

Next, we show that $\phi \times \psi$ takes $\Gamma(f)$ to $\Gamma(f)^\sigma$. Fix $(v, f(v)) \in \Gamma(f) \cap \mathrm{dom}(\phi \times \psi)$. Then

$$\begin{aligned}(\phi \times \psi)(v, f(v)) &= \big(\phi(v), \psi(f(v))\big) \\ &= \big(\phi(v), f^\sigma(\phi(v))\big) \quad \text{by equivariance} \\ &\in \Gamma(f^\sigma) \\ &= \Gamma(f)^\sigma.\end{aligned}$$

Since $\Gamma(f) \cap \mathrm{dom}(\phi \times \psi)$ is Zariski dense in $\Gamma(f)$, it follows that $\phi \times \psi$ takes all of $\Gamma(f)$ to $\Gamma(f)^\sigma$.

Finally, to show Zariski-density of the image, work over any field extension $K \supseteq k$ and let $v \in V$ be Zariski generic, so that $(v, f(v))$ is Zariski generic in $\Gamma(f)$. Then, by dominance of $\phi : V \dashrightarrow V^\sigma$, we have that $\phi(v)$ is Zariski generic in V^σ. And so, $\big(\phi(v), f^\sigma(\phi(v))\big)$ is Zariski generic in $\Gamma(f^\sigma)$. But $\big(\phi(v), f^\sigma(\phi(v))\big) = (\phi \times \psi)(v, f(v))$ and $\Gamma(f^\sigma) = \Gamma(f)^\sigma$, so that $(\phi \times \psi)(v, f(v))$ is Zariski generic in $\Gamma(f)^\sigma$, as desired. □

Fix, now, a sufficiently saturated model $(\mathcal{U}, \sigma) \models \mathrm{ACFA}$ extending (k, σ). Associated to an absolutely irreducible rational σ-variety (V, ϕ) is the quantifier-free $\mathcal{L}_\sigma$-definable set

$$(V, \phi)^\sharp := \{a \in \mathrm{dom}(\phi) : \sigma(a) = \phi(a)\}$$

with parameters from k. Existential closedness of $(\mathcal{U}, \sigma)$, along with absolute irreducibility of V, ensures that that this set is nonempty. In fact, it is Zariski dense in V. Moreover, we can associate to (V, ϕ) a rational type $p(x) \in S_{\mathrm{qf}}(k)$, the *generic quantifier-free type of* (V, ϕ) *over* k, which is determined by saying that x is Zariski generic in V over k and $x \in (V, \phi)^\sharp$. By a *generic point of* (V, ϕ) we mean a realisation of this generic type.

Every rational type arises in this way. Indeed, given $p \in S_{\mathrm{qf}}(k)$ rational, fix $a \models p$, let $V = \mathrm{loc}(a/k)$ be the Zariski locus of a over k, and take $\phi : V \dashrightarrow V^\sigma$ to be the rational map whose graph is $\mathrm{loc}(a, \sigma(a)/k)$. That this locus is the graph of a rational map is a consequence of the fact that p is a rational type. Then p is the generic quantifier-free type of (V, ϕ).

These constructions are functorial: given rational σ-varieties (V, ϕ) and (W, ψ), with generic quantifier-free types p and q, respectively, dominant equivariant rational maps $(V, \phi) \dashrightarrow (W, \psi)$ correspond (via restriction) to rational maps $p \to q$ in the sense of Section 2.1.

We will be considering algebraic families of σ-varieties, and we record for future use the fact that they can be made canonical (at least in characteristic zero):

Proposition 3.2. *Suppose* $\mathrm{char}(k) = 0$ *and* (V, ϕ) *and* (Z, ψ) *are absolutely irreducible rational* σ*-varieties over* (k, σ)*, and* $\Gamma \subseteq V \times Z$ *is an irreducible subvariety which is invariant for* $\phi \times \psi$*, and such that* $\pi_1 : \Gamma \to Z$ *is dominant. Then there exists a rational* σ*-variety* (Z_0, ψ_0) *and an equivariant dominant rational map* $\mu : (Z, \psi) \dashrightarrow (Z_0, \psi_0)$ *such that for general* $a, a' \in Z$*,*

$$\mu(a) = \mu(a') \iff \Gamma_a = \Gamma_{a'}.$$

Proof. From $\Gamma \subseteq V \times Z$, Fact 1.9 provides a dominant rational map $\mu : Z \dashrightarrow Z_0$ such that, for general $a, a' \in Z$, $\mu(a) = \mu(a')$ if and only if $\Gamma_a = \Gamma_{a'}$. It remains, therefore, to put a σ-variety structure on Z_0 such that μ is equivariant.

Let $f : Z \dashrightarrow Z_0$ be $\mu^\sigma \circ \psi$. We claim that for general $a, b \in Z$, if $\Gamma_a = \Gamma_b$ then $f(a) = f(b)$. Since $\mu^\sigma = \mu_{\Gamma^\sigma}$ is a quotient map for Γ^σ, it suffices to show that if $\Gamma_a = \Gamma_b$ then ${\Gamma^\sigma}_{\psi(a)} = {\Gamma^\sigma}_{\psi(b)}$. Fix $x \in \Gamma^\sigma_{\psi(a)}$ a Zariski generic point over $k(a, b)$. Then $(\psi(a), x) \in \Gamma^\sigma$ is Zariski generic over k, and hence, by $(\psi \times \phi)$-invariance, is of the form $(\psi(a), \phi(v))$ for some $v \in V$ such that $(a, v) \in \Gamma$. It follows that $v \in \Gamma_a = \Gamma_b$, and so $(b, v) \in \Gamma$ and Zariski generic over k. By $(\psi \times \phi)$-invariance again, $x = \phi(v) \in \Gamma^\sigma_{\psi(b)}$. As this is the case for all Zariski generic points over $k(a, b)$, it follows that ${\Gamma^\sigma}_{\psi(a)} \subseteq {\Gamma^\sigma}_{\psi(b)}$, and we conclude ${\Gamma^\sigma}_{\psi(a)} = {\Gamma^\sigma}_{\psi(b)}$, by symmetry.

It now follows from the universality of μ, given by Fact 1.9, that there is a unique dominant rational map $\psi_0 : Z_0 \to {Z_0}^\sigma$ with $\psi_0 \circ \mu = f = \mu^\sigma \circ \psi$, as required. □

3.1. **Invariant rational functions.** A special case of equivariant rational maps that is of interest are those from (V, ϕ) to the affine line equipped with the trivial dynamics, $\lambda : (V, \phi) \dashrightarrow (\mathbb{A}^1, \mathrm{id})$. These are called the *invariant rational functions* on (V, ϕ); they are those rational functions, $\lambda \in k(V)$, such that $\lambda = \lambda^\sigma \phi$.

Lemma 3.3. *Suppose a is a generic point of (V, ϕ), and $\lambda \in k(V)$. Then λ is an invariant rational function of (V, ϕ) if and only if $\lambda(a) \in \mathrm{Fix}(\sigma)$.*

Proof. Note that $\sigma(\lambda(a)) = \lambda^\sigma(\sigma(a)) = \lambda^\sigma(\phi(a))$, where the final equality is because $a \in (V, \phi)^\sharp$. Hence, if $\lambda = \lambda^\sigma \phi$ then $\lambda(a) \in \mathrm{Fix}(\sigma)$. Conversely, if $\lambda(a) \in \mathrm{Fix}(\sigma)$ then $\lambda(a) = \lambda^\sigma(\phi(a))$. But, as a is Zariski generic in V, it follows that $\lambda = \lambda^\sigma \phi$ as rational functions on V. □

We have the following geometric characterisation of nonorthogonality to the fixed field in terms of invariant rational functions:

Proposition 3.4. *Suppose (V, ϕ) is a rational σ-variety over an algebraically closed difference field (k, σ) with quantifier-free generic type p.*

(a) *p is weakly orthogonal to $\mathrm{Fix}(\sigma)$ if and only if (V, ϕ) admits no nonconstant invariant rational functions.*
(b) *p is orthogonal to $\mathrm{Fix}(\sigma)$ if and only if, for every rational σ-variety (W, ψ) over k, the invariant rational functions on $(V, \phi) \times (W, \psi)$ are all pullbacks of invariant rational functions on (W, ψ).*

Proof. Part (a) is just Proposition 2.6 together with the assumption that k is algebraically closed.

Suppose, now, that (W, ψ) is another rational σ-variety over k and f is an invariant rational function on $(V, \phi) \times (W, \psi)$. Let $a \models p$ and b generic in (W, ψ), with $a \mathop{\smile\!\!\!\!\!\!|}_k b$. Then a is generic in (V, ϕ) over $K := k(b)$, and $\lambda := f(-, b)$ is a rational function on V over K with the property that

$$\sigma(\lambda(a)) = \lambda^\sigma(\sigma(a)) = f^\sigma(\sigma(a), \sigma(b)) = \sigma(f(a, b)) = f(a, b) = \lambda(a).$$

That is, λ is an invariant rational function on (V, ϕ) over K. If p is orthogonal to $\mathrm{Fix}(\sigma)$ then the nonforking extension of p to K is weakly orthogonal to $\mathrm{Fix}(\sigma)$, and hence, by part (a), we have that $\lambda \in K^{\mathrm{alg}}$. It follows from the absolute irreducibility of V that $K = k(W)$ is relatively algebraically closed in $k(V \times W)$, and hence $\lambda \in K$.

Writing $\lambda = g(b)$ we see that g is an invariant rational function on (W, ψ) and f is the pullback of g. This proves the left-to-right implication of part (b).

For the converse, suppose p is nonorthogonal to $\mathrm{Fix}(\sigma)$, and let this be witnessed by $B \supseteq k$ and c from $\mathrm{Fix}(\sigma)$ such that $a \mathop{\smile\!\!\!\!\!\mid}_k B$ and $a \not\mathop{\smile\!\!\!\!\!\mid}_B c$. By Corollary 2.8, we can choose B of the form kb where $q := \mathrm{qftp}(b/k)$ is rational. Let (W, ψ) be a rational σ-variety over k such that q is the generic quantifier-free type of (W, ψ). The fact that $a \not\mathop{\smile\!\!\!\!\!\mid}_{kb} c$ tells us that $\mathrm{tp}(a/K)$, where $K := k(b)$, is not weakly orthogonal to $\mathrm{Fix}(\sigma)$. Hence, by Proposition 2.6, there exists $\lambda \in K(a) \cap \mathrm{Fix}(\sigma) \setminus K^{\mathrm{alg}}$. Writing $\lambda = f(a, b)$ we have that f is an invariant rational function on $(V, \phi) \times (W, \psi)$. The fact that $\lambda \notin K$ tells us that f is not the pullback of a rational function on W. $\square$

3.2. **Isotriviality.** The geometric counterpart to quantifier-free internality to the fixed field is isotriviality in the following natural sense:

Definition 3.5. Suppose (V, ϕ) is an absolutely irreducible rational σ-variety over a perfect difference field k. By a *trivialisation of* (V, ϕ) *over* k we mean

- a rational σ-variety (Z, ψ),
- an invariant subvariety Y of $(\mathbb{A}^\ell \times Z, \mathrm{id} \times \psi)$, and,
- an equivariant birational map

$$\begin{array}{ccc} (V \times Z, \phi \times \psi) & \overset{g}{\underset{\cong}{\dashrightarrow}} & (Y, \mathrm{id} \times \psi) \\ & \searrow \qquad \swarrow & \\ & (Z, \psi) & \end{array}$$

all defined over k. We say that (V, ϕ) is *isotrivial* if there exists a trivialisation.

Proposition 3.6. *Suppose* (V, ϕ) *is a rational* σ*-variety over an algebraically closed difference field* (k, σ) *with generic quantifier-free type* p*.*

(a) *If* (V, ϕ) *is isotrivial then* p *is qf-internal to* $\mathrm{Fix}(\sigma)$*.*
(b) *Suppose* $\mathrm{char}(k) = 0$*. If* p *is qf-internal to* $\mathrm{Fix}(\sigma)$ *then* (V, ϕ) *is isotrivial.*

Proof. Suppose

$$\begin{array}{ccc} (V \times Z, \phi \times \psi) & \overset{g}{\underset{\cong}{\dashrightarrow}} & (Y, \mathrm{id} \times \psi) \\ & \searrow \qquad \swarrow & \\ & (Z, \psi) & \end{array}$$

is a trivialisation of (V, ϕ) over k. Choose a generic in (V, ϕ) and b be generic in (Z, ψ), with $a \mathop{\smile\!\!\!\!\!\mid}_k b$. Hence, (a, b) is generic in $(V \times Z, \phi \times \psi)$, so that $g(a, b)$ is generic in $(Y, \mathrm{id} \times \psi)$. In particular, $g(a, b) \in (Y, \mathrm{id} \times \psi)^\sharp$, so that $g(a, b) = (c, b)$ for some $c \in \mathrm{Fix}(\sigma)^\ell$. Setting $K := k(b)$ we have that $a \mathop{\smile\!\!\!\!\!\mid}_k K$ and $a \in K(c)$, the latter witnessed by g_b^{-1}. This shows that $\mathrm{qftp}(a/k) = p$ is qf-internal to $\mathrm{Fix}(\sigma)$.

