Stochastic Optimization

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Outline

Disclaimers

The only part of my talk that is CERTAIN:

- IP/OR focus
- Incomplete

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My goal:

- Give a very basic overview of the several issues that arise when considering uncertainty

Stochastic IP

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\begin{array}{ll}
\min & c(\xi)^T x \\
\text{s.t.} & A(\xi)x \leq b(\xi) \\
 & x \in \mathbb{Z}^p \times \mathbb{R}^{n-p}
\end{array}
$$

where ξ is a random vector.

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But:

- What does this mean?
- How do we solve it?

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Typical issues:

1. Choose a desired interpretation
2. Formulate a deterministic problem that can (approximately) solve it
3. Solve the deterministic problem

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Hidden challenge: dealing with unknown probability distributions. Example:

- Computing $\mathbb{E}[\max\{x^T\xi, b\}]$ where x is an arbitrary vector and each component of ξ is an i.i.d. uniform random variable is #P-hard (Hanasusanto, Kuhn, and Wiesemann, 2016)

Outline

Two-stage stochastic programs

$$\begin{array}{ll} \min & \mathbb{E}[Q(x,\xi)] \\ \text{s.t.} & x \in X \end{array} \tag{SP}$$

where

$$Q(x,\xi) := \begin{array}{ll} \min & q(\xi)^T y \\ \text{s.t.} & W(\xi) y \le h(\xi) - T(\xi) x \end{array} \tag{REC}$$

- x are called first-stage variables
- y are called second-stage variables

Hardness

Graph reliability problem

Given directed graph $G = (V, E)$, $u, v \in V$, count how many subgraphs $H = (V, F)$ of G have a $u - v$ path

Introduce (v, u), $E' = E \cup (v, u)$ $m = |E|$

* y_{ij}, $\forall ij \in E'$

* $g_{ij} = \begin{cases} -2, & \text{w.p. } 1/2 \rightarrow \text{fail} \\ 0, & \text{w.p. } 1/2 \rightarrow \text{exists,} \end{cases}$ $\forall ij \in E$

& $r_{vu} = 1$ w.p. 1

↳ 2^m scenarios

Find max cost circulation in scenario S.

Let's say capacities of arcs are x.

If $\exists$ circul. w/ only edges in $E \Rightarrow$ cost is $\geq x$

Else ≤ 0.

$\max \; \mathbb{E}[Q(x, \xi)]$

s.t. $x = 1$

$Q(x, \xi)$ is cost of circulation.

D has reliability $\geq R$

$\Leftrightarrow z^* \geq \frac{R}{2^m}$

Sample average approximation (Kleywegt, Shapiro, and Mello, 2002)

Addresses issue 2.

Instead of solving

$$z^* := \begin{array}{ll} \min & \mathbb{E}[Q(x,\xi)] \\ \text{s.t.} & x \in X \end{array} \qquad \text{(SP)}$$

take $\xi^1, \ldots, \xi^N$ i.i.d. samples of ξ and solve

$$z^N := \begin{array}{ll} \min & \sum_{i=1}^{N} \frac{1}{N} Q(x,\xi^i) \\ \text{s.t.} & x \in X \end{array} \qquad \text{(SAA-N)}$$

Let x^N be an optimal solution to (SAA-N)

Sample average approximation (Kleywegt, Shapiro, and Mello, 2002)

Proposition

$\mathbb{E}[z^N] \leq z^*$

Let x^* opt. to (SP)

$z^* = \mathbb{E}[Q(x^*, \xi)]$

x^* feas. for (SAA-N)

$z_N \leq \frac{1}{N} \sum_{i=1}^{N} Q(x^*, \xi^i)$

$\mathbb{E}[z_N] \leq \frac{1}{N} \sum_{i=1}^{N} \underbrace{\mathbb{E}[Q(x^*, \xi^i)]}_{z^*}$

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Theorem

Assume X is compact, $Q(x, \xi)$ is real-valued for all $x \in X$ and all ξ. Then $z^N \rightarrow z^*$ and every limit point of $\{x^N\}$ solves (SP) with probability 1.

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Ahmed and Shapiro, 2002 extends it to cases with integral recourse.

Solving (SAA-LP)

$$z^N := \begin{array}{ll} \min & \sum_{i=1}^{N} \frac{1}{N} Q(x, \xi^i) \\ \text{s.t.} & Ax \le b \end{array} \tag{SAA-LP}$$

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Given a fixed ξ, different x values give different RHS for (REC).

