# STABLE DEFINABILITY AND GENERIC RELATIONS

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Abstract. An amalgamation base $p$ in a simple theory is *stably definable* if its canonical base is interdefinable with the set of canonical parameters for the $\phi$-definitions of $p$ as $\phi$ ranges through all stable formulae. A necessary condition for stably definability is given and used to produce an example of a supersimple theory with stable forking having types that are not stably definable. This answers negatively a question posed in [8]. A criterion for and example of a stably definable amalgamation base whose restriction to the canonical base is not axiomatised by stable formulae are also given. The examples involve generic relations over non CM-trivial stable theories.

## 1. Introduction and Preliminaries

In a stable theory the canonical base of a stationary type $p$ is the set of canonical parameters for the $\phi$-definitions of $p$ as $\phi$ varies among all formulae. In a simple theory, since types need no longer be definable, an alternative construction of the canonical base was found (cf. [1]). However, if the simple theory has stable forking one might expect canonical bases to have a description in the same spirit as the stable case. Indeed, the first author and A. Pillay have shown (in [8]) that stable forking for a simple theory is equivalent to the canonical base of every amalgamation base being interbounded with the set of canonical parameters of its $\phi$-definitions as $\phi$ ranges over all stable formulae. They asked whether in fact, under the additional assumption that Lascar-strong type equals strong type, interbounded can be replaced by interdefinable. That is, using the terminology introduced below, in a simple theory with stable forking (and Lstp = stp), is every amalgamation base *stably definable*? One consequence of our work here, which began as a close study of the example in Remark 2.9 of [8], is that this is not the case. Indeed, we obtain rather weak sufficient conditions for there to exist amalgamation bases that are not stably definable (Theorem 2.1 below). We also investigate an *a priori* stronger property considered in [7] and [8] (and defined as *stable determinability* below) whereby the restriction of an amalgamation base to its canonical base is axiomatised by stable formulae. It follows from [8] that under strong stable forking, stable definability and stable determinability are equivalent. We show that this is not the case merely assuming stable forking; from sufficient conditions for non stable determinability (Proposition 2.3) we are able to produce stably definable types that are not stably determinable in supersimple theories that have stable forking.

Our examples involve formulating a condition on stable theories which is strictly weaker than non CM-triviality and then adding a generic relation. This is enough

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to obtain stably definable non stably determinable types. To get non stably definable types we require additional hypotheses on the underlying stable theory. In particular, any completion of the theory of algebraically closed fields with a generic predicate has non stably definable types, and stably definable types that are not stably determinable.

We adhere closely, in convention, notation, and terminology, to [8]. While we do assume some familiarity with simplicity theory, we begin by recall a few of the key notions relevant to this paper.

Fix a complete simple theory $T$ and work in a sufficiently saturated universal domain $\overline{M} \models T$. In fact we work in $\overline{M}^{\mathrm{eq}}$ and all tuples are assumed to be (possibly infinite) tuples of imaginary elements, unless explcitly stated otherwise. Sometimes we are interested in *hyperimaginary* elements: elements of the form $a/E$ where $a$ is a tuple of imaginaries and $E(x,y)$ is a *type*-definable equivalence relation. To see how first order model theory generalises to hyperimaginaries, we suggest [1]. The theory $T$ is said to *eliminate hyperimaginaries* if every hyperimaginary is interdefinable with a set of imaginary elements.

The notion of a canonical base of a stationary type in a stable theory can be extended to simple theories. The role of stationarity is played by "amalgamation bases": A complete type $p(x)$ over a hyperimaginary parameter $e$ is called an *amalgamation base* if whenever $d$ and $f$ are hyperimaginaries that are independent over $e$ with $e \in \mathrm{dcl}(d) \cap \mathrm{dcl}(f)$, and $p_1$ and $p_2$ are nonforking extensions of $p$ to $d$ and $f$ respectively, then the union $p_1(x) \cup p_2(x)$ does not fork over $e$. For $p$ an amalgamation base the *canonical base* of $p$, which we denote by $\mathrm{Cb}(p)$, was defined in [1]. This definition is not simply a direct extension of the definition in the stable case, and we leave it to the reader to consult [1] for details. One important complication is that $\mathrm{Cb}(p)$ may only be a hyperimaginary element, even when $p(x)$ is over imaginary paramaters. Indeed, in this paper, when we assume that $T$ has elimination of hyperimaginaries it is usually so that we can treat canonical bases as ordinary (imaginary) tuples.

By a *canonical type* we mean an amalgamation base $p$ whose set of realisations coincides with that of $p|_{\mathrm{Cb}(p)}$. A key property of canonical bases is that if $p$ is a canonical type and $f$ is an automorphism of the universe, then $f$ fixes the set of realisations of $p$ set-wise if and only if it fixes $\mathrm{Cb}(p)$ pointwise.

Given an amalgamation base $p(x)$, let $\mathbb{P}_p$ denote the set of global nonforking extensions of $p|_{\mathrm{Cb}(p)}$ to $\overline{M}$. If $\phi(x,y)$ is a stable formula, then all members of $\mathbb{P}_p$ have the same $\phi$-type. This (global) $\phi$-type is definable, and its $\phi$-definition is called the *$\phi$-definition of* $p(x)$.

**Definition 1.1.** The *stable canonical base* of $p$, denoted by $\mathrm{SCb}(p)$, is the set of canonical parameters for the $\phi$-definitions of $p(x)$, as $\phi(x,y)$ ranges over all stable formulae. We say that $p$ is *stably definable* if $\mathrm{dcl}(\mathrm{SCb}(p)) = \mathrm{dcl}(\mathrm{Cb}(p))$. The theory $T$ is *stably definable* if every amalgamation base is stably definable.

Recall that $T$ is said to have *stable forking* if whenever $q(x)$ is a complete type over a set $B$, and $q$ forks over a subset $A \subseteq B$, then there is an instance of a stable formula $\phi(x,b) \in q(x)$ which forks over $A$. In [8] $T$ is said to have *strong stable forking* if whenever $q(x)$ is a complete type over a set $B$, and $q$ forks over an arbitrary set $A$ (not necessarily contained in $B$), then there is an instance of a stable formula $\phi(x,b) \in q(x)$ which forks over $A$. There are examples of simple theories without strong stable forking (e.g., psuedo-finite fields), but all known simple theories have stable forking. In [8] it was observed that stable forking is equivalent to $\mathrm{Cb}(p) \subseteq \mathrm{bdd}(\mathrm{SCb}(p))$ for all amalgamation bases $p$. In particular, stable definability implies stable forking.

**Remark 1.2.** *Any one-based theory which eliminates hyperimaginaries is stably definable.* This was proved in [7] by the first author.

A related notion is the following:

**Definition 1.3.** An amalgamation base $p(x)$ is said to be *stably determinable* if the canonical type $p|_{\mathrm{Cb}(p)}$ is axiomatised by instances of stable formulae. That is, if there exist a set of stable formulae $\{\phi_i(x, y_i) : i \in I\}$ and tuples $\{b_i : i \in I\}$ from $\overline{M}^{\mathrm{eq}}$, such that

$$a \models p|_{\mathrm{Cb}(p)} \quad \text{if and only if} \quad \models \bigwedge_{i \in I} \phi_i(a, b_i).$$

If every amalgamation base is stably determinable, then $T$ is said to be *stably determinable*.