Suppose, now, that $\mathrm{char}(k) = 0$ and p is qf-internal to $\mathrm{Fix}(\sigma)$. Let $a \models p$. Using condition (iv) of Proposition 2.10 we have $K \supseteq k$ with $a \mathop{\smile\!\!\!\!\!\mid}_k K$, and an ℓ-tuple c from $\mathrm{Fix}(\sigma)$ such that $K(a) = K(c)$. (This is where characteristic zero is being used, we do not have to take the perfect clsoure.) Moreover, part of condition (iv) of Proposition 2.10 tells us that we can take K to be of the form $K = k(b)$, where $r := \mathrm{qftp}(b/k)$ is rational. Let (Z, ψ) be the rational σ-variety over k whose quantifier-free generic type is r. Let $Y := \mathrm{loc}(c, b/k)$. It follows that Y is $(\mathrm{id} \times \psi)$-invariant in $\mathbb{A}^\ell \times Z$. Note that $\mathrm{loc}(a, b/k) = V \times Z$ as $a \mathop{\smile\!\!\!\!\!\!\mid}_k K$. Let $g : V \times Z \dashrightarrow Y$ be the birational map such that $g(-, b)$ witnesses $K(a) = K(c)$. Note that

$$(\mathrm{id} \times \psi) g(a, b) = (c, \psi(b))$$

and also that

$$\begin{aligned} g^\sigma(\phi \times \psi)(a, b) &= g^\sigma(\phi(a), \psi(b)) \\ &= g^\sigma(\sigma(a), \sigma(b)) \quad \text{as } (a, b) \text{ are } \sharp\text{-points} \\ &= \sigma(g(a, b)) \\ &= \sigma(c, b) \\ &= (c, \psi(b)). \end{aligned}$$

As (a, b) is Zariski generic in $V \times Z$ over k, this means that $(\mathrm{id} \times \psi)g = g^\sigma(\phi \times \psi)$. So g is equivariant. We have thus produced a trivialisation. □

The above proof gives us a little more that it is worth extracting for later use:

Corollary 3.7. *Suppose (V, ϕ) is a rational σ-variety over an algebraically closed difference field (k, σ) of characteristic zero. If (V, ϕ) is isotrivial then there exists a trivialisation where Z is an invariant subvariety of $(V^n \times \mathbb{A}^m, \phi \times \mathrm{id})$, for some $n, m \geq 0$, that projects dominantly onto V^n, and such that ψ is the restriction of $\phi \times \mathrm{id}$ to Z.*

Proof. Let p be the generic quantifier-free type of (V, ϕ) over k. By Proposition 3.6(a), p is qf-internal to $\mathrm{Fix}(\sigma)$. Now, the proof of Proposition 3.6(b) constructs a trivialisation of (V, ϕ) that has the additional property we are seeking. Indeed, condition (iv) of Proposition 2.10 ensures that the tuple b used in that construction is of the form $b = (a_1, \ldots, a_n, d)$ where $a_1, \ldots, a_n$ are independent realisations of p and d is an m-tuple from $\mathrm{Fix}(\sigma)$. It follows that $Z = \mathrm{loc}(b/k)$ and ψ are of the desired form. □

Question 3.8. The statement of Corollary 3.7 does not mention any model theory, but its proof goes via the model-theoretic arguments of §2.4. Is there a purely algebraic dynamics proof of this result? Such a proof might very well extend to arbitrary characteristic.

We also obtain a geometric formulation of Proposition 2.11 that may be of independent interest:

Corollary 3.9. *Suppose (V, ϕ) is a rational σ-variety over an algebraically closed difference field (k, σ). Suppose $\mathrm{char}(k) = 0$. The following are equivalent:*

(i) *There is a rational σ-variety (W, ψ) over k such that $(V, \phi) \times (W, \psi)$ admits an invariant rational function that is not the pullback of a rational function on W.*

(ii) *There is a dominant equivariant rational map $(V, \phi) \dashrightarrow (V', \phi')$ over k with (V', ϕ') isotrivial and positive-dimensional.*

Proof. This is a matter of putting together Propositions 2.11, 3.4, and 3.6.

Let p be the generic quantifier-free type of (V, ϕ). Condition (i) is equivalent to p being nonorthogonal to $\mathrm{Fix}(\sigma)$, by 3.4(b). By 2.11, this is in turn equivalent to the existence of a rational map $p \to q$ where $q \in S_{\mathrm{qf}}(k)$ is positive-dimensional rational and qf-internal to $\mathrm{Fix}(\sigma)$. Such $p \to q$ corresponds to a dominant equivariant rational map $(V, \phi) \dashrightarrow (V', \phi')$. That q is positive-dimensional is equivalent to V' being positive-dimensional, and that q is qf-internal to $\mathrm{Fix}(\sigma)$ is equivalent to $(V'\phi')$ being isotrivial. The latter is by 3.6 as we are in characteristic zero. □

Remark 3.10. Like Corollary 3.7, Corollary 3.9 does not mention any model theory, but we have given a model-theoretic proof. In this case, however, we do see an algebraic-geometric approach, along the following lines: After taking projective closures, a rational function λ on $V \times W$ induces a rational map f_λ from V to the Hilbert scheme of rational functions on W, given by $a \mapsto \lambda(a, -)$, whose image we can take to be V'. If λ is invariant for $\phi \times \psi$, and assuming that ψ is birational, we can give V' a σ-variety structure ϕ' defined by precomposition with ψ^{-1}. It is then not hard to verify that $f_\lambda : (V, \phi) \dashrightarrow (V', \phi')$ is equivariant and that (V', ϕ') is isotrivial. Finally, if λ does not arise as the pullback of a rational function on W then V' will be positive-dimensional.

4. The binding group theorems

In this section we prove Theorems 1.3 and 1.5. For that purpose we now restrict entirely to characteristic zero. Fix an algebraically closed difference field (k, σ) and work in a sufficiently saturated model $(\mathcal{U}, \sigma) \models \mathrm{ACFA}_0$ extending (k, σ), whose fixed field we denote by $\mathcal{C} := \mathrm{Fix}(\sigma)$.

Fix also a rational type $q(x) \in S_{\mathrm{qf}}(k)$ that is qf-internal to $\mathcal{C}$.

Recall from Definition 1.4 that the *quantifier-free binding group of q with respect to $\mathcal{C}$*, denote by $\mathrm{Aut}_{\mathrm{qf}}(q/\mathcal{C})$, is the (abstract) subgroup of permutations α of $q(\mathcal{U})$ satisfying:

($\star$) For any quantifier-free formula $\theta(x, y)$ over k, any tuple a of realisations of q, and any tuple c of elements of $\mathcal{C}$,

$$\models \theta(a, c) \iff \models \theta(\alpha(a), c).$$

It is not hard to see that the set of such permutations does form a subgroup.

Remark 4.1. (a) The binding group can be understood as an automorphism group for a certain auxiliary two-sorted structure $\mathcal{Q}$, whose sorts are $q(\mathcal{U})$ and $\mathcal{C}$ and where the language is made up of a predicate symbol for each relatively quantifier-free k-definable subset of $q(\mathcal{U})^n \times \mathcal{C}^m$ in $(\mathcal{U}, \sigma)$. Let $\mathrm{Aut}(\mathcal{Q}/\mathcal{C}) := \{\alpha \in \mathrm{Aut}(\mathcal{Q}) : \alpha|_{\mathcal{C}} = \mathrm{id}_{\mathcal{C}}\}$. It can be easily seen that the map $\mathrm{Aut}(\mathcal{Q}/\mathcal{C}) \to \mathrm{Aut}_{\mathrm{qf}}(q/\mathcal{C})$, given by restriction to $q(\mathcal{U})$, is an isomorphism of groups that preserves the action on $q(\mathcal{U})$.

(b) As q is rational we need only consider quantifier-free $\mathcal{L}_{\mathrm{ring}}$-formulas in ($\star$).

(c) The binding group is a birational invariant in the sense that any birational equivalence, $\gamma : q \to \widehat{q}$, between rational types, lifts canonically to an isomorphism of group actions,

$$\gamma^* : \mathrm{Aut}_{\mathrm{qf}}(q/\mathcal{C}) \to \mathrm{Aut}_{\mathrm{qf}}(\widehat{q}/\mathcal{C}),$$

given by $\alpha \mapsto \gamma\alpha\gamma^{-1}$.

We focus, first of all, on proving that $\mathrm{Aut}_{\mathrm{qf}}(q/\mathcal{C})$, along with its action on $q(\mathcal{U})$, has a quantifier-free definable avatar in $(\mathcal{U},\sigma)$. This is the main clause of Theorem 1.5, and will occupy us for most of the section. Our construction is informed by those of Hrushovski [14] and the first author [16], but with particular attention paid to the birational geometric setting in which we find ourselves.

Since q is rational it is the generic quantifier-free type of a rational σ-variety (V,ϕ) over k. Since q is qf-internal to $\mathcal{C}$, and we are in characteristic zero, Proposition 3.6 tells us that (V,ϕ) is isotrivial. In fact, by Corollary 3.7, we have a trivialisation

$$\begin{array}{ccc} (V\times\widetilde{Z},\phi\times\widetilde{\psi}) & \overset{\widetilde{g}}{\underset{\cong}{\dashrightarrow}} & (\widetilde{Y},\mathrm{id}\times\widetilde{\psi}) \\ & \searrow \quad \swarrow & \\ & (\widetilde{Z},\widetilde{\psi}) & \end{array}$$

where $\widetilde{Y}\subseteq\mathbb{A}^\ell\times\widetilde{Z}$ is invariant for $\mathrm{id}\times\widetilde{\psi}$, and $\widetilde{Z}$ is an invariant subvariety of $(V^n\times\mathbb{A}^m,\phi\times\mathrm{id})$, for some $n,m\geq 0$, that projects dominantly onto V^n, and such that $\widetilde{\psi}$ is the restriction of $\phi\times\mathrm{id}$ to Z.

We make this trivialisation more canonical by applying Proposition 3.2 to the graph of $\widetilde{g}$. Note that this graph is an invariant subvariety by equivariance – see Lemma 3.1. What 3.2 yields is a dominant equivariant $\mu:(\widetilde{Z},\widetilde{\psi})\dashrightarrow(Z,\psi)$ such that for general $e,e'\in\widetilde{Z}$, $\mu(e)=\mu(e')$ if and only if $\widetilde{g}_e=\widetilde{g}_{e'}$. It follows that $\widetilde{g}$ descends to an equivariant birational map $g:(V\times Z,\phi\times\psi)\dashrightarrow(Y,\mathrm{id}\times\psi)$ over (Z,ψ), where Y is now the invariant subvariety of $(\mathbb{A}^\ell\times Z,\mathrm{id}\times\psi)$ obtained as the (Zariski closure of the) image of $\widetilde{Y}$ under $\mathrm{id}\times\mu$. We thus obtain a trivialisation

$$\begin{array}{ccc} (V\times Z,\phi\times\psi) & \overset{g}{\underset{\cong}{\dashrightarrow}} & (Y,\mathrm{id}\times\psi) \\ & \searrow \quad \swarrow & \\ & (Z,\psi) & \end{array}$$

such that the family of birational maps $(g_e:V\dashrightarrow Y_e:e\in Z)$ is canonical in the sense that if $g_e=g_{e'}$ then $e=e'$, for general $e,e'\in Z$.

We will use both the canonicity of this family of birational maps and the fact that (Z,ψ) is the image of $(\widetilde{Z},\widetilde{\psi})\subseteq(V^n\times\mathbb{A}^m,\phi\times\mathrm{id})$.

Remark 4.2. Our dependence on characteristic zero ends here. That is, given a trivialisation of (V,ϕ) with (Z,ψ) of the above form – so both canonical and the image of of some $(\widetilde{Z},\widetilde{\psi})\subseteq(V^n\times\mathbb{A}^m,\phi\times\mathrm{id})$ – the rest of our construction of the binding group, and hence of Theorems 1.3 and 1.5 go through in any characteristic.

