A DIFFERENTIAL ANALOGUE OF THE WILD AUTOMORPHISM CONJECTURE

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ABSTRACT. A differential analogue of the conjecture of Reichstein, Rogalski, and Zhang in algebraic dynamics is here established: if X is a projective variety over an algebraically closed field of characteristic zero which admits a global algebraic vector field $v : X \to TX$ such that (X, v) has no proper invariant subvarieties then X is an abelian variety. Vector fields on abelian varieties with this property are also examined.

1. INTRODUCTION

The theory of difference fields and the theory of differential fields share many similarities, with results in one of the fields often motivating and inspiring analogues in the other. Notable examples of this principle include Picard-Vessiot theory, which was developed in the late 19th century for differential fields (see [4, Chapt. VIII]) with the analogous theory later being developed for difference fields [9]; Cantat's theorem [6] on invariant hypersurfaces, which gives a difference analogue of the Jouanolou-Ghys theorem [13, 10]; and transcendence results for special values of solutions to differential equations (see [17, Theorem 5.23]), which now have satisfying counterparts for solutions to Mahler-type difference equations [1].

In this paper, we consider a differential analogue of a dynamical conjecture due to Reichstein, Rogalski, and Zhang [15], which asserts that if an automorphism σ of a complex projective variety X has the property that σ leaves no proper Zariski closed subset invariant, then X is an abelian variety. Reichstein, Rogalski, and Zhang called such automorphisms *wild*, and it is easy to see that abelian varieties always have wild automorphisms coming from suitably chosen translation maps. A differential analogue of this conjecture can be formulated by observing that for an algebraic vector field $v : X \to TX$, we can view a subvariety Y as *invariant* for (X, v) if $v|_Y$ maps Y into TY. Indeed, this point of view is what inspired Cantat to prove his analogue of the Jouanolou-Ghys theorem. In analogy with the terminology used by Reichstein, Rogalski, and Zhang, we say that v is *wild* if there do not exist any proper closed subsets Y of X with the above invariant property.

We prove that the corresponding differential version of the conjecture of Rogalski, Reichstein, and Zhang holds.

Theorem 1.1. *Suppose X is a projective variety over an algebraically closed field of characteristic zero. If X admits a wild vector field then X is an abelian variety.*

Note that wildness precludes the existence of a nontrivial rational first integral. Smooth projective varieties X that admit a vector field having no nontrivial rational first integrals were studied in [12, §6.4], where it was shown that the Albanese map on X is surjective and its generic fibre is birationally equivalent to a homogeneous space for a connected linear algebraic group action. As it turns out, however, this fact does not play a role in our proof. Instead, we show directly in §2, using [5, Thm. 2.1] of Buium on D-varieties, that under the wildness hypothesis $\mathrm{Aut}_0(X)$ will act transitively on X, which forces $X = Y \times A$ where A is an abelian variety and Y is a homogeneous space for a connected linear algebraic group. In particular, $H^1(Y, \mathcal{O}_Y) = 0$. But Y inherits a wild vector field from X, and we show in §3 that wildness forces $h^1(Y, \mathcal{O}_Y) > 0$ unless Y is zero-dimensional. We thus conclude that $X = A$. This proof is put together in Section 4.

Examples of wild vector fields on abelian varieties exist: all nontrivial vector fields on an a simple abelian variety are wild. In general, a vector field on an abelian variety is wild if and only if it admits no nontrivial rational first integrals. These facts are established in a final section.

Remark 1.2. Suppose X is an algebraic variety over a field k of characteristic zero, and v is an algebraic vector field on X. Then, by [14, Thm. 2.1], every irreducible component of X is invariant. It follows that if v is wild then X is irreducible. Moreover, although we do not need it in the sequel, wildness also forces X to be smooth. Indeed, using [18, Thm. 12] (see also [11], which gives a generalisation), one can prove that the singular locus is an invariant subvariety.

In fact, we do not even need to assume reduced in Theorem 1.1, the result holds of all projective schemes of finite type over an algebraically closed field of characteristic zero. This is because the reduced locus is an invariant subvariety by [14, Lemma 1.8], and hence if X admits a wild vector field then it is reduced.

2. Transitivity of the automorphism group

In this section, we show that a projective variety admitting a wild vector field is a homogeneous space for its automorphism group. Our proof will use a bit of differential-algebraic geometry that we now recall.