**Remark 1.4.** It follows from results in [7] that if $p$ is stably determinable then it is stably definable. From [8] one can also conclude that under the assumption of *strong* stable forking the notions of stable determinability and stable definability coincide. It remains open as to whether, under the assumptions of strong stable forking and elimination of hyperimaginaries, every simple theory is stably definable.

The paper is organised as follows. In Section 2 we give general crieria for the existence of non stably definable and non stably determinable amalgamation bases. In Section 3 we apply these criteria to simple theories obtained by adding a generic relation to certain stable theories; thereby producing the desired counterexamples. In a final section we point out that these examples can also be found among pseudo-finite fields.

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## 2. The criteria

In this section $T$ is a complete simple theory, and $\overline{M} \models T$ is a sufficiently saturated universal domain.

**Theorem 2.1** (T eliminates hyperimaginaries)**.** *Let $p(x)$ be an amalgamation base, $c \models p$, and $e = \mathrm{Cb}(p)$. Suppose there exists $d \in \mathrm{dcl}(e)$ and $d' \neq d$, such that $\mathrm{tp}(d/ab) = \mathrm{tp}(d'/ab)$ for some $a$ and $b$ satisfying:*

(1) $\mathrm{acl}(a) = \mathrm{acl}(e)$*, and*
(2) $\mathrm{tp}(c/b)$ *is an amalgamation base and $c$ is independent of $a$ over $b$.*

*Then the following hold:*

(a) *$T$ is not stable,*
(b) *$T$ is not 1-based, and*
(c) *the type $p|_e$ is not stably definable.*

*Proof.* Part (a) follows from part (c), but we give a direct argument here. Let $f$ be an automorphism fixing $ab$ and sending $d$ to $d'$, and let $f(c) = c'$. We first point out that

$$(*) \quad \operatorname{tp}(c/ad) \cup \operatorname{tp}(c'/ad') \text{ is inconsistent.}$$

Indeed, suppose $c^*$ realises this partial type. By (1) and the fact that $c$ and $c^*$ have the same type over $a$, $e = \operatorname{Cb}(\operatorname{stp}(c^*/a))$. Hence $e \in \operatorname{dcl}(c^*a)$, and so $d \in \operatorname{dcl}(c^*a)$. But $c^*ad' \equiv c'ad' \equiv cad \equiv c^*ad$. As $d \neq d'$, this implies that $d \notin \operatorname{dcl}(c^*a)$. The contradiction proves $(*)$.

Now suppose $T$ is stable. By (1) and (2), $\operatorname{tp}(c/bad')$ and $\operatorname{tp}(c'/bad')$ are both nonforking extensions of the stationary type $\operatorname{tp}(c/b) = \operatorname{tp}(c'/b)$. Hence $\operatorname{tp}(c/bad') = \operatorname{tp}(c'/bad')$. In particular, $\operatorname{tp}(c/ad') = \operatorname{tp}(c'/ad')$ contradicting $(*)$. This proves (a).

Part (b) is also a consequence of (c) (cf. Theorem 4.3 of [7]). But we give a direct proof. Suppose $T$ is 1-based. Then $e \in \operatorname{acl}(c)$. Hence $a \in \operatorname{acl}(c)$. As $a$ is independent from $c$ over $b$, it follows that $a \in \operatorname{acl}(b)$. But then, $d$ and $d'$ are also in $\operatorname{acl}(b)$. So we have $a, d, d' \in \operatorname{acl}(b)$. In particular, $ad$ is independent of $ad'$ over $b$. Recall that $\operatorname{tp}(c/bad)$ and $\operatorname{tp}(c'/bad')$ are nonforking extensions of the amalgamation base $\operatorname{tp}(c/b) = \operatorname{tp}(c'/b)$. By the independence theorem, $\operatorname{tp}(c/bad) \cup \operatorname{tp}(c'/bad')$ is consistent, contradicting $(*)$. This proves (b).

We now proceed with the proof of part (c). We need to show that $\operatorname{SCb}(p|_e)$ and $\operatorname{Cb}(p|_e)$ are not interdefinable. Since $d \in \operatorname{dcl}(e)$, it will suffice to show that $d \notin \operatorname{dcl}(\operatorname{SCb}(p|_e))$. Suppose for a contradiction that $d \in \operatorname{dcl}(\operatorname{SCb}(p|_e))$. Then there exist stable formulae $\sigma_1(x,z), \ldots, \sigma_n(x,z)$ such that the $\sigma_i$-definition of $p|_e$ has $e_i$ as its canonical parameter, and $d \in \operatorname{dcl}(e_1, \ldots, e_n)$. Let $e_i' = f(e_i)$ for $i = 1, \ldots, n$, where $f$ is the automorphism fixing $ab$ and taking $cd$ to $c'd'$ given by (2)(ii) . The same function witnessing $d \in \operatorname{dcl}(e_1, \ldots, e_n)$ will witness $d' \in \operatorname{dcl}(e_1', \ldots, e_n')$. As $d \neq d'$, some $e_i \neq e_i'$. We may assume that $e_1 \neq e_1'$.

**Claim 2.2.** $\operatorname{tp}_{\sigma_1}(c/ae_1) \cup \operatorname{tp}_{\sigma_1}(c'/ae_1')$ *is inconsistent.*

*Proof.* Suppose $\operatorname{tp}_{\sigma_1}(c/ae_1) \cup \operatorname{tp}_{\sigma_1}(c'/ae_1')$ is consistent, and extend it to a complete $\sigma_1$-type over $\operatorname{acl}(a)$, say $r(x)$. Then $r(x)$ is a nonforking extension of both $\operatorname{tp}_{\sigma_1}(c/ae_1)$ and $\operatorname{tp}_{\sigma_1}(c'/ae_1')$.

As $e = \operatorname{Cb}(p)$ and $c \models p$, $p|_e$ has the same realisation set as $\operatorname{tp}(c/\operatorname{acl}(e))$. So by (1), the $\sigma_1$-fragment of $p|_e$ is $\operatorname{tp}_{\sigma_1}(c/\operatorname{acl}(a))$. Hence $e_1$ is the canonical base of $\operatorname{tp}_{\sigma_1}(c/\operatorname{acl}(a))$. Since $\sigma_1$ is stable, $\operatorname{tp}_{\sigma_1}(c/ae_1)$ is a stationary $\sigma_1$-type and $\operatorname{tp}_{\sigma_1}(c/\operatorname{acl}(a))$ is its unique nonforking extension to $\operatorname{acl}(a)$. So $r(x) = \operatorname{tp}_{\sigma_1}(c/\operatorname{acl}(a))$.

Similarly $e_1'$ is the canonical base of $\operatorname{tp}_{\sigma_1}(c'/\operatorname{acl}(a))$, which is therefore the unique nonforking extension of $\operatorname{tp}_{\sigma_1}(c'/ae_1')$. Hence $r(x) = \operatorname{tp}_{\sigma_1}(c'/\operatorname{acl}(a))$ as well. That is, $\operatorname{tp}_{\sigma_1}(c/\operatorname{acl}(a)) = \operatorname{tp}_{\sigma_1}(c'/\operatorname{acl}(a))$. But then their canonical bases $e_1$ and $e_1'$ must coincide, which is a contradiction. □

There is a stable formula witnessing Claim 2.2. Indeed, from Claim 2.2 there exists $\chi(x, ae_1) \in \operatorname{tp}_{\sigma_1}(c/ae_1)$ such that $\models \neg\chi(c', ae_1)$. Since $\chi(x, ae_1)$ is equivalent to a boolean combination of instances of $\sigma_1(x,z)$, and $\sigma_1(x,z)$ is stable, there exists $\psi(w) \in \operatorname{tp}(ae_1)$ such that $\chi(x,w) \wedge \psi(w)$ is stable. Setting $\xi(x,w) := \chi(x,w) \wedge \psi(w)$, we have $\models \xi(c, ae_1) \wedge \neg\xi(c', ae_1)$.