Let r be the generic quantifier-free type of (Z,ψ) over k. Let us first observe that $\mathrm{Aut}_{\mathrm{qf}}(q/\mathcal{C})$ acts on $r(\mathcal{U})$ as well:

Lemma 4.3. *There is an action of* $\mathrm{Aut}_{\mathrm{qf}}(q/\mathcal{C})$ *on* $r(\mathcal{U})$ *determined by the property that*

$$g_e(b)=g_{\alpha(e)}(\alpha(b))$$

for any $e\models r, b\models q$ *with* $b\in\mathrm{dom}(g_e)\subseteq V$, *and any* $\alpha\in\mathrm{Aut}_{\mathrm{qf}}(q/\mathcal{C})$.

Proof. Because of the the dominant equivariant rational map $\mu:(\widetilde{Z},\widetilde{\psi})\dashrightarrow(Z,\psi)$, and the nature of $(\widetilde{Z},\widetilde{\psi})$, realisations of r are of the form $\mu(\overline{a},d)$, for some $\overline{a}\models q^{(n)}$

and $d \in \mathcal{C}^m$. The action we have in mind, for $\alpha \in \mathrm{Aut}_{\mathrm{qf}}(q/\mathcal{C})$, is $\mu(\overline{a}, d) \mapsto \mu(\alpha\overline{a}, d)$. Note that ($\star$) ensures that $(\alpha\overline{a}, d)$ is again generic in $(\widetilde{Z}, \widetilde{\psi})$, and hence $\mu(\alpha\overline{a}, d)$ is again a realisation of q. This is well-defined because ($\star$) also ensures that if $\mu(\overline{a}, d) = \mu(\overline{a}', d')$ then $\mu(\alpha\overline{a}, d) = \mu(\alpha\overline{a}', d')$.

Fix $e = \mu(\overline{a}, d) \models r, b \models q$ with $b \in \mathrm{dom}(g_e)$, and $\alpha \in \mathrm{Aut}_{\mathrm{qf}}(q/\mathcal{C})$. Then $(b, e) \in (V \times Z, \phi \times \psi)^\sharp$ and hence $g(b, e) = (g_e(b), e) \in (Y, \mathrm{id} \times \psi)^\sharp$ as g is equivariant. It follows that $g_e(b) =: c \in Y(\mathcal{C})$. Applying ($\star$) to the fact that $g_{\mu(\overline{a},d)}(b) = c$ we deduce that $g_{\mu(\alpha\overline{a},d)}(\alpha(b)) = c$ as well. That is, $g_{\alpha(e)}(\alpha(b)) = c$, as desired. □

Let $f : (Y, \mathrm{id} \times \psi) \to (V \times Z, \phi \times \psi)$ be the inverse to g. So we have $f_e : Y_e \to V$ the birational inverse to g_e given by $y \mapsto \pi_1(f(y, e))$. The definable copy of $\mathrm{Aut}_{\mathrm{qf}}(q/\mathcal{C})$ that we will eventually construct will come from identifying elements of $\mathrm{Aut}_{\mathrm{qf}}(q/\mathcal{C})$ with birational maps of the form $f_{e'} \circ g_e : V \dashrightarrow V$ for certain pairs (e, e') of realisations of r.

Proposition 4.4. *Suppose e, e' realise r with $\mathrm{qftp}(e/\mathcal{C}) = \mathrm{qftp}(e'/\mathcal{C})$. Then*

$$f_{e'} \circ g_e : V \dashrightarrow V$$

is a birational map that is defined on all realisations of q, and whose restriction to $q(\mathcal{U})$, say $\alpha = \alpha_{e,e'}$, is an element of $\mathrm{Aut}_{\mathrm{qf}}(q/\mathcal{C})$.

Conversely, if $\beta \in \mathrm{Aut}_{\mathrm{qf}}(q/\mathcal{C})$ and $e \models r$ then $\mathrm{qftp}(e/\mathcal{C}) = \mathrm{qftp}(\beta(e)/\mathcal{C})$ and $\beta = \alpha_{e,\beta(e)}$. That is, $\beta = (f_{\beta(e)} \circ g_e)|_{q(\mathcal{U})}$.

Remark 4.5. Note that we are not claiming that g_e is defined on all of $q(\mathcal{U})$, just that the composition $f_{e'} \circ g_e$ is.

Proof of 4.4. First of all, we need to observe that the composition $f_{e'} \circ g_e$ makes sense. Since $g_e : V \dashrightarrow Y_e$ and $f_{e'} : Y_{e'} \dashrightarrow V$ are birational maps, it suffices to show that $Y_e = Y_{e'}$. To that end, observe that, as $e \in (Z, \psi)^\sharp$, $(Y_e)^\sigma = Y^\sigma_{\psi(e)}$. On the other hand, as Y is an invariant subvariety of $(\mathbb{A}^\ell \times Z, \mathrm{id} \times \psi)$, $Y_e \subseteq Y^\sigma_{\psi(e)}$. By dimension considerations, it follows that $(Y_e)^\sigma = Y_e$. It follows that Y_e is defined over $k(e) \cap \mathcal{C}$. Hence, $Y_e(\mathcal{C})$ is Zariski dense in Y_e, and similarly for $Y_{e'}(\mathcal{C})$. It suffices to show, therefore, that $Y_e(\mathcal{C}) = Y_{e'}(\mathcal{C})$. But this is the case as, for any $c \in \mathcal{C}^\ell$, the statement that $c \in Y_e$ is part of $\mathrm{qftp}(e/\mathcal{C})$, and by assumption $\mathrm{qftp}(e/\mathcal{C}) = \mathrm{qftp}(e'/\mathcal{C})$.

We now proceed by a series of claims.

Claim 4.6. *$f_{e'} \circ g_e$ is defined on all realisations of q_e, the nonforking extension of q to ke.*

Proof of Claim: If $a \models q_e$ then it is Zariski generic in V over $k(e)$ and hence outside the indeterminacy locus of g_e. Moreover, as $e \in (Z, \psi)^\sharp$ and g is equivariant, it follows that $c := g_e(a) \in Y_e(\mathcal{C})$. Since f_e is defined at c, so is $f_{e'}$. ✠

Let us denote by α the restriction of $f_{e'} \circ g_e$ to realisations of q_e.

Claim 4.7. *Condition ($\star$) holds of α on realisations of q_e. That is, for any quantifier-free formula $\theta(x, y)$ over k, any tuple a of realisations of q_e, and any tuple c of elements of $\mathcal{C}$,*

$$\models \theta(a, c) \iff \models \theta(\alpha(a), c).$$

In particular, if $a \models q_e$ then $\alpha(a) \models q$

Proof of Claim: Taking negations it suffices to prove the left to right direction. Let $d := g_e(a)$, which, as we saw in the proof of the last claim, is a tuple of elements of $Y_e(\mathcal{C})$. Then $f_e(d) = a$. So $\models \theta(a, c)$ tells us that $\models \theta(f_e(d), c)$, so that $\models \theta(f_{e'}(d), c)$. But $f_{e'}(d) = f_{e'}(g_e(a)) = \alpha(a)$, as desired. ✠

Claim 4.8. *Suppose $u, u' \models r$ such that $u \perp e$ and $\mathrm{qftp}(eu/\mathcal{C}) = \mathrm{qftp}(e'u'/\mathcal{C})$. Then $f_{e'} \circ g_e = f_{u'} \circ g_u$ as birational maps on V.*

Proof of Claim: We already know that $Y_e = Y_{e'}$. For the same reasons, $Y_u = Y_{u'}$. It follows that $g_u \circ f_e$ and $g_{u'} \circ f_{e'}$ are both birational maps from Y_e to Y_u. Moreover, they agree on the $\mathcal{C}$-points since $\mathrm{qftp}(eu/\mathcal{C}) = \mathrm{qftp}(e'u'/\mathcal{C})$. But as the $\mathcal{C}$-points are Zariski dense, we have $g_u \circ f_e = g_{u'} \circ f_{e'}$. Now, let a realise q_{eu}, the nonforking extension of q to keu. Note that g_u is defined at a because $a \models q_u$, and

$$\begin{aligned} g_u(a) &= g_u(f_e(g_e(a))) \\ &= g_{u'}(f_{e'}(g_e(a))) \quad \text{as } g_u \circ f_e = g_{u'} \circ f_{e'} \\ &= g_{u'}(\alpha(a)). \end{aligned}$$

Hence

$$f_{e'}(g_e(a)) = \alpha(a) = f_{u'}(g_{u'}(\alpha(a))) = f_{u'}(g_u(a)).$$

That is, $f_{e'} \circ g_e$ and $f_{u'} \circ g_u$ agree on realisations of q_{eu}, and so by Zariski-density on all of V. ✠

Claim 4.9. *$f_{e'} \circ g_e$ is defined on all realisations of q. Moreover, if $\alpha = \alpha_{e,e'}$ now denotes the restriction of $f_{e'} \circ g_e$ on $q(\mathcal{U})$, then $\alpha \in \mathrm{Aut}_{\mathrm{qf}}(q/\mathcal{C})$.*

Proof of Claim: Suppose $a \models q$. Choose $u \models r$ with $u \perp ea$, and $u' \models r$ such that $\mathrm{qftp}(eu/\mathcal{C}) = \mathrm{qftp}(e'u'/\mathcal{C})$. Then, by Claim 4.6, $f_{u'} \circ g_u$ is defined on the realisations of q_u, and hence on a, while, by Claim 4.8, $f_{e'} \circ g_e = f_{u'} \circ g_u$ as birational maps on V. So $f_{e'} \circ g_e$ is defined at a.

For the moreover clause, suppose $\bar{a}$ is a tuple of realisations of q, c a tuple of elements of $\mathcal{C}$, and $\theta(x, y)$ a quantifier-free formula over k. Now let $u \models r$ with $u \perp e\bar{a}$, and $u' \models r$ such that $\mathrm{qftp}(eu/\mathcal{C}) = \mathrm{qftp}(e'u'/\mathcal{C})$. Then, by Claim 4.7 applied to $f_{u'} \circ g_u$ restricted to q_u, we have $\models \theta(\bar{a}, c) \iff \models \theta(f_{u'}(g_u(\bar{a})), c)$. Since $f_{u'} \circ g_u = \alpha$ by Claim 4.8, this shows that $\alpha \in \mathrm{Aut}_{\mathrm{qf}}(q/\mathcal{C})$. ✠

Claim 4.9 is the main clause of the Proposition.

Finally, for the converse direction suppose $\beta \in \mathrm{Aut}_{\mathrm{qf}}(q/\mathcal{C})$ and $e \models r$. Recall, by Lemma 4.3, that $\mathrm{Aut}_{\mathrm{qf}}(q/\mathcal{C})$ acts on $r(\mathcal{U})$, and this is what we mean by $\beta(e)$. Moreover, by property $(\star)$ of $\mathrm{Aut}_{\mathrm{qf}}(q/\mathcal{C})$, we get that $\mathrm{qftp}(e/\mathcal{C}) = \mathrm{qftp}(\beta(e)/\mathcal{C})$. Hence $\alpha_{e,\beta(e)}$, which is the restriction of $f_{\beta(e)} \circ g_e$ to $q(\mathcal{U})$, is an element of $\mathrm{Aut}_{\mathrm{qf}}(q/\mathcal{C})$ by what we have just proved. We want to show it agrees with β. Fix $a \models q$ and compute

$$\begin{aligned} \alpha_{e,\beta(e)}(a) &= f_{\beta(e)}(g_e(a)) \\ &= f_{\beta(e)}(g_{\beta^{-1}\beta(e)}(\beta^{-1}\beta(a))) \\ &= f_{\beta(e)}(g_{\beta(e)}(\beta(a))) \qquad \text{by Lemma 4.3 applied to } \beta^{-1} \in \mathrm{Aut}_{\mathrm{qf}}(q/\mathcal{C}) \\ &= \beta(a) \end{aligned}$$

as desired.

This completes the proof of Proposition 4.4. □

Let $X := \{(e,e') : e,e' \models r, \mathrm{qftp}(e/\mathcal{C}) = \mathrm{qftp}(e'/\mathcal{C})\}$.
Let Λ be the set of invariant rational function on (Z,ψ), as defined in §3.1.

