Co-Dimension Two Bifurcations in PWSC Dynamical Systems

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Large Networks of Neurons

- We consider large networks of coupled, two-dimensional, integrate-and-fire networks
- The networks are given by:

$$\dot{v}_i = F(v_i) - w_i + I + gs(er - v_i) = G(v_i, w_i, s)$$
$$\dot{w}_i = \frac{1}{\tau_w}(bv_i - w_i)$$
$$\dot{s} = -\frac{s}{\tau_s} + \frac{\lambda_s}{\tau_s N}\sum_{i=1}^{N}\sum_{t<t_{i,k}}\delta(t - t_{i,k})$$
$$v(t^-) = v_{peak}, \rightarrow v(t^+) = v_{reset}, w(t^+) = w(t^-) + \frac{\lambda_w}{\tau_w}$$

where w is the adaptation variable, and v is the voltage.

- In the large network limit, one obtains the first order moment-closure simplified population density equation:

$$\frac{\partial \rho_V(v,t)}{\partial t} = -\frac{\partial}{\partial t}(G(v, \langle w\rangle, s)\rho_V(v,t)) \quad (1)$$
$$s' = -\frac{s}{\tau_s} + \frac{\lambda_s}{\tau_s}J(v_{peak}, t) \quad (2)$$
$$\langle w\rangle' = -\frac{\langle w\rangle}{\tau_w} + \frac{\lambda_w}{\tau_w}J(v_{peak}, t) \quad (3)$$

- The steady state to this system is given by:

$$\bar{s} = \lambda_s \bar{R}, \quad \bar{w} = \lambda_w \bar{R}$$
$$\bar{R} = \left(\int_{v_{reset}}^{v_{peak}} \frac{dv'}{G(v, \bar{s}, \bar{w})}\right)^{-1}$$

- After applying the general approach in [2] to linearize the system of equations, one arrives at the eigenvalue equation:

$$0 = \left(e^{\mu/R} - 1\right)\left(\mu + \frac{1}{\tau_s}\right)\left(\mu + \frac{1}{\tau_w}\right) \quad (4)$$
$$+ \left(\mu + \frac{1}{\tau_s}\right)\left(\frac{\lambda_w}{\tau_w}\mu\hat{B}(\mu)\right) \quad (5)$$
$$- \left(\mu + \frac{1}{\tau_w}\right)\left(\frac{\lambda_s}{\tau_s}\mu\hat{A}(\mu)\right) \quad (6)$$

where

$$\hat{A}(\mu) = \int_0^1 e^{\mu y/R}\frac{g(e_r - \eta^{-1}(y'))}{G_1(\eta^{-1}(y')\lambda_s R, \lambda_w R)}dy' \quad (7)$$
$$\hat{B}(\mu) = \int_0^1 e^{\mu y/\langle R\rangle}\frac{-1}{G_1(\eta^{-1}(y'), \lambda_s R, \lambda_w R)}dy' \quad (8)$$

and $y = \eta(v)$ is the Abbott-Vreeswijk transform.

- The spectral has a countable solution set μ_i, $i = 1, 2, \ldots$ where $\Re(\mu_i) < 0 \ \forall i$, and for small ϵ in addition to two eigenvalues given by the equation

$$0 = (\mu_1 + \gamma)(\mu_1 + 1) + \lambda_w(\mu_1 + \gamma)\frac{\partial\langle R_i(t)\rangle}{\partial w}$$
$$- \gamma\lambda_s(\mu_1 + 1)\frac{\partial\langle R_i(t)\rangle}{\partial s} \quad (9)$$

to lowest order in ϵ.

- Equation (9) is the eigenvalue equation for the mean-field system

$$\dot{s} = -\frac{s}{\tau_s} + \frac{\lambda_s}{\tau_s}\langle R_i(t)\rangle$$
$$\dot{w} = -\frac{w}{\tau_w} + \frac{\lambda_w}{\tau_w}\langle R_i(t)\rangle$$
$$\langle R_i(t)\rangle = \begin{cases}\left(\int_{v_{reset}}^{v_{peak}}\frac{dv}{G(v,s,w)}\right)^{-1} & H(s,w) \geq 0 \\ 0 & H(s,w) < 0\end{cases}$$
$$H(s,w) = \min_{v\in[v_{reset}, v_{peak}]} G(v,s,w) = G(v^*(s), s, w)$$