Now by (2), $c$ is independent of $a$ over $b$, and hence also $c'$ is independent of $a$ over $b$. As $e_1 \in \operatorname{SCb}(p|_e) \subset \operatorname{acl}(e) = \operatorname{acl}(a)$, it follows that both $c$ and $c'$ are individually independent of $ae_1$ over $b$. In particular, $\operatorname{tp}_\xi(c/bae_1)$ and $\operatorname{tp}_\xi(c'/bae_1)$

both do not fork over $b$. But as $\xi$ is stable and $\mathrm{tp}(c/b)$ is an amalgamation base, it follows that $\mathrm{tp}_\xi(c/b) = \mathrm{tp}_\xi(c'/b)$ is stationary. Hence, $\mathrm{tp}_\xi(c/bae_1) = \mathrm{tp}_\xi(c'/bae_1)$. But this contradicts the fact that $\models \xi(c, ae_1) \wedge \neg\xi(c', ae_1)$, completing the proof of Theorem 2.1. □

The following proposition gives a criterion for a canonical type to not be stably determinable, and is essentially extracted from the example in Remark 2.9 of [8].

**Proposition 2.3** (T eliminates hyperimaginaries)**.** *Let $p(x)$ be a canonical type, $c \models p$, and $e = \mathrm{Cb}(p)$. Suppose that for some $b$ and $c'$*

1. *$\mathrm{tp}(c/b)$ is a nonalgebraic amalgamation base,*
2. *$c' \models \mathrm{tp}(c/b)$ and $c$ is independent of $c'$ over $b$,*
3. *$c' \not\models p$, and*
4. *$cc'$ is independent of $e$ over $b$.*

*Then $p$ is not stably determinable.*

*Proof.* Since $p$ is canonical, the set of realisations of $p(x)$ coincides with that of $\mathrm{tp}(c/e)$. As $c' \not\models p$, there is $\xi(x, s) \in \mathrm{tp}(c/e)$, such that $\models \neg\xi(c', s)$.

**Claim 2.4.** *There is an infinite indiscernible sequence $(c_i : i \in \mathbb{Z})$, with $c_0 = c$, such that $c_i \models p$ for all $i \geq 0$ but $\models \neg\xi(c_i, s)$ for all $i < 0$.*

*Proof.* Since $c$ and $c'$ are independent realisations of $\mathrm{tp}(c/b)$, and this type is an amalgamation base, there is an infinite $b$-indiscernible sequence passing through $(c, c')$. We index this sequence thus:

$$(\dots, c^{-2}, c'^{-2}, c^{-1}, c'^{-1}, c = c^0, c' = c'^0, c^1, c'^1, c^2, c'^2 \dots).$$

Note that the sequence of *pairs* $(\dots, c^{-2}c'^{-2}, c^{-1}c'^{-1}, cc', c^1c'^1, c^2c'^2, \dots)$ is also $b$-indiscernible. On the other hand, $cc'$ is independent of $e$ over $b$. It follows that fixing $cc'$ we can move $(\dots, c^{-2}c'^{-2}, c^{-1}c'^{-1}, cc', c^1c'^1, c^2c'^2, \dots)$ by an automorphism in such a way that it becomes $eb$-indiscernible. Relabelling we may assume that $(\dots, c^{-2}c'^{-2}, c^{-1}c'^{-1}, cc', c^1c'^1, c^2c'^2 \dots)$ is $eb$-indiscernible. In particular $c^i \models p$ but $\models \neg\xi(c'^i, s)$ for all $i$. Hence the $b$-indiscernible subsequence of the original sequence given by $(\dots, c'^{-2}, c'^{-1}, c, c^1, c^2, \dots)$ has the required properties (setting $c_i := c'^i$ for $i < 0$ and $c_i := c^i$ for $i \geq 0$). □

Now suppose that $p$ *is* stably determinable and seek a contradiction. As $p$ is canonical this means that $p(x)$ has the same set of realisations as $\bigwedge_k \phi_k(x, a_k)$, where each $\phi_k(x, z_k)$ is stable. By compactness and the fact that $\xi(x, s) \in p$, some finite conjunction of the $\phi_k$'s, say $\phi(x, a) := \bigwedge_{k=1}^{n} \phi_k(x, a_k)$, implies $\xi(x, s)$. By Claim 2.4, $\models \neg\phi(c_i, a)$ for all $i < 0$ while $\models \phi(c_i, a)$ for all $i \geq 0$ (since $\models \neg\xi(c_i, s)$ for all $i < 0$ but $c_i \models p$ for all $i \geq 0$). That is, $(c_i : i \in \mathbb{Z})$ and $a$ witness the instability of $\phi(x, z)$ – which is a contradiction. This proves Proposition 2.3. □

## 3. Generic predicates over stable theories

Let $T^-$ be a complete stable theory admitting quantifier elimination and eliminating $\exists^\infty$, in a language $\mathcal{L}^-$. Let $\mathcal{L} = \mathcal{L}^- \cup \{R\}$, where $R$ is a new binary[1]

[1] In what follows we could just as well work with an $n$-ary predicate symbol for any $n \geq 2$.

predicate symbol. By results in [3], $T^-$ has a model companion in $\mathcal{L}$. This model companion, $T^-_R$, is axiomatised by $T^-$ together with an axiom for each $\mathcal{L}^-$- formula $\phi(x_1, y_1, \ldots, x_n, y_n, \overline{z})$ and each subset $I \subseteq \{1, \ldots, n\}$, stating that: *for all* $\overline{c}$*, if there exist distinct pairs* $(a_1, b_1), \ldots, (a_n, b_n) \notin \operatorname{acl}^-(\overline{c})$ *with* $\models \phi(a_1, b_1, \ldots, a_n, b_n, \overline{c})$*, then there exist* $x_1, y_1 \ldots, x_n, y_n$ *such that:*

$$\models \phi(x_1, y_1, \ldots, x_n, y_n, \overline{c}) \wedge \bigwedge_{i \in I} R(x_i, y_i) \wedge \bigwedge_{j \notin I} \neg R(x_j, y_j).$$

Moreover, the completions of $T^-_R$ are given by describing $R$ on $\operatorname{acl}^-(\emptyset)$.

**Remark 3.1.** We have been intentionally ambigious about what sorts the pairs come from. Indeed, we want $R$ to be a binary relation on all of $(\mathcal{L}^-)^{\mathrm{eq}}$. This can be done as follows: For every pair of sorts, $S$ and $S'$ from $(\mathcal{L}^-)^{\mathrm{eq}}$, let $R_{SS'}$ be a new unary predicate on $S \times S'$. The model companion is obtained by adding the above axioms for each $R_{SS'}$. Since all variables belong to particular sorts, by an abuse of notation, we may (and will) use $R$ to represent all of these new predicates at once.