Proposition 4.10. *Fix $e,e' \models r$. Then $(e,e') \in X$ if and only if $\lambda(e) = \lambda(e')$ for all $\lambda \in \Lambda$. In particular, X is quantifier-free-type-definable over k.*

Proof. Recall that Λ is the set of rational functions λ on Z which, when evaluated at some (equivalently any) $e \models r$, lands in the fixed field. The desired result is then just Proposition 2.4; namely, the fact that $\mathrm{qftp}(e/\mathcal{C})$ is isolated by $\mathrm{qftp}(e/k, k(e)\cap\mathcal{C})$, for any e realising a rational type. □

Next, let E to be the equivalence relation on X given by

$$(e,e')E(u,u') \iff f_{e'} \circ g_e = f_{u'} \circ g_u \text{ as birational transformations of } V.$$

As E is relatively definable (even relatively $\mathcal{L}_{\mathrm{ring}}$-definable), we have that X/E is type-definable. This will be our type-definable copy of $\mathrm{Aut}_{\mathrm{qf}}(q/\mathcal{C})$.

The following summarises what we have so far:

Proposition 4.11. *There is a type-definable group structure, $\mathcal{G}$, on X/E, and a type-definable group action of $\mathcal{G}$ on $q(\mathcal{U})$, such that the groups $\mathcal{G}$ and $\mathrm{Aut}_{\mathrm{qf}}(q/\mathcal{C})$, along with their actions on $q(\mathcal{U})$, are isomorphic.*

More precisely, the association $(e,e') \mapsto \alpha_{e,e'}$ given by Proposition 4.4 induces a bijection $\iota : X/E \to \mathrm{Aut}_{\mathrm{qf}}(q/\mathcal{C})$ with the following additional properties:

(a) *Let $R_1 \subseteq X^3$ be the relatively $\mathcal{L}_{\mathrm{ring}}$-definable ternary relation given by: $\big((e_1,e_1'),(e_2,e_2'),(e_3,e_3')\big) \in R_1$ if and only if*

$$(f_{e_1'} \circ g_{e_1}) \circ (f_{e_2'} \circ g_{e_2}) = f_{e_3'} \circ g_{e_3}$$

as birational transformations of V. Then R_1/E makes X/E into a group, $\mathcal{G}$, such that $\iota : \mathcal{G} \to \mathrm{Aut}_{\mathrm{qf}}(q/\mathcal{C})$ is an isomorphism of groups.

(b) *Let $R_2 \subseteq X \times q(\mathcal{U})^2$ be the relatively $\mathcal{L}_{\mathrm{ring}}$-definable relation:*

$$\big((e,e'),a,b\big) \in R_2 \iff (f_{e'} \circ g_e)(a) = b.$$

Then, modulo E, the relation R_2 induces a group action of $\mathcal{G}$ on $q(\mathcal{U})$ such that $(\iota,\mathrm{id}) : (\mathcal{G}, q(\mathcal{U})) \to (\mathrm{Aut}_{\mathrm{qf}}(q/\mathcal{C}), q(\mathcal{U}))$ is an isomorphism of group actions.

Proof. Suppose $(e,e'),(u,u') \in X$. Then

$$\begin{aligned} \alpha_{e,e'} = \alpha_{u,u'} &\iff f_{e'} \circ g_e|_{q(\mathcal{U})} = f_{u'} \circ g_u|_{q(\mathcal{U})} \quad \text{by construction of } \alpha \\ &\iff f_{e'} \circ g_e = f_{u'} \circ g_u \text{ on } V \quad \text{as } q(\mathcal{U}) \text{ is Zariski dense in } V \\ &\iff (e,e')E(u,u'). \end{aligned}$$

This shows that we have an induced injective map $\iota : X/E \to \mathrm{Aut}_{\mathrm{qf}}(q/\mathcal{C})$. It is surjective as any $\beta \in \mathrm{Aut}_{\mathrm{qf}}(q/\mathcal{C})$ is of the form $\alpha_{e,\beta(e)}$, for any $e \models r$, by the converse direction of Proposition 4.4.

Parts (a) and (b) follow rather easily. Fix $(e_1,e_1'),(e_2,e_2'),(e_3,e_3') \in X$. Then, as $q(\mathcal{U})$ is Zariski dense in V,

$$\big((e_1,e_1'),(e_2,e_2'),(e_3,e_3')\big) \in R_1 \iff (f_{e_1'} \circ g_{e_1}) \circ (f_{e_2'} \circ g_{e_2}) = f_{e_3'} \circ g_{e_3} \text{ on } q(\mathcal{U}).$$

By Proposition 4.4, this says that

$$\big((e_1,e_1'),(e_2,e_2'),(e_3,e_3')\big) \in R_1 \iff \alpha_{(e_1,e_1')}\alpha_{(e_2,e_2')} = \alpha_{(e_3,e_3')} \text{ in } \mathrm{Aut}_{\mathrm{qf}}(q/\mathcal{C}).$$

Since $\iota : X/E \to \mathrm{Aut}_{\mathrm{qf}}(q/\mathcal{C})$ is induced by $(e, e') \mapsto \alpha_{e,e'}$, this shows that R_1/E makes X/E into a group such that ι becomes an isomorphism of groups.

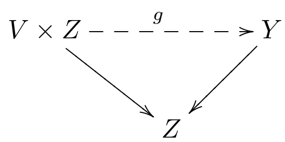
Fix $(e, e') \in X$ and $a, b \models q$. Then, by Proposition 4.4, and construction,

$$\big((e, e'), a, b\big) \in R_2 \iff \alpha_{e,e'}(a) = b.$$

This shows that R_2 induces an action of $\mathcal{G} = X/E$ on $q(\mathcal{U})$ that is isomorphic (via ι) to the action of $\mathrm{Aut}_{\mathrm{qf}}(q/\mathcal{C})$ on $q(\mathcal{U})$. This proves part (b). □

4.1. **Defining $\mathcal{G}$.** All that remains of the main clause of Theorem 1.5 is to show that $\mathcal{G}$ is quantifier-free definable. Note that we do not even know yet that it is quantifier-free-type-definable: the quotient of a quantifier-free definable set by an $\mathcal{L}_{\mathrm{ring}}$-definable equivalence relation need not be quantifier-free. However, we will show eventually, in Proposition 4.25 below, that $\mathcal{G}$ is actually the set of $\sharp$-points of some σ-variety structure on an algebraic group.

To that end, we first show how to construct an algebraic group from the purely algebraic (canonical) family



of birational maps on V. This part of the construction occurs entirely in ACF.

Fix a nonempty Zariski open subset $Z_0 \subseteq Z$ such that:

- $Z_0 \subseteq \pi_2(\mathrm{dom}(g))$ so that $g_e : V \dashrightarrow Y_e$ is a birational map with inverse $f_e : Y_e \to V$, for each $e \in Z_0$, and
- if $e, e' \in Z_0$ and $g_e = g_{e'}$ then $e = e'$. It follows in this case that $f_e = f_{e'}$ implies $e = e'$ as well.

Set

$$T := \{(e, e') \in Z_0 \times Z_0 : Y_e = Y_{e'}\}.$$

Note that $f_{e'} \circ g_e$ is a $k(e, e')$-birational transformation of V for any $(e, e') \in T$. So E extends naturally from X to T. That is, we now denote by E the equivalence relation on T given by

$$(e, e')E(u, u') \iff f_{e'} \circ g_e = f_{u'} \circ g_u,$$

set

$$W := T/E,$$

and denote by

$$\pi : T \to W$$

the quotient map. We denote by

$$1 \in W$$

the element given by $1 := \pi(u, u)$ for any $u \in Z_0$. It corresponds, of course, to the identity birational transformation of V. We also have

$$\mathrm{inv} : W \to W$$

given by $\mathrm{inv}\, \pi(u, u') = \pi(u', u)$.

Note that $Z_0, T, E, W, \pi, 1$, and inv are all $\mathcal{L}_{\mathrm{ring}}$-definable over k.

Remark 4.12. If $\pi(u,u')=\pi(u,u'')$ then $u'=u''$. Indeed, by definition $f_{u'}\circ g_u = f_{u''}\circ g_u$ on V, and hence, as g_u is birational, $f_{u'}=f_{u''}$, which in turn implies that $u'=u''$ by the canonicity of the family. If $\pi(u,u')=w$ we write wu for u' (hence $u=\mathrm{inv}(w)u'$). In particular, u and u' are $\mathcal{L}_{\mathrm{ring}}$-interdefinable over $k(w)$. We note also that since $w=\pi(u,wu)$, if $w_1u=w_2u$ for some $w_1,w_2\in W$, then $w_1=w_2$.

Consider the subset $H_0\subseteq W$ made up of those $w\in W$ such that wu exists for $u\in Z$ generic over w. That is, to be more precise,

> for any (equivalently some) Zariski generic $u\in Z$ over $k(w)$ there is $u'\in Z_0$ with $(u,u')\in T$ and $w=\pi(u,u')$.

Note that H_0 is $\mathcal{L}_{\mathrm{ring}}$-definable; we can quantify over Zariski generic $u\in Z$ using definability of types in ACF. Indeed, if we let $r_0(u)$ be the Zariski generic type of Z (a stationary $\mathcal{L}_{\mathrm{ring}}$-type in ACF over k), and we let $\phi(u,w)$ be the $\mathcal{L}_{\mathrm{ring}}$-formula saying that $w\in W$ and there is $u'\in Z_0$ with $(u,u')\in T$ and $w=\pi(u,u')$, then H_0 is defined by the ϕ-definition of r_0.

Lemma 4.13. *Suppose $w\in H_0$ and $u\in Z$ is Zariski generic over $k(w)$. Then wu is also Zariski generic in Z over $k(w)$.*

Proof. This follows immediately from Remark 4.12. □

One consequence of this is that we can switch the order of the defining condition of H_0, that is: $w\in H_0$ if and only if

> for any (equivalently some) Zariski generic $u'\in Z$ over $k(w)$ there is $u\in Z_0$ with $(u,u')\in T$ and $w=\pi(u,u')$.

and again u is completely determined by u', and is Zariski generic over $k(w)$.

We obtain an $\mathcal{L}_{\mathrm{ring}}$-definable group:

Proposition-Definition 4.14. $(H_0,1,\cdot,\mathrm{inv})$ *is an $\mathcal{L}_{\mathrm{ring}}$-definable group where we define $w_1\cdot w_2$ to be the unique $w_3\in H_0$ with the property that $w_i=\pi(u_i,u_i')$, for $i=1,2,3$, for some (equivalently any) $(u_i,u_i')\in T$ such that*

$$(f_{u_1'}\circ g_{u_1})\circ(f_{u_2'}\circ g_{u_2})=f_{u_3'}\circ g_{u_3}$$

on V.

Proof. Let $u\in Z$ be Zariski generic over k. As 1 is a k-point we have that u is Zariski generic over $k(1)=k$ and $1=\pi(u,u)$, witnessing that $1\in H_0$.

To see that H_0 is preserved by inv, fix $w\in H_0$ and u Zariski generic in Z over $k(w)$. By Lemma 4.13, wu is also Zariski generic over $k(w)$. On the other hand, $k(\mathrm{inv}(w))\subseteq k(w)$ as inv is ACF-definable over k. So $\mathrm{inv}(w)=\pi(wu,u)$ witnesses that $\mathrm{inv}(w)\in H_0$.

Finally, it remains to show that if $w_1,w_2\in H_0$ then there is $w_3\in H_0$ satisfying $w_1\cdot w_2=w_3$. (Uniqueness is immediate by the nature of the equivalence relation E, $\mathcal{L}_{\mathrm{ring}}$-definability is clear from the definitions, as is the fact that the group axioms are satisfied.) Let $u\in Z$ be Zariski generic over $k(w_1,w_2)$. We have $w_1=\pi(u,w_1u)$ and $w_2=\pi(\mathrm{inv}(w_2)u,u)$. Now, as $f_u=g_u^{-1}$,

$$(f_{w_1u}\circ g_u)\circ(f_u\circ g_{\mathrm{inv}(w_2)u})=f_{w_1u}\circ g_{\mathrm{inv}(w_2)u},$$

and hence $w_1\cdot w_2=\pi(\mathrm{inv}(w_2)u,w_1u)=:w_3$. By the definition of $\cdot$, and uniqueness, we get that w_3 is in the $\mathcal{L}_{\mathrm{ring}}$-definable closure of $k(w_1,w_2)$, so in the perfect closure of this field. By 4.12, we know that $\mathrm{inv}(w_2)u$ is Zariski generic in Z over $k(w_1,w_2)$, and hence over $k(w_3)$. That is, $w_3=\pi(\mathrm{inv}(w_2)u,w_1u)$ witnesses that $w_3\in H_0$. □

The $\mathcal{L}_{\mathrm{ring}}$-definable group H_0 was constructed purely out of the algebraic family of birational maps on V. However, so far, there is no reason why it should be nontrivial. In fact it is:

Proposition 4.15. *$\mathcal{G}$ is a subgroup of H_0.*

Proof. We only need to show that $\mathcal{G} \subseteq H_0$, as the group structures were defined identically on both of them – see Proposition 4.11(a).