Value function of an LP:

$$V(d) := \begin{array}{ll} \min & g^T y \\ \text{s.t.} & Dy \le d \end{array}$$

is a PWL convex function

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$$= \begin{array}{ll} \max & [h(\xi) - T(\xi)]^T w \\ \text{s.t.} & W(\xi)^T w = q(\xi) \\ & w \leq 0 \end{array} = \max_{k=1,\dots,K(\xi)} [h(\xi) - T(\xi)]^T w(\xi)^k$$

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which is solved in a cutting plane fashion.

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Assumes second stage problem always has an optimal solution and does not work if second stage variables are integer.

Multistage stochastic programs

Generalizes two-stage to multiple, where uncertainty is revealed in stages and decisions can be made in each stage considering all previous uncertainty realizations as deterministic.

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Much more complex, with tons of research questions on their own.

- Stochastic Dual Dynamic Programming: Pereira and Pinto, 1991
  Example performance:
  Matos, Philpott, and Finardi, 2015: 300 power plants, 120 stages, 20 realizations per stage: 2% gap within 2 hours.
- Stochastic Dual Dynamic Integer Programming: Zou, Ahmed, and Sun, 2019
  Example performance:
  9 stages, 2% gap in 3h.

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Addresses issue 3

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Let $\mathcal{N} = \{P_1, \ldots, P_L\}$ be a partition of $1, \ldots, N$.

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Song and Luedtke, 2015: Propose a way to iteratively refine $\mathcal{N}$ until we reach optimal solution to original problem.

Stochastic cutting planes (Bertsimas and Li, 2022)

Addresses issue 3

Idea: Adapt cutting plane approach to Stochastic MINLP problems giving high probability of getting good solution with much lower number of cuts.

Other measures of risk

Adresses issue 1

Expected value is risk neutral:

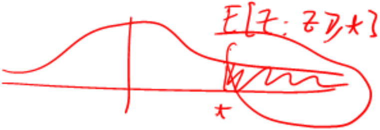
Which is better:

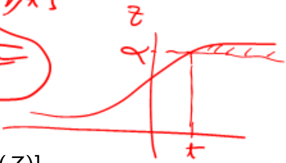
- A solution that 90% of the time has cost 1,000 and 10% of the time has cost 0
- A solution that 100% of the time has cost 901

depends on how risk-averse you are.

Ruszczyński and Shapiro, 2006: Investigate optimization of risk functions (not recommend for reading group).

Other measures of risk





Adresses issue 1

- Value at Risk: $V@R_\alpha(Z) := \inf\{t : \mathbb{P}[Z \leq t] \geq \alpha\}$
- Conditional Value-at-Risk: $CV@R_\alpha(Z) := \mathbb{E}[Z | Z \geq V@R_\alpha(Z)]$

Lemma (Rockafellar and Uryasev, 2002)

If $\mathbb{P}[Z = V@R(Z)] = 0$ then $CV@R_\alpha(Z) = \min_{\gamma \in \mathbb{R}} \left\{ \gamma + \frac{1}{1-\alpha} \mathbb{E}[Z - \gamma]_+ \right\}$

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If $Z = Q(x, \xi)$ with ξ having finite support distribution $\mathbb{P}(\xi = \xi^i) = p_i, i = 1, \ldots, N$ then:

$$\begin{array}{ll} \min & CV@R_\alpha(Q(x,\xi)) \\ \text{s.t.} & x \in X \end{array} = \begin{array}{ll} \min & \gamma + \frac{1}{1-\alpha} \sum_{i=1}^{N} p_i w_i \\ \text{s.t.} & x \in X \\ & w_i \geq Q(x, \xi^i) - \gamma, i = 1, \ldots, N \\ & \gamma \in \mathbb{R}, w \geq 0 \end{array} \tag{1}$$

Outline

Chance constraint

(Joint) chance-constraint:

$$z_\epsilon^* := \begin{array}{ll} \min & c^T x \\ \text{s.t.} & x \in X \\ & \mathbb{P}\{f_i(x,\xi) \le 0, \forall i = 1,\dots,m\} \ge 1-\epsilon \end{array} \tag{CC}$$

Individual chance-constraint: $m = 1$ (our focus next)

Tractable approximation (Nemirovskii and Shapiro, 2006)

Suppose $\psi : \mathbb{R} \to \mathbb{R}$ is:

- Nonnegative
- nondecreasing
- convex
- $\psi(a) > \psi(0) = 1, \forall z > 0.$

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If Z is a random variable and $t > 0$

$$\mathbb{E}[\psi(tZ)] \geq \mathbb{E}[\mathbb{I}_{[0,+\infty]}(tZ)] = \mathbb{P}[tZ \geq 0] = \mathbb{P}[Z \geq 0] \geq \mathbb{P}[Z > 0]$$

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Consider

$$S = \{x : \mathbb{P}\{f(x,\xi) \leq 0\} \geq 1 - \epsilon\} = \{x : \mathbb{P}\{f(x,\xi) > 0\} \leq \epsilon\}$$

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Therefore, if we can guarantee:

$$\mathbb{E}[\psi(tf(x,\xi))] \leq \epsilon$$

then chance constraint is satisfied.