**Fact 3.2** (cf. [3])**.** *Let* $T$ *be any completion of* $T^-_R$*, and* $M \models T$ *saturated.*

- (a) *Algebraic closure in the sense of* $\mathcal{L}$ *and* $\mathcal{L}^-$ *coincide.*
- (b) *Given tuples* $a, b$*, and a set* $A$*,* $\operatorname{tp}(a/A) = \operatorname{tp}(b/A)$ *if and only if there is an* $\mathcal{L}$*-isomorphism from* $\operatorname{acl}(A, a)$ *to* $\operatorname{acl}(A, b)$ *taking* $a$ *to* $b$ *and fixing* $A$ *pointwise.*
- (c) *Given a tuple* $a$ *and sets* $B \subseteq A$*,* $\operatorname{tp}(a/A)$ *forks over* $B$ *if and only if* $\operatorname{tp}^-(a/A)$ *forks over* $B$*. In particular,* $T$ *is simple and has stable forking.*
- (d) *The independence theorem holds over algebraically closed sets.*

**Remark 3.3.** *Every completion of* $T^-_R$ *eliminates hyperimaginaries.* Indeed, from Fact 3.2(d), it follows that $\mathrm{Lstp} = \mathrm{stp}$ over all sets. This together with stable forking implies elimination of hyperimaginaries (cf Lema 3.3 of [8], for example).

**Lemma 3.4.** *Suppose* $M^-$ *is a saturated model of* $T^-$*, and* $T$ *is any completion of* $T^-_R$*. Suppose* $A$ *is a small algebraically closed substructure of* $M^-$*, and consider any binary relation* $r$ *on* $A$ *such that* $r|_{\operatorname{acl}^-(\emptyset)}$ *is compatible with what is dictated by* $T$*. Then* $r$ *can be extended to a binary relation on* $M^-$ *such that* $M := (M^-, r) \models T$*.*

*Proof.* Let $F$ be a small model of $T^-$ containing $A$, and extend $r$ to $F$ in any way. By model companionship $(F, r)$ can be extended to a model $N \models T^-_R$. By choice of $r$ on $\operatorname{acl}^-(\emptyset)$, $N \models T$. Now take an elementary extension/substructure $K$ of $N$ containing $F$ of cardinality $\operatorname{card}(M^-)$. By saturation of $M^-$ there is an $\mathcal{L}^-$-isomorphism from $K^- := K|_{\mathcal{L}^-}$ to $M^-$ over $A$. Let $r$ on $M^-$ be the image of $R^K$ under this isomorphism. □

We use $\mathrm{Cb}^-$ to mean the canonical base in the sense of $\mathcal{L}^-$.

**Lemma 3.5.** *Suppose* $T$ *is a completion of* $T^-_R$*,* $M$ *is a saturated model of* $T$*, and* $c, a \in M^{\mathrm{eq}}$*. Then*

- (a) $\mathrm{Cb}^-(c/a) \subseteq \mathrm{SCb}(c/a)$ *and*
- (b) $\mathrm{Cb}^-(c/a)$ *is interalgebraic with* $\mathrm{Cb}(c/a)$*.*

*Proof.* Part (a) follows from the stability of $T^-$: every $\mathcal{L}^-$ formula $\phi$ is stable, and a code in $M^-$ for the $\phi$-definition of the $\phi$-type of $c$ over $\operatorname{acl}^-(a) = \operatorname{acl}(a)$ remains a code in $M$.

For part (b), let $C := \mathrm{Cb}^-(c/a)$ and let $p(x) := \mathrm{stp}(c/a)$. It suffices to show that $p$ does not fork over $C$. If $p$ forks over $C$ then, by Fact 3.2(c), $p|_{\mathcal{L}^-}$ forks over $C$ – contradicting the fact that $C = \mathrm{Cb}^-(p|_{\mathcal{L}^-})$. □

3.1. **Non Stable Determinability.** In this section we obtain conditions on $T^-$ that ensure that completions of $T^-_R$ will have stably definable, non stably determinable, types.

**Proposition 3.6.** *Let $M^- \models T^-$ be saturated, and suppose there exist (possibly infinite) tuples $c, a, b = \mathrm{acl}(b)$ such that*

(i) *$c$ is independent of $a$ over $b$,*
(ii) *$c, a \notin b$, and*
(iii) *$\mathrm{Cb}^-(c/a) = a$.*

*Then for any completion $T$ of $T^-_R$ there is an expansion of $M^-$ to a model $M \models T$, such that $\mathrm{stp}(c/a)$ is stably definable but not stably determinable.*

*Proof.* Write $c = (c_1, c_2, \dots)$ and $a = (a_1, a_2, \dots)$. Choose $c' \models \mathrm{tp}^-(c/b)$ with

$$(*) \quad c' \text{ independent of } ca \text{ over } b.$$

Let $A$ be an algebraically closed substructure containing $c, c', a, b$, and expand $M^-$ to a model $M$ of $T$ such that every pair from $A$ with no component in $\mathrm{acl}(\emptyset)$ is $R$-related except for $(c'_1, a_1)$. This is possible by Lemma 3.4.

Let $p = \mathrm{stp}(c/a)$. Note that by assumption (iii) and Lemma 3.5(b), $p$ is a canonical type. We show it is stably definable. Let $f$ be any automorphism fixing $a$. Then, as $\mathrm{Cb}^-(c/a) = a$, we have that $\mathrm{stp}^-(c/a) = \mathrm{stp}^-(f(c)/a)$. So there is an $\mathcal{L}^-$-isomorphism, $g$, fixing $\mathrm{acl}(a)$ pointwise and taking $c$ to $f(c)$. By our choice of $R$ on $\mathrm{acl}(ca)$ – namely that every pair not both of whose components are in $\mathrm{acl}(\emptyset)$ is $R$-related – $g$ restricts to an $\mathcal{L}$-isomorphism from $\mathrm{acl}(ca)$ to $\mathrm{acl}(f(c)a)$ over $\mathrm{acl}(a)$. Hence $f(c) \models p$ by Fact 3.2(b). We have shown that $\mathrm{Cb}(p) \subseteq \mathrm{dcl}(a)$. But by assumption (iii) and Lemma 3.5(a), this implies that $\mathrm{Cb}(p) \subseteq \mathrm{dcl}(\mathrm{SCb}(p))$. That is, $p$ is stably definable.

To show that $p$ is not stably determinable, we now check conditions (1)–(4) of Proposition 2.3.

(1) $\mathrm{tp}(c/b)$ *is a nonalgebraic amalgamation base*: It is an amalgamation base by Fact 3.2(d) and because $b = \mathrm{acl}(b)$; and it is nonalgebraic by (ii).
(2) $\mathrm{tp}(c'/b) = \mathrm{tp}(c/b)$: By (i) and (ii), $e_1 \notin \mathrm{acl}(cb)$, and so all pairs from $\mathrm{acl}(cb) \cap M^2 \setminus \mathrm{acl}(\emptyset) \cap M^2 \subset R^M$. Similarly, by $(*)$ and (ii), $e_1 \notin \mathrm{acl}(c'b)$. Hence $\mathrm{acl}(c'b) \cap M^2 \setminus \mathrm{acl}(\emptyset) \cap M^2 \subset R^M$ also. It follows by Fact 3.2(b) that $\mathrm{tp}(c'/b) = \mathrm{tp}(c/b)$.
(3) $c' \not\models p$: This is because $M \models \neg R(c'_1, a_1)$ while $M \models R(c_1, a_1)$.
(4) *$cc'$ is independent of $e = \mathrm{Cb}(c/a)$ over $b$*: By $(*)$, $cc'$ is independent of $a$ over $b$. But, as $a = \mathrm{Cb}^-(c/a)$ we have that $a$ is interalgebraic with $e$ by Lemma 3.5(b).