Suppose $w = \pi(e, e') \in \mathcal{G}$ where $(e, e') \in X$. By Proposition 4.4 we know that $\beta := \alpha_{e,e'} := (f_{e'} \circ g_e)|_{q(\mathcal{U})}$ is an element of $\mathrm{Aut}_{\mathrm{qf}}(q/\mathcal{C})$. Fix $u \models r$ Zariski generic in Z over $k(w)$. The converse direction of Proposition 4.4 tells us that $\beta = \alpha_{u,\beta(u)}$ as well. Hence $(f_{e'} \circ g_e)|_{q(\mathcal{U})} = (f_{\beta(u)} \circ g_u)|_{q(\mathcal{U})}$, which implies that $f_{e'} \circ g_e = f_{\beta(u)} \circ g_u$ on V. That is, $w = \pi(e, e') = \pi(u, \beta(u))$, and the latter witnesses that $w \in H_0$. $\square$

In fact, $\mathcal{G}$ lands in a much smaller subgroup of H_0.

Definition 4.16. For each rational function $\lambda \in k(Z)$, let H_λ be the set of those $w \in H_0$ such that $\lambda(u) = \lambda(wu)$ for some (equivalently any) Zariski generic $u \in Z$ over $k(w)$. For a subset $A \subseteq k(Z)$, we let $H_A = \bigcap_{\lambda \in A} H_\lambda$.

Proposition 4.17. *For each $\lambda \in k(Z)$, H_λ is an $\mathcal{L}_{\mathrm{ring}}$-definable subgroup of H_0 over k. Moreover, $\mathcal{G} \leq H_\Lambda$, where Λ, recall, is the ring of invariant rational functions on (Z, ψ).*

Proof. To see that $\mathcal{G} \subseteq H_\Lambda$, fix $w \in \mathcal{G}$ and $u \models r$ Zariski generic over $k(w)$. We have just seen, in the proof of Proposition 4.15, that $wu = \beta(u)$ for some $\beta \in \mathrm{Aut}_{\mathrm{qf}}(q/\mathcal{C})$. In particular, $\mathrm{qftp}(u/\mathcal{C}) = \mathrm{qftp}(wu/\mathcal{C})$ and so $\lambda(u) = \lambda(wu)$, for all $\lambda \in \Lambda$. This witnesses that $w \in H_\Lambda$.

To see that H_λ is a subgroup of H_0, for any $\lambda \in k(Z)$, we just follow the proof of Proposition 4.14. Namely, given $w_1, w_2 \in H_\lambda$, let $u \in Z$ be Zariski generic over $k(w_1, w_2)$. As $w_1 \in H_\lambda$, we have $w_1 = \pi(u, w_1 u)$ and $\lambda(u) = \lambda(w_1 u)$. Similarly we have $w_2 = \pi(\mathrm{inv}(w_2)u, u)$ and $\lambda(u) = \lambda(\mathrm{inv}(w_2)u)$. Hence $w_1 \cdot w_2 = \pi(\mathrm{inv}(w_2)u, w_1 u)$, both $w_1 u$ and $\mathrm{inv}(w_2)u$ are Zariski generic over $k(w_1, w_2)$, and $\lambda(w_1 u) = \lambda(\mathrm{inv}(w_2)u)$. This witnesses that $w_1 \cdot w_2 \in H_\lambda$. It is also clear that H_λ is preserved by inv. $\square$

By the descending chain condition for groups definable in ACF, H_Λ is also $\mathcal{L}_{\mathrm{ring}}$-definable group. We will show that $\mathcal{G}$ is quantifier-free definable in ACFA by endowing H_0 with an $\mathcal{L}_{\mathrm{ring}}$-definable dynamical structure, $\rho_0 : H_0 \to H_0^\sigma$, and then showing that $\mathcal{G} = (H_0, \rho)^\sharp \cap H_\Lambda$.

We need two preparatory lemmas that have to do with the transform of the situation by σ. Note that we have $Y^\sigma \subseteq \mathbb{A}^\ell \times Z^\sigma$ a family of subvarieties of $\mathbb{A}^\ell$, and

$$\begin{array}{ccc} V^\sigma \times Z^\sigma & \overset{g^\sigma}{\dashrightarrow} & Y^\sigma \\ & \searrow \quad \swarrow & \\ & Z^\sigma & \end{array}$$

a family of birational maps on V^σ, both parameterised by Z^σ. We also have $T^\sigma \subseteq Z_0^\sigma \times Z_0^\sigma$ and $\pi^\sigma : T^\sigma \to W^\sigma = T^\sigma / E^\sigma$.

Lemma 4.18. *Suppose $e \in Z_0$ is in the domain of ψ. Then $Y_e = Y^\sigma_{\psi(e)}$.*

Proof. We use the fact that Y is an invariant subvariety of $(A^\ell \times Z, \psi \times \mathrm{id})$. Since, for any $x \in \mathbb{A}^\ell$, we have that $(x, e) \in \mathrm{dom}(\psi \times \mathrm{id})$, the invariance tells us that

$$\begin{aligned} x \in Y_e &\implies (e, x) \in Y \\ &\implies (x, \psi(e)) \in Y^\sigma \\ &\implies x \in Y^\sigma_{\psi(e)}. \end{aligned}$$

That is, $Y_e \subseteq Y^\sigma_{\psi(e)}$. But as $e \in Z_0$, we have that Y_e is birationally equivalent to V and $Y^\sigma_{\psi(e)}$ is birationally equivalent to V^σ, so that these Zariski closed subsets of $\mathbb{A}^\ell$ are irreducible and have the same dimension. It must therefore be that $Y_e = Y^\sigma_{\psi(e)}$. □

Lemma 4.19. *Suppose $u, u' \in Z$ are Zariski generic over k, and $(u, u') \in T$. Then*

$$\phi \circ f_{u'} \circ g_u = f^\sigma_{\psi(u')} \circ g^\sigma_{\psi(u)} \circ \phi$$

as rational maps $V \dashrightarrow V^\sigma$.

Proof. There are various things to check to even make sense of the statement. First of all, as $u, u' \in Z$ is Zariski generic over k we have that $u, u' \in Z_0$ so that $g_u, f_{u'}$ are well-defiined birational maps. Moreover, $u, u' \in \mathrm{dom}(\psi)$ and, as $\psi : Z \dashrightarrow Z^\sigma$ is a dominant rational map, we get that $\psi(u), \psi(u') \in Z_0^\sigma$ so that $g^\sigma_{\psi(u)}, f^\sigma_{\psi(u')}$ are also well-defined birational maps. Finally, to compose things, we need to know that $Y^\sigma_{\psi(u)} = Y^\sigma_{\psi(u')}$. This follows by Lemma 4.18 since $Y_u = Y_{u'}$.

The identity itself follows readily from the fact that

$$\begin{aligned} g^\sigma_{\psi(u)} \circ \phi &= g_u, \quad \text{and} \\ f^\sigma_{\psi(u')} &= \phi \circ f_{u'} \end{aligned}$$

as rational functions on V and $Y_{u'}$, respectively. These, in turn, follow from the fact that $g : (Z \times V, \psi \times \phi) \dashrightarrow (Y, \psi \times \mathrm{id})$, and its inverse f, are equivariant. □

Remark 4.20. One consequence of the above proof that is worth pointing out is that $\psi : Z \to Z^\sigma$ is necessarily generically injective (and hence birational in characteristic zero). Indeed, we saw that $g^\sigma_{\psi(u)} \circ \phi = g_u$ for any $u \in \mathrm{dom}(\psi)$, hence, if $u_1, u_2 \in Z_0$ with $\psi(u_1) = \psi(u_2)$ then $g_{u_1} = g_{u_2}$, and so, by canonicity, $u_1 = u_2$.

We now enrich H_0 with dynamics.

Definition 4.21. Let $\rho_0 : H_0 \to H_0^\sigma$ be defined as follows: Given $w = \pi(u, u')$ in H_0, where u, u' are Zariski generic over k, set $\rho_0(w) := \pi^\sigma(\psi(u), \psi(u'))$.

Proposition 4.22. *ρ_0 is a well-defined $\mathcal{L}_{\mathrm{ring}}$-definable group isomorphism over k.*

Proof. Note, first of all, that such u, u' exist by definition of H_0 and Remark 4.12. Also, as we have already seen, Zariski genericity ensures that $\psi(u), \psi(u') \in Z_0^\sigma$ and $(\psi(u), \psi(u')) \in T^\sigma$. Hence $\pi^\sigma(\psi(u), \psi(u'))$ makes sense. But there are various things to check:

(1) $\pi^\sigma(\psi(u), \psi(u'))$ depends only on w and not on the choice of u, u'. Suppose $e, e' \in Z$ is another choice of Zariski generic points over k with $\pi(e, e') = w$. Then $f_{e'} \circ g_e = f_{u'} \circ g_u$. Hence

$$\begin{aligned} f^\sigma_{\psi(u')} g^\sigma_{\psi(u)} \phi &= \phi f_{u'} g_u \quad \text{by Lemma 4.19} \\ &= \phi f_{e'} g_e \\ &= f^\sigma_{\psi(e')} g^\sigma_{\psi(e)} \phi \quad \text{by Lemma 4.19 again.} \end{aligned}$$

As ϕ is dominant, it follows that

$$f^\sigma_{\psi(e')} \circ g^\sigma_{\psi(e)} = f^\sigma_{\psi(u')} \circ g^\sigma_{\psi(u)},$$

which says exactly that $(\psi(e), \psi(e'))E^\sigma(\psi(u), \psi(u'))$, namely that

$$\pi^\sigma(\psi(e), \psi(e')) = \pi^\sigma(\psi(u), \psi(u')),$$

as desired.

(2) ρ_0 is injective. This is just a matter of noticing that each of the steps in the proof of (1) above are reversible.

(3) $\rho_0(w) \in H_0^\sigma$. The defining condition for H_0^σ is obtained by applying σ to the defining condition for H_0. That is, $x \in W^\sigma$ is in H_0^σ if and only if

For any (equivalently some) Zariski generic $y \in Z^\sigma$ over $k(x)$ there is $y' \in Z_0^\sigma$ with $(y, y') \in T^\sigma$ and $x = \pi(y, y')$.

Now, we could have chosen u to be Zariski generic in Z over $k(w)$. In which case, $\psi(u)$ is Zariski generic in Z^σ over $k(w)$. At this point, we already know that ρ_0 is an $\mathcal{L}_{\text{ring}}$-definable function over k, so that $\rho_0(w)$ is in the perfect closure of $k(w)$. Hence $\psi(u)$ is Zariski generic in Z^σ over $k(\rho_0(w))$. So $\rho_0(w) = \pi^\sigma(\psi(u), \psi(u'))$ witnesses that $\rho_0(w) \in H_0^\sigma$.

(4) ρ_0 is a group homomorphism. Here the group structure on H_0^σ is the one obtained by transforming the group structure on H_0 by σ. Suppose $w_1 \cdot w_2 = w_3$ in H_0. Write $w_i = \pi(u_i, u_i')$ where $u_i, u_i' \in Z$ are Zariski generic over k, for $i = 1, 2, 3$. So $(f_{u_1'} \circ g_{u_1}) \circ (f_{u_2'} \circ g_{u_2}) = f_{u_3'} \circ g_{u_3}$ on V. It follows that

$$\phi \circ f_{u_1'} \circ g_{u_1} \circ f_{u_2'} \circ g_{u_2} = \phi \circ f_{u_3'} \circ g_{u_3}.$$

Now, applying Lemma 4.19 repeatedly, we deduce that

$$f^\sigma_{\psi(u_1')} \circ g^\sigma_{\psi(u_1)} \circ f^\sigma_{\psi(u_2')} \circ g^\sigma_{\psi(u_2)} \circ \phi = f^\sigma_{\psi(u_3')} \circ g^\sigma_{\psi(u_3)} \circ \phi.$$

As ϕ is dominant, we get

$$f^\sigma_{\psi(u_1')} \circ g^\sigma_{\psi(u_1)} \circ f^\sigma_{\psi(u_2')} \circ g^\sigma_{\psi(u_2)} = f^\sigma_{\psi(u_3')} \circ g^\sigma_{\psi(u_3)}$$

which says that $\rho_0(w_1) \cdot \rho_0(w_2) = \rho_0(w_3)$, as desired.