Tractable approximation (Nemirovskii and Shapiro, 2006)

If f is convex on x, leads to conservative convex constraint approximation of chance-constraint:

$$\inf_{t>0}\left[t\mathbb{E}[\psi(\frac{1}{t}f(x,\xi))] - t\epsilon\right] \leq 0$$

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Ahmed et al., 2017 give a different way to approximate chance-constraints.
Xie and Ahmed, 2020 give a bi-criteria approximation for covering programs

Scenario approximation (Calafiore and Campi, 2005)

Take samples $\xi^1, \ldots, \xi^N$ and solve:

$$\begin{array}{ll} \min & c^T x \\ \text{s.t.} & x \in X \\ & f(x, \xi^i) \leq 0, \forall i = 1, \ldots, N \end{array}$$

Theorem

Assume X convex, $f(x, \xi)$ convex on x for every ξ. Let $\delta > 0$.
If $N \geq \frac{2}{\epsilon} \log\left(\frac{1}{\delta}\right) + 2n + \frac{2n}{\epsilon} \log(\frac{2}{\epsilon})$ then, with confidence $1 - \delta$, the optimal solution x^* to the scenario approximation problem will satisfy chance-constraint.

Sample average approximation (Luedtke and Ahmed, 2008)

Take samples $\xi^1, \dots, \xi^N$ and solve the chance-constrained problem with confidence α over finite distribution.

$$z^*_{N,\alpha} := \begin{array}{ll} \min & c^T x \\ \text{s.t.} & x \in X \\ & \sum_{i=1}^{N} \frac{1}{N} \mathbb{I}(f(x, \xi^i) \le 0) \ge 1 - \alpha \end{array} \quad \text{(SAA-CC)}$$

Assume that (CC) has an optimal solution.

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Theorem

If $\alpha > \epsilon$ then

$$\mathbb{P}[z^*_{N,\alpha} \leq z^*_\epsilon] \geq 1 - exp\{-2N(\alpha - \epsilon)^2\}$$

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Theorem

Suppose X is finite and $\alpha < \epsilon$. Then, all feasible solutions to the sample problem at level α are feasible to the nominal problem at level ϵ with probabaility at least

$$1 - |X| exp\{-2N(\epsilon - \alpha)^2\}$$

Outline

Distributional robustness

Main idea:

- Often times, exact probability distribution is not really known.
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Example: We want to solve

$$\min\{\max_{\mathbb{P}\in B_W(\mathbb{Q})} \mathbb{E}^{\mathbb{P}}[Q(x,\xi)] : x \in X\}$$

where $\mathbb{Q}$ is a given probability distribution, and $B_W(\mathbb{Q})$ is a set of probability distributions that are "like" $\mathbb{Q}$.

Distributional robustness (Esfahani and Kuhn, 2018)

Distance between two probability distributions:

$$d_W(\mathbb{Q}_1, \mathbb{Q}_2) := \inf \left\{ \mathbb{E}^{\mathbb{Q}} ||\xi^1 - \xi^2|| : \begin{array}{l} \mathbb{Q} \text{ is a joint distribution of } \xi^1 \text{ and } \xi^2 \\ \text{with marginals } \mathbb{Q}_1 \text{ and } \mathbb{Q}_2 \end{array} \right\}$$

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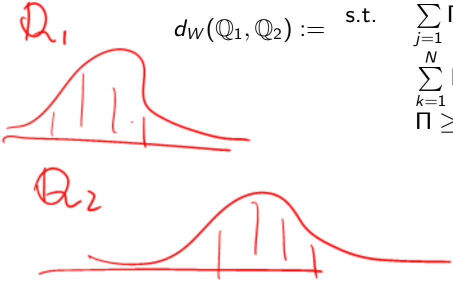
If $\mathbb{Q}_i$ $(i = 1, 2)$ are finite discrete distributions with N possible realizations $\xi^i_1, \ldots, \xi^i_k$ and probabilities $q^i_1, \ldots, q^i_N$, then

$$d_W(\mathbb{Q}_1, \mathbb{Q}_2) := \begin{array}{ll} \min & \sum_{j=1}^{N} \sum_{k=1}^{N} ||\xi^1_j - \xi^2_k|| \Pi_{jk} \\ \text{s.t.} & \sum_{j=1}^{N} \Pi_{jk} = q^2_k, \forall k = 1, \ldots, N \\ & \sum_{k=1}^{N} \Pi_{jk} = q^1_j, \forall j = 1, \ldots, N \\ & \Pi \geq 0 \end{array}$$