Hence Proposition 2.3 applies, and $p$ is not stably determinable. □

Toward an application of the above proposition, recall the following notion introduced by Hrushovski in [5]:[2]

[2]We were also informed by Pillay's reformulation of non CM-triviality as *2-ampleness* in [9].

**Definition 3.7.** A stable theory is *CM-trvial* if all of its models satisfy the following condition: For all algebraically closed $A, B, C$; if $\mathrm{acl}(A \cup C) \cap \mathrm{acl}(A \cup B) = A$ then $\mathrm{Cb}(C/A) \subset \mathrm{acl}(\mathrm{Cb}(C/A \cup B))$.

**Corollary 3.8.** *If $T^-$ is not CM-trivial then any completion $T$ of $T_R^-$ has a stably definable but non stably determinable canonical type.*

*Proof.* We will show that non CM-triviality implies the existence of $c, a, b$ satisfying conditions (i)–(iii) of Proposition 3.6. Let $M^-$ be a saturated model of $T^-$. As $T^-$ is not CM-trivial, there exist $A \subset B$ and $c$ with

(1) $\mathrm{acl}(cA) \cap \mathrm{acl}(B) = \mathrm{acl}(A)$ while
(2) $\mathrm{Cb}^-(c/A)$ is not contained in $\mathrm{acl}(\mathrm{Cb}^-(c/B))$.

Let $b := \mathrm{acl}(\mathrm{Cb}^-(c/B))$ and $a := \mathrm{Cb}^-(c/A)$. Note that by elimination of hyperimaginaries (Remark 3.3) these are (possibly infinite) tuples from $(M^-)^{\mathrm{eq}}$.

Now $c$ is independent of $\mathrm{acl}(B)$ over $b$, and hence in particular of $a$ over $b$. It is clear that $a = \mathrm{Cb}^-(c/a)$. It remains to show, therefore, that $c, a \notin b$. By (2), $a \notin b$. If $c \in b$ then by (1), $c \in \mathrm{acl}(A)$; and so $c = \mathrm{Cb}^-(c/A)$ and $c = \mathrm{Cb}^-(c/B)$ – contradicting (2). Hence, by Proposition 3.6, for any completion $T$ of $T_R^-$, $p = \mathrm{stp}(c/a)$ is stably definable but not stably determinable. □

**Remark 3.9.** It follows that if $T^-$ is non CM-trivial (or more generally, satisfies the hypotheses of Proposition 3.6) then $T$ does not have strong stable forking (cf. Proposition 2.5 of [8]).

In particular, any completion of the theory of algebraically closed fields in any fixed characteristic equipped with a generic predicate has stably definable but non stably determinable types. For a very different example we can take $T^-$ to be the *free pseudospace* constructed by Pillay and Baudisch [2]; which is a non CM-trivial stable theory that does not interpret a field.

**Remark 3.10.** Let us, provisionally, call a stable theory *weakly* 1*-based* if there does not exist $c, a$, and $b = \mathrm{acl}(b)$ satisfying conditions (i)–(iii) of Proposition 3.6. It is not hard to see that 1-based theories are weakly 1-based in this sense. On the other hand, as we saw in the proof of Corollary 3.8, weakly 1-based theories are CM-trivial. So

$$\text{1-based} \implies \text{weakly 1-based} \implies \text{CM-trivial}$$

The question arises as to whether these implications are strict. The second is strict: Hrushovski's example of a stable $\omega$-categorical psuedoplane (cf. Wagner's [10] treatment of this example) is CM-trivial but it is not weakly 1-based – this is witnessed by any triple of distinct elements $c, a, b$ where $c$ is related to $b$, $a$ is related to $b$, and $c$ and $a$ are independent. However, we do not know an example of a weakly 1-based theory that is not 1-based.

### 3.2. **Non Stable Definability.**

We now investigate how generic predicates can be used to produce non stably definable types.

**Proposition 3.11.** *Let $M^- \models T^-$ be saturated, and suppose $c, a, b = \mathrm{acl}(b)$ is a witness to the weak non CM-triviality of $T^-$. That is,*

(i) *$c$ is independent of $a$ over $b$,*
(ii) *$c, a \notin b$,*

(iii) $\mathrm{Cb}^-(c/a) = a$.

*Suppose moreover that* $a = (a_1, a_2, \dots)$ *is such that*

(iv) $a_1$ *is independent of* $b$ *while* $a_i \in \mathrm{dcl}(a_1 b)$ *for* $i > 1$, *and*

(v) $\mathrm{acl}(a_1) \setminus \mathrm{dcl}(a_1 \mathrm{acl}(\emptyset)) \neq \emptyset$.

*Then for any completion* $T$ *of* $T_R^-$ *there is an expansion of* $M^-$ *to a model* $M \models T$, *such that* $\mathrm{stp}(c/a)$ *is not stably definable.*

*Proof.* Let $d \in \mathrm{acl}(a_1) \setminus \mathrm{dcl}(a_1 \mathrm{acl}(\emptyset))$ and let $d_1, \dots, d_n$ be the $a_1$-conjugates of $d$ that are distinct from $d$. Since $d \notin \mathrm{dcl}(a_1 \mathrm{acl}(\emptyset))$, for some $i = 1, \dots, n$, $d' = d_i$ is an $a_1 \mathrm{acl}(\emptyset)$-conjugate of $d$. Write $c = (c_1, c_2, \dots)$.

Let $A$ be an algebraically closed substructure containing $c, a, b$, and expand $M^-$ to a model $M$ of $T$ such that $M \models \bigwedge_{j=1}^{n} \neg R(c_1, d_j)$ but all other pairs from $A$ with no component in $\mathrm{acl}(\emptyset)$ are $R$-related. This is possible by Lemma 3.4.

Let $p = \mathrm{stp}(c/a)$, $e = \mathrm{Cb}(p)$. By assumption (iii) and Lemma 3.5(b), $\mathrm{acl}(a) = \mathrm{acl}(e)$ and so $p$ is a canonical type. We wish to apply Theorem 2.1 to the data $(p, c, e, d, d', a, b)$ to conclude that $p$ is not stably definable. We already have conditions (1) and (2); namely that $\mathrm{acl}(a) = \mathrm{acl}(e)$ and that $\mathrm{tp}(c/b)$ is an amalgamation base (as $b = \mathrm{acl}(b)$) and $c$ is independent of $a$ over $b$. It remains to show that $d \in \mathrm{dcl}(e)$ and that $\mathrm{tp}(d/ab) = \mathrm{tp}(d'/ab)$; which we do in the following claims:

**Claim 3.12.** $d \in \mathrm{dcl}(e)$

*Proof.* First note that $a \in \mathrm{dcl}(e)$. Indeed

$$a = \mathrm{Cb}^-(c/a) \subseteq \mathrm{SCb}(c/a) \subseteq \mathrm{dcl}(\mathrm{Cb}(c/a)) = \mathrm{dcl}(e),$$

where the first containment is by Lemma 3.5(a).