(5) ρ_0 is an isomorphism. We have already seen that it is injective. As H_0^σ is an $\mathcal{L}_{\text{ring}}$-definable group of the same Morley rank and degree as H_0, any injective $\mathcal{L}_{\text{ring}}$-definable homomorphism $H_0 \to H_0^\sigma$ is surjective. □

Proposition 4.23. *For each each $\lambda \in \Lambda$, the subgroup H_λ of H_0 is ρ_0-invariant, in the sense that $\rho_0(H_\lambda) \subseteq H_\lambda^\sigma$.*

Proof. Let $w \in H_\lambda$ and u Zariski generic in Z over $k(w)$. So $w = \pi(u, wu)$, and hence $\rho_0(w) = \pi^\sigma(\psi(u), \psi(wu))$ by definition. Note that $\psi(u)$ is Zariski generic in Z^σ over $k(\rho_0(w))$ since ψ is dominant and ρ_0 is $\mathcal{L}_{\text{ring}}$-definable. Hence, to show that $\rho_0(w) \in H_\lambda^\sigma$ it suffices to show that $\lambda^\sigma\psi(u) = \lambda^\sigma\psi(wu)$. But $\lambda = \lambda^\sigma\psi$ as λ is an invariant rational function on (Z, ψ), and $\lambda(u) = \lambda(wu)$ as $w \in H_\lambda$. □

Definition 4.24. Let $H := H_\Lambda$ and $\rho := \rho_0|_H : H \to H^\sigma$.

By the equivalence of categories between $\mathcal{L}_{\text{ring}}$-definable groups and algebraic groups, we can give H the structure of a (possibly not connected) algebraic group over k such that ρ is an isomorphism of algebraic groups. That is, (H, ρ) is a *σ-group* in the sense of [17], except that (H, ρ) isn't technically a σ-variety as we have defined it, because H is not necessarily irreducible. Nevertheless, much of our terminology about rational σ-varieties makes sense and can be used profitably in this setting.

Proposition 4.25. $\mathcal{G} = (H, \rho)^\sharp := \{w \in H : \sigma(w) = \rho(w)\}$

Proof. We already know from Proposition 4.17 that $\mathcal{G} \leq H$. To see that $\mathcal{G} \subseteq (H_0, \rho_0)^\sharp$, fix $w = \pi(e, e') \in \mathcal{G}$ where $e, e' \in X$. In particular, $e, e' \models r$, so they are Zariski generic over k and $\sigma(e) = \psi(e)$ and $\sigma(e') = \psi(e')$. Hence

$$\begin{aligned} \rho_0(w) &= \pi^\sigma(\psi(e), \psi(e')) \\ &= \pi^\sigma(\sigma(e), \sigma(e')) \\ &= \sigma(\pi(e, e')) \\ &= \sigma(w) \end{aligned}$$

as desired.

For the converse, suppose $w \in (H, \rho)^\sharp$. Fix $e \models r$ Zariski generic in Z over $k(w)$. Then $w = \pi(e, we)$. Now,

$$\begin{aligned} \pi^\sigma(\sigma(e), \sigma(we)) &= \sigma(w) \\ &= \rho_0(w) \quad \text{as } w \in (H_0, \rho_0)^\sharp \\ &= \pi^\sigma(\psi(e), \psi(we)) \\ &= \pi^\sigma(\sigma(e), \psi(we)) \quad \text{as } e \in (Z, \psi)^\sharp \end{aligned}$$

Remark 4.12, applied to π^σ, implies that $\sigma(we) = \psi(we)$. Hence $we \in (Z, \psi)^\sharp$ as well, so that $we \models r$. As $w \in H_\Lambda$ we have that $\lambda(e) = \lambda(we)$, for all $\lambda \in \Lambda$. Hence, by Proposition 4.10, $(e, we) \in X$. So $w \in X/E = \mathcal{G}$. □

In particular, we have now shown that $\mathcal{G}$ is a quantifier-free definable group. That, together with 4.11, completes the proof of the main clause of Theorem 1.5.

4.2. **Proof of Theorem 1.3.** Let us recall the notation of that theorem. We denote by $\mathrm{Bir}(V) = \mathrm{Bir}_{\mathcal{U}}(V)$ the group of all birational transformations of V over $\mathcal{U}$. We set $\mathbb{V} := (V, \phi)$ and consider the collection $\mathcal{I}_{\mathbb{V}}$ of all irreducible invariant subvarieties of $(V^r \times \mathbb{A}^s, \phi \times \mathrm{id})$ over k that project dominantly onto each copy of V, as $r, s \in \mathbb{N}$ vary. Setting $\mathbb{L} = (\mathbb{A}^1, \mathrm{id})$, we then consider the subgroup $\mathrm{Bir}(\mathbb{V}/\mathbb{L}) = \mathrm{Bir}_{\mathcal{U}}(\mathbb{V}/\mathbb{L})$ of $\mathrm{Bir}(V)$ made up of those birational transformations that preserve each element of $\mathcal{I}_{\mathbb{V}}$. More precisely, those $\delta \in \mathrm{Bir}(V)$ such that, for each $X \subseteq V^r \times \mathbb{A}^s$ in $\mathcal{I}_{\mathbb{V}}$,

- $X \cap (\mathrm{dom}(\delta)^r \times \mathbb{A}^s)$ is nonempty, and
- $\delta(X) \subseteq X$.

Here, δ acts diagonally on V^r and trivial on $\mathbb{A}^s$.

Theorem 1.3 asserts that $\mathrm{Bir}(\mathbb{V}/\mathbb{L})$ is an algebraic group of birational transformations over k. This is what we want to prove.

Observe that we have already constructed an algebraic group of birational transformations of V, namely H. Indeed, there is a rational map

$$\theta : H \times V \dashrightarrow V$$

such that for every $w \in H$ and $(u, u') \in T$ with $\pi(u, u') = w$, we have the birational transformation

$$\theta_w := f_{u'} \circ g_u : V \dashrightarrow V.$$

By definition of the group structure given in Definition 4.14 we have that

$$\theta_1 = \mathrm{id}_V, \quad \text{and}$$

$$\theta_{w_1} \circ \theta_{w_2} = \theta_{w_1 \cdot w_2} \quad \text{for all } w_1, w_2 \in H.$$

So $w \mapsto \theta_w$ makes H a subgroup of $\mathrm{Bir}(V)$, as Definition 1.1 requires.

Lemma 4.26. $\mathrm{Bir}(\mathbb{V}/\mathbb{L}) \leq H$. *That is, if* $\delta \in \mathrm{Bir}(\mathbb{V}/\mathbb{L})$ *then there is* $w \in H$ *such that* $\delta = \theta_w$.

Proof. Recall that our canonical trivialisation

$$\begin{array}{ccc} (V \times Z, \phi \times \psi) & \overset{g}{\underset{\cong}{\dashrightarrow}} & (Y, \mathrm{id} \times \psi) \\ & \searrow \quad \swarrow & \\ & (Z, \psi) & \end{array}$$

was induced by a trivialisation

$$\begin{array}{ccc} (V \times \widetilde{Z}, \phi \times \widetilde{\psi}) & \overset{\widetilde{g}}{\underset{\cong}{\dashrightarrow}} & (\widetilde{Y}, \mathrm{id} \times \widetilde{\psi}) \\ & \searrow \quad \swarrow & \\ & (\widetilde{Z}, \widetilde{\psi}) & \end{array}$$

via a dominant equivariant $\mu : (\widetilde{Z}, \widetilde{\psi}) \to (Z, \psi)$. The original trivialisation had the property that $\widetilde{Z}$ is an invariant subvariety of $(V^n \times \mathbb{A}^m, \phi \times \mathrm{id})$ over k, for some $n, m \geq 0$, that projects dominantly onto V^n, and $\widetilde{\psi}$ is the restriction of $\phi \times \mathrm{id}$ on $V^n \times \mathbb{A}^m$. In particular, $\widetilde{Z} \in \mathcal{I}$.

Let Γ be the graph of $\widetilde{g}$ over $\widetilde{Z}$. That is,

$$\Gamma := \{(a, z, \widetilde{g}_z(a)) : a \in V, z \in \widetilde{Z}\} \subseteq V \times \widetilde{Z} \times \mathbb{A}^\ell \subseteq V^{n+1} \times \mathbb{A}^{m+\ell}.$$

Because $\widetilde{g}$ is equivariant and dominant, Γ is invariant for $\phi \times \widetilde{\psi} \times \mathrm{id}$ on $V \times \widetilde{Z} \times \mathbb{A}^\ell$, by Lemma 3.1, and hence for $\phi \times \mathrm{id}$ on $V^{n+1} \times \mathbb{A}^{m+\ell}$. Moreover, as Γ projects dominantly onto $V \times \widetilde{Z}$, it projects dominantly onto each of the $n+1$ copies of V. Hence, $\Gamma \in \mathcal{I}_{\mathbb{V}}$.

Let $L \supseteq k$ be a (finitely generated) field extension over which δ is defined. Fix $u \in Z$ Zariski generic over L, and write $u = \mu(b)$ where $b \in \widetilde{Z}$ is Zariski generic over L. In particular, $b = (\overline{a}, d)$ where $\overline{a} \in V^n$ is Zariski generic over L and $d \in \mathbb{A}^m$. Since $\delta \in \mathrm{Bir}(\mathbb{V}/\mathbb{L})$ and $\widetilde{Z} \in \mathcal{I}_{\mathbb{V}}$, we have that $\delta b := (\delta\overline{a}, d)$ is again Zariski generic in $\widetilde{Z}$ over L. Fix $a \in V$ Zariski generic over $L(b)$. Then, as $\Gamma \in \mathcal{I}_{\mathbb{V}}$, we have that $(\delta a, \delta b, \widetilde{g}_b(a)) \in \Gamma$, so that $\widetilde{g}_{\delta b}(\delta a) = \widetilde{g}_b(a)$. By Zariski genericity of a, we conclude that $\widetilde{g}_{\delta b} \circ \delta = \widetilde{g}_b$. Applying μ, it follows that $g_{\mu(\delta b)} \circ \delta = g_u$ as rational maps on V. Letting $u' := \mu(\delta b)$ and $w = \pi(u, u') \in W$, we have that $\delta = f_{u'} \circ g_u = \theta_w$.

We claim that $w \in H_0$. That is, we claim that $w = \pi(e, e')$ where e (an in fact e') are Zariski generic in Z over $k(w)$. (See page 27 to recall the definition of H_0.) To see this let e, e' realise the same $\mathcal{L}_{\mathrm{ring}}$-type over L but independent from w over L. Since $\theta_w = \delta$ is over L, we still have that $f_{e'} \circ g_e = \theta_w$, and so $\pi(e, e') = w$. But now, as u was chosen Zariski generic over L, we have that e is Zariski generic over $k(w)$.

Finally, we need to show that $w \in H$. That is, we need to show that, for every invariant rational function $\lambda \in k(Z)$, $\lambda(e) = \lambda(e')$. Note that there is $(\overline{a}, d) \in \widetilde{Z}$

generic over L such that $e = \mu(\overline{a}, d)$ and $e' = \mu(\delta\overline{a}, d)$. Indeed, this was the case for u, u' by construction, and is part of the $\mathcal{L}_{\mathrm{ring}}$-type over L. Pulling back by μ, it suffices to show that for every invariant rational function $\lambda \in k(\widetilde{Z})$, $\lambda(\overline{a}, d) = \lambda(\delta\overline{a}, d)$. We may assume that $\lambda \notin k$. So $\lambda : (\widetilde{Z}, \widetilde{\psi}) \dashrightarrow (\mathbb{A}, \mathrm{id})$ is dominant and equivariant, and hence its graph

$$\Gamma(\lambda) \subseteq \widetilde{Z} \times \mathbb{A} \subseteq V^n \times \mathbb{A}^{m+1}$$

is an element of $\mathcal{I}_{\mathbb{V}}$, by Lemma 3.1. It follows, since $\delta \in \mathrm{Bir}(\mathbb{V}/\mathbb{L})$, that

$$\delta((\overline{a}, d), \lambda(\overline{a}, d)) = ((\delta\overline{a}, d), \lambda(\overline{a}, d)) \in \Gamma(\lambda).$$

This means that $\lambda(\overline{a}, d) = \lambda(\delta\overline{a}, d)$, as desired. □

Proof of Theorem 1.3 (conclusion). Let $G := \{w \in H : \theta_w \in \mathrm{Bir}(\mathbb{V}/\mathbb{L})\}$. Note that G is a (possibly not connected) algebraic subgroup of H over k. Indeed, the preservation of any fixed $X \in \mathcal{I}_{\mathbb{V}}$ is a Zariski closed condition on w. Lemma 4.26, identifies $\mathrm{Bir}(\mathbb{V}/\mathbb{L})$ with G and thus completes the proof of Theorem 1.3. □

4.3. **The rest of Theorem 1.5.**

We now make the connection between $\mathrm{Aut}_{\mathrm{qf}}(q/\mathcal{C})$ and $\mathrm{Bir}(\mathbb{V}/\mathbb{L})$, so between $\mathcal{G}$ and G, as called for by the "in fact" clause of Theorem 1.5.