Fix an algebraically closed field k of characteristic zero. We can embed k as the subfield of constants of a differential field (F, δ) that is *differentially closed*, meaning that every finite system of differential-polynomial equations and inequations over F that has a solution in some differential field extension of (F, δ) already has a solution in (F, δ). (Here we are using that k is algebraically closed.) Let us fix such (F, δ). We will need the notion of a *D-variety over* (F, δ) in the sense of Buium [5]: an algebraic variety Y over F equipped with a derivation on $\mathcal{O}_Y$ that extends δ on F. Such a derivation induces, and indeed is determined by, a regular section to the *prolongation* $\tau Y \to Y$, a certain torsor of the tangent space $TY \to Y$ induced by δ. In affine co-ordinates $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, the defining equations for τY are of the form

$$f^\delta(x) + \sum_{i=1}^n \frac{\partial f}{\partial x_i}(x) y_i = 0$$

as $f \in F[x]$ ranges among a generating set for the ideal of (the corresponding affine chart of) Y, and f^δ is the polynomial obtained from f by applying δ to the

coefficients. Given a D-variety (Y, s), a D-*subvariety* is a subvariety $Y' \subseteq Y$ such that $s(Y') \subseteq \tau(Y')$. For a more detailed (but still brief) introduction to these notions, including additional references, see [3, §2.1].

If Y happens to be defined over the constants of (F, δ), namely k, that is, if $Y = X_F$ is the base extension of a variety X over k, then $\tau Y = (TX)_F$. In that case, every algebraic vector field v on X gives rise to a D-variety over (F, δ), namely (X_F, v_F). For any subvariety $Z \subseteq X$, the extension Z_F is a D-subvariety of (X_F, v_F) if and only if Z is an invariant subvariety of (X, v).

Proposition 2.1. *Let X be an irreducible projective variety over k, let v be an algebraic vector field on X, and let $Z \subseteq X$ be a subvariety. If Z is $\mathrm{Aut}_0(X)$-invariant then Z is an invariant subvariety of (X, v).*

Proof. Let (F, δ) be a differentially closed field whose field of constants is k. Let X_F denote the base extension to F and view (X_F, v_F) as a D-variety over (F, δ). In this case though, as X_F is in fact defined over the constant field, $\tau(X_F) = (TX)_F$. To show that $Z \subseteq X$ is an invariant subvariety of (X, v) it suffices to show that Z_F is a D-subvariety of (X_F, v_F).

By a theorem of Buium [5, Thm 2.1], because X_F is projective and defined over the constants, (X_F, v_F) is isomorphic to the trivial D-variety $(X_F, 0)$. That is, there is $\sigma \in \mathrm{Aut}_0(X)(F)$ such that the following diagram commutes:

$$
\begin{array}{ccc}
(TX)_F & \xrightarrow{\ \tau(\sigma)\ } & (TX)_F \\
{\scriptstyle 0}\Big\uparrow & & \Big\uparrow{\scriptstyle v_F} \\
X_F & \xrightarrow{\ \ \sigma\ \ } & X_F
\end{array}
$$

The map σ is constructed as follows. If G is any connected algebraic group over the constants then we have a *logarithmic-derivative on G* which is a certain δ-rational surjective crossed homomorphism $\log_\delta : G(F) \to \mathrm{Lie}(G)(F)$. It is in order to have surjectivity that we required (F, δ) to be differentially closed. The fibre of $\log_\delta$ above 0 is $G(k)$ and all the other fibres are left cosets of $G(k)$. Now v corresponds to a point in $\mathrm{Lie}(\mathrm{Aut}_0(X))(k)$, and so we can take $\sigma \in \mathrm{Aut}_0(X)(F)$ such that $\log_\delta(\sigma^{-1}) = v$. This choice of σ will make the above diagram commute, see [5, page 53] for a detailed argument.

Note that Z_F is a D-subvariety of $(X_F, 0)$ as Z is invariant for $(X, 0)$. It follows that $\sigma(Z_F)$, the image of Z_F under σ, is a D-subvariety of (X_F, v_F). But $\sigma(Z_F) = Z_F$ by $\mathrm{Aut}_0(X)$-invariance. So Z_F is a D-subvariety of (X_F, v_F) and hence Z is an invariant subvariety of (X, v). $\qquad\square$

Corollary 2.2. *Suppose X is a projective variety over k admitting a wild algebraic vector field. Then $\mathrm{Aut}_0(X)$ acts transitively on X.*

Proof. By Remark 1.2, X is irreducible. Let v be a wild algebraic vector field on X. Letting Z be the Zariski closure of the orbit of any k-point of X, we see that Z is $\mathrm{Aut}_0(X)$-invariant, and hence an invariant subvariety of (X, v) by Proposition 2.1. By wildness, $Z = X$. So every orbit of $\mathrm{Aut}_0(X)$ is Zariski dense in X, which forces the action to be transitive. $\qquad\square$

Note that, in particular, such X are smooth.