Suppose $d \notin \mathrm{dcl}(e)$. Then there is an automorphism $g$ fixing $e$ and moving $d$. Since $a \in \mathrm{dcl}(e)$, $g$ fixes $a$, and hence $g(d) = d_j$ for some $j = 1, \dots, n$. Now $R(x_1, d) \wedge \neg R(x_1, d_j) \in p$ by choice of $M$. Hence $g$ cannot fix the set of realisations of $p$. But this contradicts the fact that $p$ is a canonical type and $e = \mathrm{Cb}(p)$. □

**Claim 3.13.** $\mathrm{tp}(d/ab) = \mathrm{tp}(d'/ab)$

*Proof.* Note that $a_1$ is independent of $b$ and $d \in \mathrm{acl}(a_1)$. So $da_1$ is independent of $b$. Similarly, $d'a_1$ is independent of $b$. But by choice of $d'$, $\mathrm{stp}^-(da_1) = \mathrm{stp}^-(d'a_1)$. Hence, by stationarity, $\mathrm{tp}^-(da_1/b) = \mathrm{tp}^-(d'a_1/b)$. Now (iv) implies that $\mathrm{tp}^-(d/ab) = \mathrm{tp}^-(d'/ab)$.

On the other hand, $c \notin \mathrm{acl}(ab)$, by (i) and (ii). Hence, every pair from $\mathrm{acl}(ab)$ with at least one component not in $\mathrm{acl}(\emptyset)$, is $R$-related. This, together with the fact that $\mathrm{tp}^-(d/ab) = \mathrm{tp}^-(d'/ab)$, implies that $\mathrm{tp}(d/ab) = \mathrm{tp}(d'/ab)$. □

This proves Proposition 3.11. □

For the rest of this section we will discuss the following application of Proposition 3.11.

**Example 3.14.** *Let* $T^- = \mathrm{ACF}_p$ *where* $p$ *is either* $0$ *or prime. Then any completion* $T$ *of* $T_R^-$ *is non stably definable.*

*Proof.* Let $K \models \mathrm{ACF}_p$ be saturated and choose $a_1, a_2, a_3, b_1, b_2$ algebraically independent transcendental elements. Let $b_3 := a_1 + a_2 b_1$ and $b_4 := a_2 b_2 + a_3$. Set $a := (a_1, a_2, a_3)$ and $b := \mathrm{acl}(b_1, b_2, b_3, b_4)$. Letting $P_a \subset K^3$ be the plane defined by the equation

$$X_3 = a_1 X_1 + a_2 X_2 + a_3,$$

and $L_b \subset K^3$ the line defined by the equations

$$\begin{aligned} X_2 &= b_1 X_1 + b_2 \\ X_3 &= b_3 X_1 + b_4 \end{aligned}$$

it is not hard to see that $L_b$ lies on $P_a$. Moreover, the field generated by $a$ is the minimal field of definition for $P_a$ and the field generated by $b_1, \ldots, b_4$ is the minimal field of definition for $L_b$.

Choose $c \in L_b$ such that $c \notin \mathrm{acl}(ab)$. We aim to show that $c, a, b$ satisfies (i)–(v) of Proposition 3.11. For (i), we note that since $c \in L_b$ and $c \notin \mathrm{acl}(ab)$, $1 = \dim(c/b) = \dim(c/ab)$ – so that $c$ is independent of $a$ over $b$. For (ii) it remains to check that $a$ is not in $b$: but if it were then $a_1, a_2, a_3, b_1, b_2 \in \mathrm{acl}(b_1, \ldots, b_4)$ which contradicts the fact that the transcendence degree of $a_1, a_2, a_3, b_1, b_2$ is 5 by choice.

**Claim 3.15.** *$c$ is a generic point in $P_a$ over* $\mathrm{acl}(a)$*.*

*Proof.* Let $V \subseteq P_a$ be the $\mathrm{acl}(a)$-locus of $c$ in the sense of algebraic geometry. As $c \notin \mathrm{acl}(ab)$, and $c \in L_b \cap V$, we must have that $L_b \subseteq V$. But $L_b \neq V$, else $L_b$ would be defined over $\mathrm{acl}(a)$ and so $(b_1, \ldots. b_4)$ would be contained in $\mathrm{acl}(a)$, which contradicts our choice. Hence, since $V$ is irreducible, $V = P_a$. That is, $c$ is generic in $P_a$ over $\mathrm{acl}(a)$. □

By Claim 3.15 together with the fact that canonical bases coincide (up to interdefinability) with minimal fields of definition, we have that $\mathrm{Cb}^-(c/a) = a$ – that is, we have established (iii).

We check (iv): First by choice of $b_3$ and $b_4$ it is clear that $a_2, a_3 \in \mathrm{dcl}(a_1 b)$. Moreover, this implies that $a_1 \notin b$, else so would $a_2$ and $a_3$ –which contradicts (ii). Hence, $a_1$ is independent of $b$.

Finally, for (v), we can take $d$ to be a square root of $a_1$ if $p \neq 2$ and a cube root of $a_1$ if $p = 2$.

Hence, by Proposition 3.11, for any completion $T$ of $T_R^-$, there is an expansion of $K$ to a model of $T$ in which $p = \mathrm{stp}(c/a)$ is not stably definable. □

In particular, there exist supersimple theories with stable forking that are not stably definable. This answers in the negative a question from [8].

On the other hand, it is shown in [7] that in any supersimple theory the canonical base of any amalgamation base $p$ *is* interdefinable with the set of canonical parameters for the $\psi$-definitions of $p(x)$ as $\psi(x, y)$ range over all *$p$-stable* [3] formulae. Hence, in the above example there must exist a $p$-stable formula which is not stable. We will exhibit such a formula.

Recovering the notation of Example 3.14, let $M = (K, R)$ be the expansion of $K$ to a model of $T$, given by the proof of Proposition 3.11, in which $p = \mathrm{stp}(c/a)$

[3] Recall that $\psi(x, y)$ is *$p$-stable* if all members of $\mathbb{P}_p$ have the same $\psi$-type, in which case this (global) $\psi$-type is definable, and its $\psi$-definition is what we mean by the *$\psi$-definition of* $p(x)$.

is not stably definable. For concreteness, assume $\mathrm{char}(K) \neq 2$. Let $x = (x_1, x_2, x_3)$ and $w = (w_1, w_2, w_3)$ and consider the formula

$$\psi(x, w) \ := \ [x_3 = (w_1)^2 x_1 + w_2 x_2 + w_3] \wedge R(x_1, w_1).$$

Letting $\hat{a} = (d = \sqrt{a_1}, a_2, a_3)$, note that $\psi(x, \hat{a})$ says "$x \in P_a$ and $R(x_1, d)$". In particular, $\psi(x, \hat{a}) \in p(x)$.

**Remark 3.16.** The formula $\psi(x, w)$ is unstable but $p$-stable.

*Proof.* Suppose $\psi(x, w)$ is stable. Let $c' = f(c)$ where $f$ is an automorphism which fixes $ba$ pointwise and takes $d$ to the other square root of $a_1$, $d'$. Hence $\models \neg R(c'_1, d)$ and so $\models \psi(c, \hat{a}) \wedge \neg\psi(c', \hat{a})$. On the other hand, both $c$ and $c'$ are independent of $ab$ over $b$ (since $1 \geq \mathrm{tr.deg.}(c/b) \geq \mathrm{tr.deg.}(c/ab) \geq 1$). In particular, $\mathrm{tp}_\psi(c/\operatorname{acl}(ab))$ and $\mathrm{tp}_\psi(c'/\operatorname{acl}(ab))$ do not fork over $b$. But $\mathrm{tp}_\psi(c/b) = \mathrm{tp}_\psi(c'/b)$ is stationary as $\psi$ is stable and $b = \operatorname{acl}(b)$. Hence, $\mathrm{tp}_\psi(c/\operatorname{acl}(a, b)) = \mathrm{tp}_\psi(c'/\operatorname{acl}(a, b))$. But this contradicts the fact that $\models \psi(c, \hat{a}) \wedge \neg\psi(c', \hat{a})$. So $\psi(x, w)$ is unstable.