Lemma 4.27. $\mathcal{G} \leq G$.

Proof. Fix $w \in \mathcal{G}$. We need to show that θ_w preserves each member of $\mathcal{I}_{\mathbb{V}}$. Let $X \subseteq V^r \times \mathbb{A}^s$ be in $\mathcal{I}_{\mathbb{V}}$. So, we have an induced rational dynamics

$$\varphi := (\phi \times \mathrm{id})|_X$$

on X, such that the first r co-ordinate projections $(X, \varphi) \to (V, \phi)$ are dominant equivariant maps. Hence, if $b = (a_1, \ldots, a_r, c_1, \ldots, c_s)$ is a generic point of (X, φ) over k, then each $a_i \models q$. Since $w \in G$, we have that θ_w restricts to an element of $\mathrm{Aut}_{\mathrm{qf}}(q/\mathcal{C})$. Hence θ_w is defined at b and the defining condition $(\star)$ of the binding group ensures that $\theta_w(b) \in X$. It follows that $(\theta_w \times \mathrm{id})$ preserves X, as desired. □

Next we need to show that $\rho : H \to H^\sigma$ restricts to $G \to G^\sigma$. This will follow from the following:

Lemma 4.28. *$\theta : (H \times V, \rho \times \phi) \dashrightarrow (V, \phi)$ is equivariant in the sense that $\phi\theta = \theta^\sigma(\rho \times \phi)$ as rational maps $H \times V \dashrightarrow V^\sigma$.*

Proof. It suffices to show that, for each $w \in H$,

$$\phi\theta_w = \theta^\sigma_{\rho(w)}\phi$$

as rational maps $V \dashrightarrow V^\sigma$. Letting $w = \pi(u, u')$ where $u, u' \in Z$ are Zariski generic over k, we have that $\rho(w) = \pi^\sigma(\psi(u), \psi(u'))$ by how ρ is defined in Definition 4.21. So we have that $\theta_w = f_{u'}g_u$ and $\theta^\sigma_{\rho(w)} = f^\sigma_{\psi(u')}g^\sigma_{\psi(u)}$ and we are trying to prove that

$$\phi f_{u'}g_u = f^\sigma_{\psi(u')}g^\sigma_{\psi(u)}\phi,$$

which is exactly Lemma 4.19. □

Proposition 4.29. *$\rho : H \to H^\sigma$ restricts to an isomorphism $G \to G^\sigma$.*

Proof. Fixing $w \in G$ we need to show that $\rho(w) \in G^\sigma$. That is, given $X \subseteq V^r \times \mathbb{A}^r$ in $\mathcal{I}_{\mathbb{V}}$, we need to show that $\theta^\sigma_{\rho(w)}$ preserves X^σ. Fix $(a,d) \in X$ Zariski generic over $k(w)$, where $a = (a_1, \dots, a_r)$ is tuple of generic points of V and$d \in \mathbb{A}^s$. Since ϕ restricts to a dominant rational map from X to X^σ, we have that $(\phi a, d) \in X^\sigma$ is Zariski generic over $k(w)$. And, because $\theta^\sigma_{\rho(w)}\phi = \phi\theta_w$ by Lemma 4.28, we get

$$\theta^\sigma_{\rho(w)}(\phi a, d) = (\phi\theta_w a, d).$$

Since $w \in G$, $\theta_w \in \mathrm{Bir}(\mathbb{V}/\mathbb{L})$, and so $(\theta_w a, d) \in X$. Hence $(\phi\theta_w a, d) \in X^\sigma$. We have shown that $\theta^\sigma_{\rho(w)}(\phi a, d) \in X^\sigma$ for a Zariski generic point $(\phi a, d) \in X^\sigma$ over $k(w)$. This implies that $\theta^\sigma_{\rho(w)}(X^\sigma) \subseteq X^\sigma$, as desired. □

Proof of Theorem 1.5 (conclusion). It remains only to show that $\mathcal{G} = (G, \rho|_G)^\sharp$. But Proposition 4.25 tells us that $\mathcal{G} = (H, \rho)^\sharp$ and Lemma 4.27 tell us that $\mathcal{G} \leq G$. From this the result follows. □

5. Some applications

In this final section we describe some applications of our binding group theorems.

Fix an algebraically closed difference field (k, σ) of characteristic zero, and a sufficiently saturated model $(\mathcal{U}, \sigma) \models \mathrm{ACFA}_0$ extending (k, σ). Let $\mathcal{C} := \mathrm{Fix}(\sigma)$. As mentioned in Remark 4.2, our only use of characteristic zero is to deduce that an isotrivial σ-variety has a trivialisation of a particularly useful form.

5.1. **The autonomous case.**

Here we recover the main results of [2] by restricting to the autonomous case. In this section we assume, therefore, that $k \subset \mathcal{C}$.

Following [8] and [2] we say that a rational dynamical system (V, ϕ) is *translational* if ϕ comes from the action of an algebraic group; that is, if there is a faithful algebraic group action $\theta : H \times V \to V$ over k such that ϕ agrees with the $\theta(h, -)$ for some $h \in H(k)$. The following corollary of our binding group theorems recovers [2, Corollary A].

Theorem 5.1. *Every isotrivial rational dynamical system over an algebraically closed field of characteristic zero is, up to birational equivalence, translational.*

Proof. Suppose (V, ϕ) is an isotrivial rational dynamical system over an algebraically closed field k of characteristic zero. Let $q \in S_{\mathrm{qf}}(k)$ be the quantifier-free generic type of (V, ϕ). It is qf-internal to $\mathcal{C}$.

We claim that $\phi|_{q(\mathcal{U})} \in \mathrm{Aut}_{\mathrm{qf}}(q/\mathcal{C})$. Indeed, if $a \in (V, \phi)^\sharp \cap \mathrm{dom}(\phi)$ then

$$\begin{aligned} \sigma(\phi(a)) &= \phi^\sigma(\sigma(a)) \\ &= \phi(\sigma(a)) \quad \text{as } \phi^\sigma = \phi \text{ as } k \subseteq \mathcal{C} \\ &= \phi(\phi(a)) \quad \text{as } a \in (V, \phi)^\sharp, \end{aligned}$$

which shows that $\phi(a) \in (V, \phi)^\sharp$. Moreover, as $\phi : V \dashrightarrow V$ is dominant it takes Zariski generic points to Zariski generic points. Hence, if $a \models q$ then $\phi(a) \models q$. On the other hand, a consequence of isotriviality is that ϕ is birational. Hence, $\phi|_{q(\mathcal{U})}$ is at least a permutation of $q(\mathcal{U})$. To show that it is in $\mathrm{Aut}_{\mathrm{qf}}(q/\mathcal{C})$ consider a quantifier-free $\mathcal{L}_{\text{ring}}$-formula $\psi(x, y)$ over k, any tuple a of realisations of q, and any tuple c of elements of $\mathcal{C}$, and observe that

$$\begin{aligned} \models \psi(a,c) \quad &\iff \quad \models \psi^{\sigma}(\sigma(a),\sigma(c)) \;\text{ as } \sigma \text{ is an } \mathcal{L}_{\text{ring}}\text{-automorphism} \\ &\iff \quad \models \psi(\phi(a),c) \;\text{ as } a \text{ is from } (V,\phi)^{\sharp} \text{ and } c \text{ is from } \mathcal{C}. \end{aligned}$$

This proves that $\phi|_{q(\mathcal{U})} \in \operatorname{Aut}_{\text{qf}}(q/\mathcal{C})$. (As q is rational it suffices to verify $(\star)$ for $\mathcal{L}_{\text{ring}}$-formulas.)

We have an algebraic group of birational transformations $\theta : G \times V \dashrightarrow V$ over k given to us by Theorem 1.3. Theorem 1.5 gives us a (possibly reducible) rational dynamics $\rho : G \to G$ such that θ is equivariant with respect to $\phi \times \rho$ and ϕ, see Lemma 4.28. Moreover, the conclusion of Theorem 1.5 is that we can identify $\operatorname{Aut}_{\text{qf}}(q/\mathcal{C})$ with $(G, \rho)^{\sharp}$ and its action restricted to $q(\mathcal{U})$.

We need to upgrade θ to an honest regular algebraic group action by automorphisms. This can be done because θ makes V into a pre-homogeneous variety for G, in the sense of Weil [23]. It follows, by Weil's group-chunk theorem, that there is a birational map

$$\gamma : V \dashrightarrow \widehat{V}$$

and an algebraic group action

$$\widehat{\theta} : G \times \widehat{V} \to \widehat{V}$$

such that, for each $w \in G$,

$$\widehat{\theta}_w = \gamma\theta_w\gamma^{-1}.$$

See, for example, [25, Theorem 4.9] for a modern treatment that does not assume the connectedness of G. This is a faithful action: if $\widehat{\theta}_w = \mathrm{id}_{\widehat{V}}$ then $\theta_w = \mathrm{id}_V$ which implies $w = 1$. Next, using γ, we can transport the rational dynamics on V onto $\widehat{V}$ by setting

$$\widehat{\phi} := \gamma\phi\gamma^{-1} : \widehat{V} \dashrightarrow \widehat{V}.$$

It is $(\widehat{V}, \widehat{\phi})$ that we will show is translational.

We claim, first, that $\widehat{\theta} : (G \times \widehat{V}, \rho \times \widehat{\phi}) \to (\widehat{V}, \widehat{\phi})$ is equivariant. Indeed,

$$\begin{aligned} \widehat{\phi}\widehat{\theta}_w &= (\gamma\phi\gamma^{-1})(\gamma\theta_w\gamma^{-1}) \\ &= \gamma\phi\theta_w\gamma^{-1} \\ &= \gamma\theta_{\rho(w)}\phi\gamma^{-1} \quad \text{by equivariance of } \theta \\ &= \gamma\theta_{\rho(w)}(\gamma)^{-1}\widehat{\phi} \\ &= \widehat{\theta}_{\rho(w)}\widehat{\phi} \end{aligned}$$

for each $w \in G$, as desired.

By construction, $\gamma : (V, \phi) \dashrightarrow (\widehat{V}, \widehat{\phi})$ is now an equivariant birational map. It therefore restricts to a birational equivalence between q and the quantifier-free generic type $\widehat{q}$ of $(\widehat{V}, \widehat{\phi})$. We obtain an induced isomorphism

$$\gamma^* : \operatorname{Aut}_{\text{qf}}(q/\mathcal{C}) \to \operatorname{Aut}_{\text{qf}}(\widehat{q}/\mathcal{C}),$$

see Remark 4.1(c). So, $\widehat{\theta}$ restricts to an action of $G := (G, \rho)^{\sharp}$ on $\widehat{q}(\mathcal{U})$ that is isomorphic to the action of $\operatorname{Aut}_{\text{qf}}(\widehat{q}/\mathcal{C})$ on $\widehat{q}(\mathcal{U})$. Since $\phi|_{q(\mathcal{U})} \in \operatorname{Aut}_{\text{qf}}(q/\mathcal{C})$ we also have $\widehat{\phi}|_{\widehat{q}(\mathcal{U})} \in \operatorname{Aut}_{\text{qf}}(\widehat{q}/\mathcal{C})$. Hence $\widehat{\phi}|_{\widehat{q}(\mathcal{U})} = \widehat{\theta}(h, -)|_{\widehat{q}(\mathcal{U})}$ for some $h \in (G, \rho)^{\sharp}$.