3. Lifting derivations

Fix an algebraically closed field k of characteristic zero. In this section, we show that if X is a smooth projective variety over k with trivial Albanese, then every algebraic vector field on X can be lifted to a k-linear derivation of a homogeneous coordinate ring of X.

Given $U \subset X$ an open subset, a vector field $v \in H^0(U, TU)$, and a line bundle $\mathcal{L}$ on U, we define a *lift* $\widetilde{v}$ of v to $\mathcal{L}$ as a k-linear map $\widetilde{v} \colon \mathcal{L} \to \mathcal{L}$ satisfying

$$\widetilde{v}(fs) = v(f)s + f\widetilde{v}(s),$$

for local sections f and s of $\mathcal{O}_U$ and $\mathcal{L}$ respectively. Here we are identifying v with the derivation on $\mathcal{O}_X$ it induces. Notice that if $v \in H^0(X, TX)$ is a global section, then the local lifts $\widetilde{\mathcal{V}}_v$ of v form a subsheaf of $\mathcal{H}om_k(\mathcal{L}, \mathcal{L})$.

A more general version of this result is given in [8]. We include a proof of our case of interest for completeness.

Proposition 3.1. *Let X be a smooth variety over k, $\mathcal{L}$ a line bundle on X, and $v \in H^0(X, TX)$. Then $\widetilde{\mathcal{V}}_v$ is an $\mathcal{O}_X$-torsor. Hence, the obstruction to the existence of a global lift lives in $H^1(X, \mathcal{O}_X)$.*

Proof. First, there is a natural action of $\mathcal{O}_X$ on $\widetilde{\mathcal{V}}_v$ given as follows. If $f \in \mathcal{O}_X(U) \simeq \mathrm{Hom}(\mathcal{L}|_U, \mathcal{L}|_U)$ and $\widetilde{v} \colon \mathcal{L}|_U \to \mathcal{L}|_U$ is a lift of $v|_U$, then $f + v|_U$ is also a lift of $v|_U$.

Next, lifts of v exist locally: over any open subset $U \subset X$ where $\mathcal{L}|_U \simeq \mathcal{O}_U$, we may choose the lift of $v|_U$ to be $v|_U$ itself. Finally, if $V \subset X$ is any open and $\widetilde{v}_1$ and $\widetilde{v}_2$ are two lifts of $v|_V$, then $(\widetilde{v}_1 - \widetilde{v}_2)(fs) = f \cdot (\widetilde{v}_1 - \widetilde{v}_2)(s)$ where f is any section of $\mathcal{O}_V$ and s is any section of $\mathcal{L}|_V$. Therefore,

$$\widetilde{v}_1 - \widetilde{v}_2 \in \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}|_V, \mathcal{L}|_V) = \mathcal{O}_V,$$

which proves that the lifts of v form an $\mathcal{O}_X$-torsor.

To finish the proof, we recall that $\mathcal{O}_X$-torsors are classified by $H^1(X, \mathcal{O}_X)$, so the class $[\widetilde{\mathcal{V}}_v] \in H^1(X, \mathcal{O}_X)$ vanishes if and only if $\widetilde{\mathcal{V}}_v$ is the trivial torsor, which by definition, means $\widetilde{\mathcal{V}}_v$ has a global section. So $v \in H^0(X, TX)$ has a global lift if and only if $[\widetilde{\mathcal{V}}_v] \in H^1(X, \mathcal{O}_X)$ vanishes. $\square$

The following result is also proved in [7, Thm. 2] over $\mathbb{C}$ using analytic techniques. We present a very different algebraic proof.

Corollary 3.2. *Let X be a smooth projective variety over k of dimension at least one, and let $v \in H^0(X, TX)$. If $H^1(X, \mathcal{O}_X) = 0$ then v is not wild.*