Large Networks Continued

- For small $H(s,w)$, one can prove that

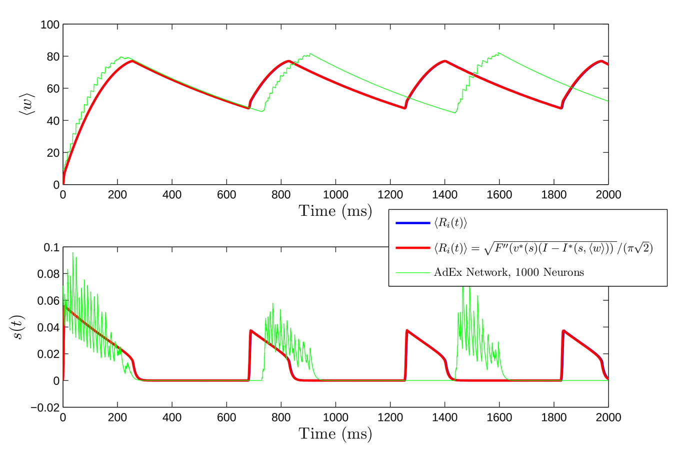
$$\langle R_i(t)\rangle \sim \sqrt{\frac{F''(v^*(s))}{2}}\frac{\sqrt{H(s,w)}}{\pi}$$

- Thus, we consider the system

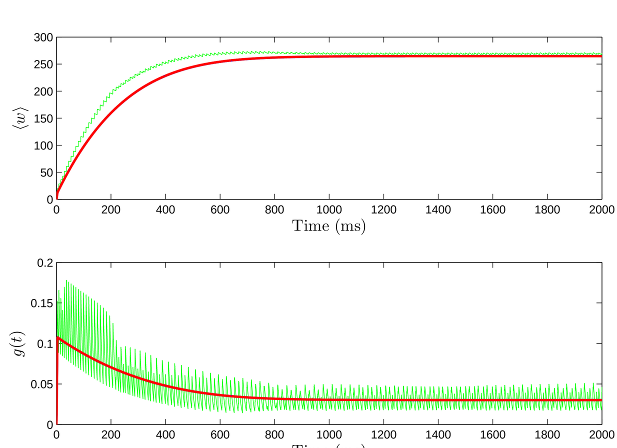
$$\dot{s} = -\frac{s}{\tau_s} + \frac{\lambda_s}{\tau_s}\langle R_i(t)\rangle \quad (10)$$
$$\dot{w} = -\frac{w}{\tau_w} + \frac{\lambda_w}{\tau_w}\langle R_i(t)\rangle \quad (11)$$
$$\langle R_i(t)\rangle = \begin{cases}\sqrt{\frac{F''(v^*(s))}{2}}\frac{\sqrt{H(s,w)}}{\pi} & H(s,w) \geq 0 \\ 0 & H(s,w) < 0\end{cases} \quad (12)$$

for analysis.

- This system is the slow system of (1)-(3) when $\tau_s, \tau_w \gg 1$ and $0 \leq H(s,w) \ll 1$. The stability and existence of the asynchronous state(s) of (1)-(3) is determined by the stability of the steady state(s) of (10)-(12)



a: Bursting AdEx Network



b: Tonic Firing AdEx Network

Figure 2: Shown above is a network of 1000 AdEx neurons (gren) in comparison to the mean-field system with $(\langle R\rangle = \int_{v_{reset}}^{v_{peak}}\frac{dv}{G(v,s,w)})^{-1}$ (blue) and the system with $\langle R\rangle$ from equation (12)(red)

Existence and Steady States

- By expanding in small s (which based on the steady state conditions of (10)-(12) is equivalent to $0 \leq H(s,w) \ll 1$, one can show that there are up to two equilibria, $e_\pm = (s_\pm, w_\pm)$:

$$s_\pm = M(g)(g - g^*) \pm \sqrt{M(g)^2(g - g^*)^2 \pm \tilde{I}}$$
$$w_\pm = \frac{\tau_w w_{jump}}{\tau_s s_{jump}}s_\pm = \eta s_\pm$$

- A third solution is the non-firing solution, $s = \langle w\rangle = 0$
- The equilibria $(s_\pm, \eta s_\pm)$ undergo a **saddle-node bifurcation** when

$$\tilde{I} = -M(g)(g - g^*)^2 + O((g - g^*)^3)$$

Non-Smooth Bifurcations

- The saddle-node bifurcation is generic for $g > g^*$
- $(s_+, \eta s_+)$ undergoes a **Hopf bifurcation** when $\tilde{I} = -2M(g)N(g)(g - g^*)(g - \bar{g}) + O((g - \bar{g})^2)$
- The Hopf bifurcation is generic for $g > \bar{g}$ with first Lyapunov coefficient given by:

$$l_1(0) = \frac{3}{82(e_r - v^*(0))^6}\frac{\lambda_w^2}{\lambda_s^2(g - \bar{g})^2}\epsilon^3 + O(\epsilon^2)$$

as $l_1(0) > 0$ for $g > \bar{g}, \tau_w, \tau_s \gg 1$, the Hopf bifurcation is subcritical