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$$d_W(\mathbb{Q}_1, \mathbb{Q}_2) := \inf \left\{ \mathbb{E}^{\mathbb{Q}} ||\xi^1 - \xi^2|| : \begin{array}{c} \mathbb{Q} \text{ is a joint distribution of } \xi^1 \text{ and } \xi^2 \\ \text{with marginals } \mathbb{Q}_1 \text{ and } \mathbb{Q}_2 \end{array} \right\}$$

If $\mathbb{Q}_i$ $(i = 1, 2)$ are finite discrete distributions with N possible realizations $\xi_1^i, \ldots, \xi_k^i$ and probabilities $q_1^i, \ldots, q_N^i$, then

$$d_W(\mathbb{Q}_1, \mathbb{Q}_2) := \begin{array}{ll} \min & \sum_{j=1}^N \sum_{k=1}^N ||\xi_j^1 - \xi_k^2|| \Pi_{jk} \\ \text{s.t.} & \sum_{j=1}^N \Pi_{jk} = q_k^2, \forall k = 1, \ldots, N \\ & \sum_{k=1}^N \Pi_{jk} = q_j^1, \forall j = 1, \ldots, N \\ & \Pi \geq 0 \end{array}$$

We can then define: $B_W(\mathbb{Q}) := \{\mathbb{P} : d_W(\mathbb{P}, \mathbb{Q}) \leq \epsilon\}$
and rewrite our problem as a tractable optimization problem.

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If $\mathbb{Q}_i$ $(i = 1, 2)$ are finite discrete distributions with N possible realizations $\xi^i_1, \ldots, \xi^i_k$ and probabilities $q^i_1, \ldots, q^i_N$, then

$$d_W(\mathbb{Q}_1, \mathbb{Q}_2) := \begin{array}{ll} \min & \sum_{j=1}^{N} \sum_{k=1}^{N} ||\xi^1_j - \xi^2_k|| \Pi_{jk} \\ \text{s.t.} & \sum_{j=1}^{N} \Pi_{jk} = q^2_k, \forall k = 1, \ldots, N \\ & \sum_{k=1}^{N} \Pi_{jk} = q^1_j, \forall j = 1, \ldots, N \\ & \Pi \geq 0 \end{array}$$

We can then define: $B_W(\mathbb{Q}) := \{\mathbb{P} : d_W(\mathbb{P}, \mathbb{Q}) \leq \epsilon\}$
and rewrite our problem as a tractable optimization problem.

Chen, Kuhn, and Wiesemann, 2018 deals with distributionally robust chance-constraints
Rahimian and Mehrotra, 2019 provides a review of DRO.

Outline

Conclusion

Takeaways:

- Tons of possible ways to interpret an optimization problem with uncertainty
- Theoretical and computational challenges are plentiful
- Mix of statistics, optimization, engineering, economics

Conclusion

Takeaways:

- Tons of possible ways to interpret an optimization problem with uncertainty
- Theoretical and computational challenges are plentiful
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Missing:

- Tons of details
- Lots more references
- Specific ideas to particular combinatorial optimization problems
- Combinatorial/Approximation algorithms

Papers suggested

- Ahmed and Shapiro, 2002: SAA two-stage with integer recourse
- Song and Luedtke, 2015: Aggregate scenarios for two-stage
- Bertsimas and Li, 2022: Stochastic cutting planes
- Zou, Ahmed, and Sun, 2019: Multistage stochastic programs with integer variables
- Luedtke and Ahmed, 2008: SAA chance-constraint
- Ahmed et al., 2017, Xie and Ahmed, 2020 and Ahmed and Xie, 2018: Approximations of chance-constraint
- Chen, Kuhn, and Wiesemann, 2018, Esfahani and Kuhn, 2018: DRO
- Barrera et al., 2016: SAA chance-constraint and sampling rare events
- Jiang and Guan, 2018: SAA with DRO and risk-averse

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