Proof. Let $\mathcal{L}$ be a very ample invertible sheaf. By Proposition 3.1, we can lift v to $\mathcal{L}$. Such a lift then defines a degree-preserving graded derivation δ of the homogenous coordinate ring

$$R := \bigoplus_{n \geq 0} H^0(X, \mathcal{L}^{\otimes n}).$$

Note that $X = \mathrm{Proj}(R)$. Let $f \in H^0(X, \mathcal{L}) \setminus \{0\}$ be an eigenvector for δ. Then the homogeneous ideal fR defines a δ-invariant ideal of R. We claim that fR gives rise to an invariant codimension one subvariety of (X, v). Indeed, since $\mathcal{L}$ is very ample, R is generated in degree one, and, since R has Krull dimension at least two, Krull's principal ideal theorem implies that there is some $g \in H^0(X, \mathcal{L})$ that is not in the radical of fR. The degree zero piece of $R[1/g]$ can be identified with $\mathcal{O}(U)$ for

some dense Zariski open $U \subseteq X$, and δ induces a derivation of $R[1/g]$ that agrees with the derivation given by v. That f was an eigenvector for δ in R ensures that $I := fg^{-1}\mathcal{O}(U)$ is a δ-invariant ideal. As $\mathcal{O}(U)$ extends $\mathbb{Q}$, the radical of I is also δ-invariant (see, for example, [14, Lemma 1.8]). It therefore defines an invariant subvariety of (U, v), whose Zariski closure in X will be an invariant subvariety Z of (X, v). It remains only to verify that Z is nonempty, but this follows from the fact that I is proper: else there is some $n \geq 1$ and some $h \in H^0(X, \mathcal{L}^{\otimes n})$ such that $(fg^{-1})(hg^{-n}) = 1$, which then gives that $fh = g^{n+1}$ implying that g in the radical of fR, a contradiction. $\qquad\square$

4. Proof of Theorem 1.1

We can now give a short proof of our main result. Suppose X is a projective variety over an algebraically closed field k and v is a wild algebraic vector field on X. By Corollary 2.2, $\mathrm{Aut}_0(X)$ acts transitively on X. Then by [16, Theorem 5.2] we can identify X with $Y \times A$, where A is the Albanese of X and $Y = H/P$ with H a connected affine algebraic group and P a parabolic subgroup, and $\mathrm{Aut}_0(X) \cong \mathrm{Aut}_0(Y) \times A$. In particular, $H^1(Y, \mathcal{O}_Y) = 0$. Since $H^0(X, TX) = \mathrm{Lie}(\mathrm{Aut}_0(X))$, it follows that v can be written uniquely as $v_1 + v_2$ where we may identify v_1 with a vector field v_1' on Y and v_2 with a vector field v_2' on A. If Y has dimension at least one, then, by Corollary 3.2, there is an invariant proper subvariety Z for (Y, v_1') and by construction $Z \times A$ is invariant for both (X, v_1) and (X, v_2) and hence is invariant for (X, v), thus contradicting wildness. It follows that Y is a point and so X is abelian, as desired. $\qquad\square$

5. Wild vector fields on abelian varieties

We first point out that on simple abelian varieties every nontrivial algebraic vector field is wild. Fix an algebraically closed field k of characteristic zero.

Proposition 5.1. *Suppose A is a simple abelian variety over k. Every nontrivial algebraic vector field on A is wild.*

Proof. Let (F, δ) be a differentially closed field with field of constants k. Buium's theorem [5, Thm 2.1] gives us $\sigma \in \mathrm{Aut}_0(A)(F)$ such that $\sigma : (A_F, v_F) \to (A_F, 0)$ is an isomorphism of D-varieties. But as $\mathrm{Aut}_0(A) = A$, we have that σ is just translation by some $P \in A(F)$. Suppose $X \subseteq A$ is an invariant subvariety for v. Then $\sigma(X_F) = P + X_F$ is a D-subvariety of the trivial D-variety $(A_F, 0)$. But every D-subvariety of $(A_F, 0)$ is defined over the constant field k. Since X_F is also defined over k, we have that for any field-automorphism $\alpha \in \mathrm{Aut}(F/k)$, $\alpha(P) - P \in \mathrm{Stab}(X_F)$. If $\mathrm{Stab}(X_F)$ is finite then this tells us that P has a finite orbit under $\mathrm{Aut}(F/k)$, implying that $P \in A(k)$. It would follow that translation by P is an isomorphism between the vector fields (A, v) and $(A, 0)$ over k, which is ony possible if $v = 0$ already. Hence, $\mathrm{Stab}(X_F)$ is a positive-dimensional algebraic subgroup of A. Now, by simplicity, this would mean that $\mathrm{Stab}(X_F) = A_F$, which in turn implies that $X = A$. $\qquad\square$

Regarding general abelian varieties, we have the following classification of wild vector fields:

Proposition 5.2. *Suppose v is an algebraic vector field on an abelian variety A over k. Then v is wild if and only if it admits no nontrivial rational first integrals.*

Proof. Suppose f is a nontrivial rational first integral of (A, v). Let δ_v denote the derivation on $k(A)$ induced by v, this means that $f \in k(A) \setminus k$ with $\delta_v(f) = 0$. In that case, any level set of f over k is an invariant subvariety of (A, v), and by nontriviality some level set will be proper. Hence (A, v) is not wild.