Now suppose $\psi(x, w)$ is not $p$-stable. By a criteria given in [7], there exists a tuple $e = (e_1, e_2, e_3)$ and a $\mathrm{Cb}(p)$-indiscernible sequence $(c^i : i \in \mathbb{Z})$ of realisations of $p|_{\mathrm{Cb}(p)}$ such that $c^i$ is independent of $e$ over $\mathrm{Cb}(p)$ for all $i \in \mathbb{Z}$, and $\models \psi(c^i, e)$ if and only if $i \geq 0$. We may assume that $c^0 = c$, and so $\models \psi(c, e)$. Letting $e' = ((e_1)^2, e_2, e_3)$, we have

(i) $c \in P_{e'}$,
(ii) $c$ is independent of $e'$ over $\mathrm{Cb}(p)$, and
(iii) $\models R(c_1, e_1)$.

As $c$ is a generic point of $P_a$ over $\mathrm{Cb}(p)$, (i) and (ii) imply that $P_a = P_{e'}$, and so $a = e'$. That is, $a_2 = e_2$, $a_3 = e_3$, and $a_1 = (e_1)^2$. Since $\models \neg R(c_1, d')$, (iii) implies that $e_1 = d$. Hence $e = \hat{a}$ and so $\psi(x, e) \in p(x)$. Since $\models \neg\psi(c^{-1}, e)$, we have that $c^{-1}$, which is a realisation of $p|_{\mathrm{Cb}(p)}$, does not realise $p$. This contradicts the fact that $p$ is a canonical type. □

To see explicitly how $\psi(x, w)$ is responsible for the non stable definability of $p(x)$, it is worth noting that the $\psi$-definition of $p$ is the formula "$w = \hat{a}$", and that the canonical parameter of this formula is $\hat{a} = (d, a_1, a_3)$ itself, which we know by the proof of Theorem 2.1 is in $\mathrm{Cb}(p)$ but not in $\mathrm{SCb}(p)$. To see that "$w = \hat{a}$" is the $\psi$-definition of $p$, suppose $\psi(x, e)$ is in some (equivalently all) $\mathbf{q} \in \mathbb{P}_p$. Then "$x \in P_{e'}$" is in $\mathbf{q}$, where $e' = (e_1^2, e_2, e_3)$. Since $c$ is generic in $P_a$, this implies that $P_{e'} = P_a$, and so $e' = a$. Hence either $e = \hat{a}$ or $e = (d', a_2, a_3)$, where $d'$ is the other square root of $a_1$. The latter is impossible as it would imply that $\neg R(x_1, e_1) \in \mathbf{q}$ (since $\models \neg R(c_1, d')$) while we already know that $R(x_1, e_1) \in \mathbf{q}$ (since $\psi(x, e) \in \mathbf{q}$). Hence $e = \hat{a}$.

**Remark 3.17.** This example also yields a concrete instance of a tuple $x$ and sets $E$ and $F$, such that $\mathrm{tp}^-(x/F)$ does not fork over $E$ while $\mathrm{tp}(x/F)$ *does* fork over $E$. This despite Fact 3.2(c) – the point being that here $E \not\subseteq F$. Indeed, since $p = \mathrm{tp}(c/\operatorname{acl}(a))$ is not stably definable it is not stably determinable and hence $\mathrm{tp}^-(c/\operatorname{acl}(a)) \not\vdash p$. Hence there exists a realisation $c_\circ$ of $\mathrm{tp}^-(c/\operatorname{acl}(a))$ such that $c_\circ \not\models p$. We can find a $c_\circ$-indiscernible sequence $(a^i)$ in the type of $a$ (with $a = a^0$) such that $\bigcap_i P_{a^i} = \{c_\circ\}$. Hence $\bigcup_i p_i$ is inconsistent, where $p_i$ is the conjugate of $p$ under the automorphism taking $a$ to $a^i$. So $p = \mathrm{tp}(c/\operatorname{acl}(a))$ forks over $c_\circ$. But $\mathrm{tp}^-(c/\operatorname{acl}(a))$ does not fork over $c_\circ$ in the sense of $T^-$ as it is realised by $c_\circ$.

**Remark 3.18.** It is important that we work with a plane rather than a line. In fact, if $\psi'(x,w) := [x_2 = (w_1)^2 x_1 + w_2] \wedge R(x_1, w_1)$ then it is not hard to see that $\psi'(x,w)$ *is* stable. Indeed, suppose the instability of $\psi'(x,w)$ were witnessed by infinite sequences $(c^i : i \in \mathbb{N})$ and $(e^j : j \in \mathbb{N})$ such that $\models \psi'(c^i, e^j)$ if and only if $i > j$. Then the line given by $X_2 = (e_1^0)^2 X_1 + e_2^0$ and the line given by $X_2 = (e_1^1)^2 X_1 + e_2^1$ share infinitely many common points (namely $c^2, c^3, \dots$) and hence coincide. But then $e^0 = e^1$, which is a contradiction.

## 4. Psuedo-finite fields

In this final section we point out that the above techniques also work in psuedo-finite fields to produce both non stably definable types and stably definable non stably determinable types. The key observation, due to Duret, is that if $k$ is a psuedo-finite field, $q$ is a prime number different than the characteristic of $k$, and $k$ contains the $q$th roots of unity, then the formula $\exists z(z^q = x + y) \wedge (x \neq y)$ defines a random graph in $k$. This random graph plays the role of the generic predicate of the previous section, while the role of $T^-$ is played by the quantifier-free fragment of the theory of $k$ in the language of rings.

Here are some facts about psuedo-finite fields that we will use freely.

**Fact 4.1** (cf. [6])**.** *Let $T = \mathrm{Th}(k, +, -, \times, 0, 1)$ where $k$ is a psuedo-finite field, and work in a sufficiently saturated elementary extension $F \succeq k$.*

(a) *For any subfield $L$ containing $k$, $\mathrm{acl}(L) = L^{\mathrm{alg}} \cap F$.*
(b) *Given tuples $u, v$ and a subfield $L$ containing $k$, $\mathrm{tp}(u/L) = \mathrm{tp}(v/L)$ if and only if there is an field-isomorphism from $L(u)^{\mathrm{alg}} \cap F$ to $L(v)^{\mathrm{alg}} \cap F$ taking $u$ to $v$ and fixing $L$ pointwise.*
(c) *$T$ is supersimple (and hence eliminates hyperimaginaries). Moreover, non-forking in $F$ is characterised by non-forking in $F^{\mathrm{alg}}$: given a tuple $u$ and subfields $K \subseteq L$ containing $k$, $\mathrm{tp}(u/L)$ does not fork over $K$ if and only if $\mathrm{tr.deg.}(K(u)/K) = \mathrm{tr.deg.}(L(u)/L)$.*
(d) *The independence theorem holds over algebraically closed sets.*

For the rest of this section, let us fix a psuedo-finite field $k$ containing the algebraic closure of the prime field $\mathbb{F}$. Let $T = \mathrm{Th}(k)$ and work in a sufficiently saturated elementary extension $F \succeq k$. In what follows we will work over $\mathbb{F}^{\mathrm{alg}}$ (by naming the elements of $\mathbb{F}^{\mathrm{alg}}$ for example). Fix a prime $q \neq \mathrm{char}(k)$, and let $R(x, y)$ denote the relation on $F$ defined by $\exists z(z^q = x + y) \wedge (x \neq y)$.