As $\widehat{\phi}, \widehat{\theta}, \widehat{q}$ are over k, it follows that $h \in H(k)$. Finally, as $\widehat{q}(\mathcal{U})$ is Zariski dense in $\widehat{V}$, we obtain that $\widehat{\phi} = \widehat{\theta}(h, -)$. Hence $(\widehat{V}, \widehat{\phi})$ is translational. □

From this we can also recover [2, Corollary B].

Corollary 5.2. *Suppose $k \subseteq \mathcal{C}$ is algebraically closed and $p \in S_{\mathrm{qf}}(k)$ is a rational type. If p is nonorthogonal to $\mathcal{C}$ then $p^{(2)}$ is not weakly orthogonal to $\mathcal{C}$.*

Geometric formulation: *Suppose (V, ϕ) is a rational dynamical system over k. If $(V, \phi) \times (W, \psi)$ admits an invariant rational function that is not the pullback of a rational function on W, for some rational σ-variety (W, ψ) over k, then (V^2, ϕ) admits a nonconstant invariant rational function. In particular, if some cartesian power of (V, ϕ) admits a nonconstant invariant rational function then already (V^2, ϕ) does.*

Proof. By Proposition 2.11 there is a nonalgebraic rational type $q \in S_{\mathrm{qf}}(k)$ that is qf-internal to $\mathcal{C}$, and a rational map $p \to q$ over k. And it suffices to show that $q^{(2)}$ is not weakly orthogonal to $\mathcal{C}$. But, by Theorem 5.1, since we are in the autonomous situation, after possibly replacing q with a birationally equivalent quantifier-free type, we may assume that q is the generic quantifier-free type of some translational rational dynamical system (V, ϕ) over k. Now, if (V, ϕ) admits a nonconstant invariant rational function, then q is already not weakly orthogonal to $\mathcal{C}$ by Proposition 3.4(a), and we are done. So we may assume that (V, ϕ) admits no nonconstant invariant rational functions. It is shown in [2, Proposition 3.2] that any positive-dimensional translational dynamical system with no nonconstant invariant rational functions will have the property that its second cartesian power *does* admit a nonconstant invariant rational function. This means that $q^{(2)}$ is not weakly orthogonal to $\mathcal{C}$, as desired.

The geometric formulation is obtained by applying the theorem to the generic quantifier-free type p of (V, ϕ). The assumption on (V, ϕ) tells us that p is nonorthogonal to $\mathcal{C}$, this is Proposition 3.4(b). Hence $p^{(2)}$ is not weakly orthogonal to $\mathcal{C}$, which by Proposition 3.4(a), implies that (V^2, ϕ) admits a nonconstant invariant rational function.

For the "in particular" clause, suppose (V^n, ϕ) admits a nonconstant invariant rational function, and n is least such. Then $(W, \psi) := (V^{n-1}, \phi)$ admits no nonconstant invariant rational functions, and hence those on $(V^n, \phi) = (V, \phi) \times (W, \psi)$ are not pullbacks from (W, ψ), and hence (V^2, ϕ) admits a nonconstant invariant rational function. □

5.2. **Dixmier-Moeglin and Zariski dense orbits.** Now we drop the autonomous assumption, and exhibit some new applications. So (k, σ) is an arbitrary algebraically closed difference field of characteristic zero.

Given a rational σ-variety (V, ϕ) over k, it is natural to ask for conditions that would force there to exist only finitely many maximal proper invariant subvarieties over k. A necessary condition is that (V, ϕ) admit no nonconstant invariant rational function, as such a rational function would, by taking appropriate level sets, give rise to infinitely many codimension 1 invariant subvarieties. The question of whether this condition is sufficient is sometimes called the Dixmier-Moeglin equivalence problem in algebraic dynamics, see [3, Conjecture 8.5] and also [19] for a survey of Dixmier-Moeglin type problems. Actually, admitting no nonconstant invariant rational functions is *not* sufficient in general (even in the autonomous case), with counterexamples given by Henon automorphisms of the affine plane (see [3, Theorem 8.8]). But we will show that it is sufficient in the (possibly nonautonomous) isotrivial case.

But first we need a lemma that is central to how we use binding groups.

Lemma 5.3. *Suppose $q \in S_{\mathrm{qf}}(k)$ is rational and $q^{(\ell)}$ is weakly orthogonal to $\mathcal{C}$. Then $\mathrm{Aut}_{\mathrm{qf}}(q/\mathcal{C})$ acts transitively on $q^{(\ell)}(\mathcal{U})$.*

Proof. First of all, Proposition 2.4 tells us that $\mathrm{qftp}(a/k, k(a)\cap\mathcal{C})$ isolates $\mathrm{qftp}(a/\mathcal{C})$, for any $a \models q^{(\ell)}$. But, as k is algebraically closed, weak orthogonality implies that $k(a) \cap \mathcal{C} \subseteq k$. Hence, if $a_1, a_2 \models q^{(\ell)}$ then $\mathrm{qftp}(a_1/\mathcal{C}) = \mathrm{qftp}(a_2/\mathcal{C})$. Now, recall, from Remark 4.1(a) that we have the two-sorted auxiliary structure $\mathcal{Q}$ whose sorts are $q(\mathcal{U})$ and $\mathcal{C}$ and where the language is made up of a predicate symbol for each relatively quantifier-free k-definable subset of $q(\mathcal{U})^n \times \mathcal{C}^m$ in $(\mathcal{U}, \sigma)$, for any $n, m \geq 0$. That $\mathrm{qftp}(a_1/\mathcal{C}) = \mathrm{qftp}(a_2/\mathcal{C})$ means that, in the structure $\mathcal{Q}$, $\mathrm{tp}_{\mathcal{Q}}(a_1/\mathcal{C}) = \mathrm{tp}_{\mathcal{Q}}(a_2/\mathcal{C})$. Now, because $\mathcal{Q}$ is sufficiently saturated and $\mathcal{C}$ is stably embedded in $\mathcal{Q}$ – both properties inherited from $(\mathcal{U}, \sigma)$ – this means that there is $\alpha \in \mathrm{Aut}(\mathcal{Q}/\mathcal{C}) = \mathrm{Aut}_{\mathrm{qf}}(q/\mathcal{C})$ such that $\alpha(a_1) = a_2$, as desired. □

Theorem 5.4. *Suppose $q \in S_{\mathrm{qf}}(k)$ is rational and qf-internal to $\mathcal{C}$. If q is weakly orthogonal to $\mathcal{C}$ then it is isolated by a quantifier-free formula.*

Geometric formulation: *Suppose (V, ϕ) is an isotrivial rational σ-variety over k with no nonconstant invariant rational functions. Then (V, ϕ) has only finitely many maximal proper invariant subvarieties k.*

Proof. Theorem 1.5, together with Lemma 5.3, gives us a definable group acting relatively definably and transitively on $q(\mathcal{U})$. Hence $q(\mathcal{U})$, being an orbit of this definable group action, is itself a definable set. By compactness it is defined by some formula in q.

To deduce the geometric formulation we let q be the generic quantifier-free type of (V, ϕ). Isotriviality implies that q is qf-internal to $\mathcal{C}$ (Proposition 3.6), and that there are no nonconstant invariant rational functions implies that q is weakly orthgonal to $\mathcal{C}$ (Proposition 3.4(a)). So, applying the theorem to q, we have that $S := q(\mathcal{U})$ is definable. Now, note that S is the complement in $(V, \phi)^\sharp$ of the union of all $(W, \phi|_W)^\sharp$ as you range over all proper invariant subvarieties W of (V, ϕ). This is because for any proper subvariety $W \subset V$, the Zariski closure of $W \cap (V, \phi)^\sharp$ is invariant for (V, ϕ). It follows from definability of S that the above mentioned union is equal to a finite sub-union, which implies that only finitely many of the W are maximal. □

If we restrict attention to the autonomous case, this theorem says something about the Zariski dense orbit conjecture: *if $\phi : V \dashrightarrow V$ is a dominant rational self map such that (V, ϕ) has no nonconstant invariant rational functions then there exists a k-point of V whose orbit under ϕ is Zariski dense in V*. When k is uncountable this is a theorem of Amerik and Campana [1]. For countable k it is open in general, though resolved in various cases, including when V is a smooth projective surface and ϕ is regular [24]. The following case (of an isotrivial automorphism) does not seem to have been addressed in the literature.

Corollary 5.5. *Suppose $\phi : V \to V$ is an automorphism of an algebraic variety over k such that (V,ϕ) is isotrivial. If (V,ϕ) admits no nonconstant invariant rational functions then there is $a \in V(k)$ such that the orbit of a under ϕ is Zariski dense in V.*

Proof. By Theorem 5.4, (V,ϕ) has only finitely many maximal proper invariant subvarieties over k. Let W be the union of these. Let a be any k-point of V that is outside W. (This exists by the irreducibility of V and the fact that k is algebraically closed.) Then the Zariski closure of the orbit of a is invariant and defined over k, but not contained in W, and hence equal to all of V. □

5.3. **Bounding nonorthogonality.** Our goal in this final subsection is prove a version of Corollary 5.2 above for general (so possibly nonautonomous) σ-varieties.

It is well known, in stable theories, that a complete type is nonorthogonal to a definable set if and only if some Morley power of it is not weakly orthogonal. The version for rational types nonorthogonal to the fixed field in ACFA appeared as Corollary 2.8 above. The question of how high a Morley power one must take was raised in the eighties (see [13]) and has been addressed in various settings recently, especially for differential-algebraic geometry, see [12, 15, 10]. The main tool in these recent works has been the binding group action. Now that we have an appropriate quantifier-free binding group theorem we obtain the same bound for rational types in ACFA_0.

Theorem 5.6. *Suppose $p \in S_{\mathrm{qf}}(k)$ is rational. If p is nonorthogonal to $\mathcal{C}$ then $p^{(n)}$ is not weakly orthogonal to $\mathcal{C}$ where $n = \dim(p) + 3$.*

Geometric formulation: *Suppose (V,ϕ) is a rational σ-variety over k such that, for some rational σ-variety (W,ψ) over k, the cartesian product $(V,\phi)\times(W,\psi)$ admits an invariant rational function that is not the pullback of a rational function on W. Then (V^n,ϕ) admits a nonconstant invariant rational function for $n = \dim V + 3$. In particular, if some cartesian power of (V,ϕ) admits a nonconstant invariant rational function then already $(V^{\dim V+3},\phi)$ does.*

Proof. Suppose $p \in S_{\mathrm{qf}}(k)$ where k is an algebraically closed difference field. By Proposition 2.11 there is a nonalgebraic rational type $q \in S_{\mathrm{qf}}(k)$ that is qf-internal to $\mathcal{C}$, and a rational map $p \to q$ over k. Let $m := \dim(q) \leq \dim(p)$. It suffices to show that $q^{(m+3)}$ is not weakly orthogonal to $\mathcal{C} := \mathrm{Fix}(\sigma)$. We assume that $q^{(m+3)}$ is weakly orthogonal to $\mathcal{C}$, and seek a contradiction.

By Theorems 1.3 and 1.5, as well as the Weil-group-chunk argument appearing in the proof of Theorem 5.1, after possibly replacing q with something birationally equivalent, q is the generic quantifier-free type of an isotrivial rational σ-variety (V,ϕ) over k, which admits an equivariant faithful algebraic group action

$$\theta : (G \times V, \rho \times \phi) \to (V,\phi),$$

for some algebraic group G with $\rho : G \to G^{\sigma}$ an isomorphism, all over k, and such that θ restricts to an action of $\mathcal{G} = (G,\rho)^{\sharp}$ on $q(\mathcal{U})$ that is isomorphic to the action of $\mathrm{Aut}_{\mathrm{qf}}(q/\mathcal{C})$ on $q(\mathcal{U})$. By Lemma 5.3 we have that the diagonal action of $\mathcal{G}$ on $q^{(m+3)}(\mathcal{U})$ is transitive. It follows that the diagonal action of G on V^{m+3} is generically transitive, in the sense that it has a Zariski dense orbit. Since $m = \dim V$, this is ruled out by the truth of the Borovik-Cherlin conjecture in ACF_0, which implies that the maximum *generic transitivity degree* of an algebraic group action is $\dim V + 2$. See [12, Theorem 6.3].

The geometric formulation follows exactly as in Corollary 5.2, by applying the theorem to the generic quantifier-free type of (V, ϕ) and using Proposition 3.4. The "in particular" clause also follows exactly as in Corollary 5.2. □

In the differential case, where the above bound of $\dim(p) + 3$ was established in [12] and [15, §5], it is known to be sharp. We ask for the same here:

Question 5.7. Is the bound in Theorem 5.6 sharp?

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