For the converse we use the truth of the differential Dixmier-Moeglin equivalence for *isotrivial* algebraic vector fields, as studied in [3]. That (A, v) is isotrivial is what Buium's theorem says. That (A, v) admits no nontrivial rational first integrals means that it is δ-*rational*, and hence by [3, Fact 2.9] it is δ-*locally closed*.[1] That is, there exists a maximum proper D-subvariety $X \subset A$. But as v is translation invariant on A (all algebraic vector fields on an abelian variety are translation invariant), every translate of X by a k-point of A will also be a proper D-subvariety, and hence will be contained in X by maximality. The only way this is possible is if X is empty – which says precisely that (A, v) is wild. $\qquad\square$

The above proposition is the differential analogue of [2, Theorem 1.7] which treats the algebraic dynamics of dynamical degree one self-maps on semiabelian varieties.

Acknowledgment

References

[1] Boris Adamczewski and Colin Faverjon. Méthode de Mahler: relations linéaires, transcendance et applications aux nombres automatiques. *Proc. Lond. Math. Soc. (3)*, 115(1):55–90, 2017.

[2] Jason Bell and Dragos Ghioca. A conjecture strengthening the Zariski dense orbit problem for birational maps of dynamical degree one. *Canad. Math. Bull.*, pages 1–15, 2022. doi:10.4153/S0008439522000479.

[3] Jason Bell, Omar León Sánchez, and Rahim Moosa. D-groups and the Dixmier-Moeglin equivalence. *Algebra Number Theory*, 12(2):343–378, 2018.

[4] Armand Borel. *Essays in the history of Lie groups and algebraic groups*, volume 21 of *History of Mathematics*. American Mathematical Society, Providence, RI; London Mathematical Society, Cambridge, 2001.

[5] Alexandru Buium. *Differential function fields and moduli of algebraic varieties*. Springer-Verlag, 1986.

[6] Serge Cantat. Invariant hypersurfaces in holomorphic dynamics. *Math. Res. Lett.*, 17(5):833–841, 2010.

[7] James Carrell, David Lieberman. Holomorphic vector fields and Kähler manifolds. *Invent. Math.* 21, 303–309. 1973.

[8] James Carrell, David Lieberman. Vector fields and Chern numbers. *Math. Ann.* 225 no. 3, 263—273, 1977.

[9] Charles H. Franke. Picard-Vessiot theory of linear homogeneous difference equations. *Trans. Amer. Math. Soc.*, 108:491–515, 1963.

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[1]For the model theorist, this is simply the fact that as the generic type of (A, v) is internal to the constants, being weakly orthogonal to the constants implies that the type is isolated.

[10] Étienne Ghys. A propos d'un théoreme de j.-p. jouanolou concernant les feuilles fermées des feuilletages holomorphes. *Rendiconti del Circolo Matematico di Palermo*, 49(1):175–180, 2000.

[11] Robert Hart. Derivations on commutative rings. J. London Math.Soc. (2):8, 171–175, 1974.

[12] Rémi Jaoui and Rahim Moosa. Abelian reduction in differential-algebraic and bimeromorphic geometry. Preprint, arXiv:2207.07515.

[13] Jean-Pierre Jouanolou. Hypersurfaces solutions d'une équation de pfaff analytique. *Mathematische Annalen*, 232(3):239–245, 1978.

[14] Irving Kaplansky. *An introduction to differential algebra.* Actualités scientifiques et industrielles. Hermann, 1957.

[15] Zinovy Reichstein, Daniel Rogalski, and James J. Zhang. Projectively simple rings. *Adv. Math.*, 203(2):365–407, 2006.

[16] Carlos Sancho de Salas. Complete homogeneous varieties: structure and classification. *Trans. Amer. Math. Soc.*, 355(9):3651–3667, 2003.

[17] Igor R. Shafarevich, editor. *Number theory. IV*, volume 44 of *Encyclopaedia of Mathematical Sciences*. Springer-Verlag, Berlin, 1998. Transcendental numbers, A translation of ıt Number theory. 4 (Russian), Ross. Akad. Nauk, Vseross. Inst. Nauchn. i Tekhn. Inform., Moscow, Translation by N. Koblitz, Translation edited by A. N. Parshin and I. R. Shafarevich.

[18] Abraham Seidenberg, Differential ideals in rings of finitely generated type. *Amer. J. Math.*, 89:22—42, 1967.

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