**Fact 4.2** (cf. Lemme 6.2 and Corollaire 4.3 of [4])**.** *$R$ is a random graph on $F$. That is, given two disjoint finite sets of distinct elements $\{u_i : i \in I\}$ and $\{u_j : j \in J\}$, there exists $v \in F$ such that*

$$\models \bigwedge_{i \in I} R(v, u_i) \wedge \bigwedge_{j \in J} \neg R(v, u_j).$$

We now follow the construction of Example 3.14 to produce non stably definable types and stably definable but non stably determinable types. Our assumptions that $k$ contains $\mathbb{F}^{\mathrm{alg}}$ and that $F$ is saturated ensure that there are subfields of $F$ that are algebraically closed and of infinite transcendence degree. Choose $a_1, a_2, a_3, b_1, b_2 \in F$ algebraically independent such that $\mathbb{F}(a_1, a_2, a_3, b_1, b_2)^{\mathrm{alg}}$ is contained in $F$, and let $b_3 = a_1 + a_2 b_1$ and $b_4 = a_2 b_2 + a_3$. Set $a = (a_1, a_2, a_3)$ and $b = (b_1, b_2, b_3)$. Note that $\mathbb{F}(a, b)^{\text{alg}} \subset F$. Let $P_a$ be the plane defined by

$$X_3 = a_1 X_1 + a_2 X_2 + a_3,$$

and $L_b$ the line in $P_a$ defined by

$$X_2 = b_1 X_1 + b_2$$
$$X_3 = b_3 X_1 + b_4.$$

4.1. **A non stably definable type.** Let $d, d'$ be the distinct square roots of $a_1$. Use Fact 4.2 and saturation to find $c_1 \in F \setminus \mathbb{F}(a, b)^{\text{alg}}$ such that $F \models R(c_1, d) \wedge \neg R(c_1, d')$. Setting $c_2 = b_1 c_1 + b_2$ and $c_3 = b_3 c_1 + b_4$ we obtain a point $c := (c_1, c_2, c_3) \in L_b(F) \subset P_a(F)$. Consider $p := \text{stp}(c/a)$.

- *$p$ is a canonical type and* $\text{acl}(a) = \text{acl}(\text{Cb}(p))$: Exactly as in Claim 3.15 of Example 3.14, $c$ is a generic point in $P_a$ over $\mathbb{F}(a)^{\text{alg}}$. Using Fact 4.1(c), it is then not hard to see that $P_a$ is an irreducible component of $\text{Cb}(p)$-locus of $c$. Since $a$ generates the minimal field of definition of $P_a$, it follows that $a \in \text{Cb}(p)^{\text{alg}}$. Hence $\text{acl}(a) = \text{acl}(\text{Cb}(p))$ and $p$ is a canonical type.
- $d \in \text{Cb}(p)$: Using automorphisms and Fact 4.1(b) as in Claim 3.12.
- *$tp(c/\text{acl}(b))$ is an amalgamation base*: By 4.1(d).
- *$c$ is independent of $a$ over* $\text{acl}(b)$: Again following Example 3.14, but this time using 4.1(c).
- $\text{tp}(d/\text{acl}(b)a) = \text{tp}(d'/\text{acl}(b)a)$ : Note that $d$ and $d'$ have the same field-type over $\text{acl}(b)a = \mathbb{F}(b)^{\text{alg}}(a_1)$, and $\mathbb{F}(a, b)^{\text{alg}} \subset F$. Now apply 4.1(b).

Hence, by Theorem 2.1, $p$ is not stably definable.

4.2. **A stably definable, non stably determinable type.** We keep $a, b$ as above but now choose $c_1 \in F \setminus \mathbb{F}(a, b)^{\text{alg}}$ such that $\mathbb{F}(a, b, c_1)^{\text{alg}} \subset F$. Letting $c_2 = b_1 c_1 + b_2$ and $c_3 = b_3 c_1 + b_4$ we obtain $c := (c_1, c_2, c_3) \in L_b(F) \subset P_a(F)$ with $F \models R(c_1, a_1)$. Let $p := \text{stp}(c/a)$. As before, $c$ is generic in $P_a$ over $\mathbb{F}(a)^{\text{alg}}$ and hence $\text{acl}(a) = \text{acl}(\text{Cb}(p))$ and $p$ is a canonical type.

We show that $p$ is stably definable. Indeed, since all quantifier-free formulas are stable and $a$ generates the minimal field of definition of $P_a$, $a \in \text{dcl}(\text{SCb}(p))$. Hence to show that $\text{Cb}(p) \subset \text{dcl}(\text{SCb}(p))$ it suffices to show that if $f$ is any automorphism of $F$ fixing $a$, then $f(c) \models p$. But clearly $c$ and $f(c)$ have the same field-type over $\mathbb{F}(a)^{\text{alg}}$ (as they are both generic points in the plane). And so, since $\mathbb{F}(a, c)^{\text{alg}} \subset F$ by choice, $c$ and $f(c)$ have the same type over $\mathbb{F}(a)^{\text{alg}}$ by 4.1(b).

Now choose $c'_1$ with $\mathbb{F}(b, c'_1)^{\text{alg}} \subset F$ but $F \models \neg R(c'_1, a_1)$. We can do this as follows: Working in the ambient (saturated) algebraically closed field $F^{\text{alg}}$, let $K := \mathbb{F}(b, a_1)^{\text{alg}}$ and $L := \mathbb{F}(b, t)^{\text{alg}}$ where $t \in F^{\text{alg}}$ is transcendental over $\mathbb{F}(b, a_1)$. Then $t + a_1$ is in $KL$ but does not have any $q$th-roots in $KL$. Let $\sigma$ be an automorphism of $(KL)^{\text{alg}}$ fixing $KL$ pointwise, but strictly permuting the $q$th roots of $t + a_1$. Then by extending $\sigma$ to a generic automorphism of $F^{\text{alg}}$ (i.e., so that $(F^{\text{alg}}, \sigma)$ is a model of $\text{ACFA}_p$) we see that some psuedo-finite field $G$ contains $KL$ but does not contain any $q$th root of $t + a_1$ (take $G := \text{Fix}(\sigma)$). As $K \subset F \cap G$ is algebraically closed, we can embedd $G$ into $F$ over $K$. Hence, there exists $c'_1 \in F$ with $\mathbb{F}(b, c'_1)^{\text{alg}} \subset F$ but $F \models \neg R(c'_1, a_1)$ (namely, the image of $t$ under such an embedding). Note that in particular, $c'_1 \notin \mathbb{F}(a, b, c)^{\text{alg}} \subset F$.

Setting $c' := (c'_1, b_1 c'_1 + b_2, b_3 c'_1 + b_4)$ we have that $c' \models \text{stp}(c/b)$ and $c'$ is independent of $ca$ over $b$. Moreover, since $\text{acl}(a) = \text{acl}(\text{Cb}(p))$, $c'$ is independent of $c\,\mathrm{Cb}(p)$ over $b$. That is, $cc'$ is independent of $\mathrm{Cb}(p)$ over $b$, $c' \models \mathrm{stp}(c/b)$, and $c' \not\models p$. It follows by Proposition 2.3 that $p$ is not stably determinable.

We have shown:

**Example 4.3.** *Suppose $k$ is a psuedo-finite field containing the algebraic closure of the prime field. Then $T = \mathrm{Th}(k)$ has non stably definable types and stably definable types that are not stably determinable.*

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