- Bifurcation curves are valid for $g > g^*$ (saddle-node) and $g > \bar{g}$ (Hopf). The points (I_{rh}, g^*) and $(I_{rh}, \bar{g})$ are cp-dimension 2 non-smooth bifurcation points.

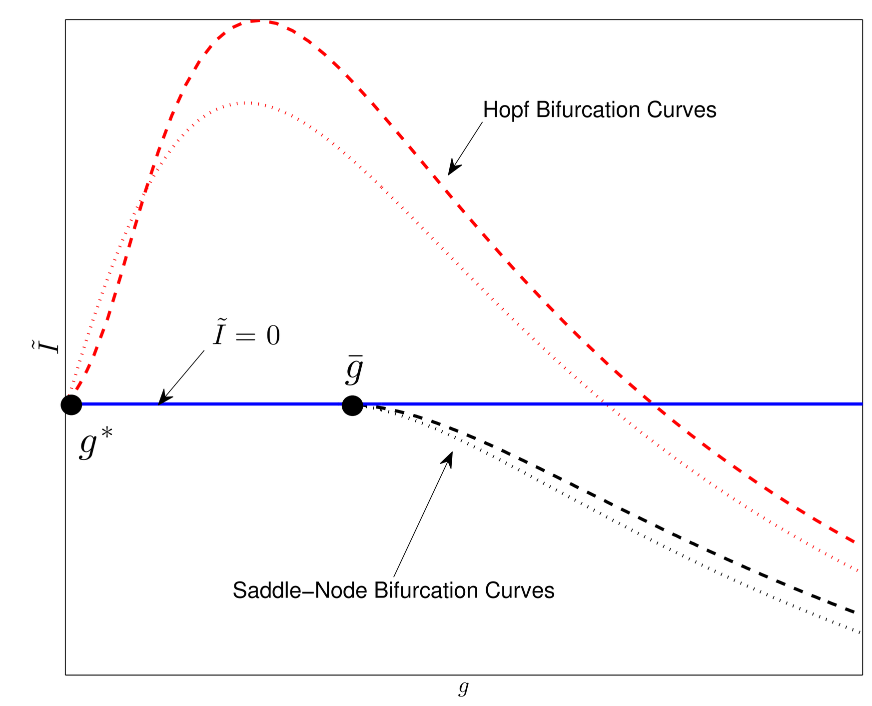
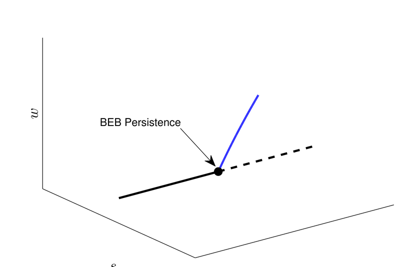
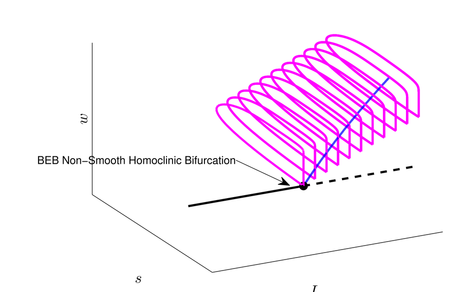


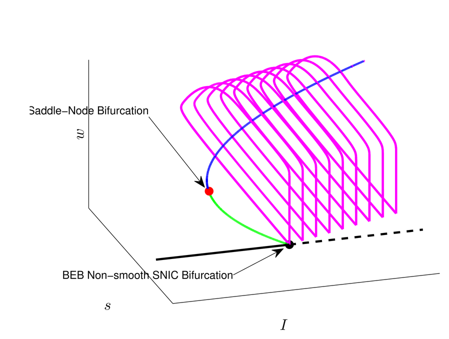
Figure 3: The saddle-node (black) and Hopf bifurcation curves (red) for the full mean-field system (dashed) and the reduced mean-field system (dotted) for a network of Izhikevich neurons. The full mean field system's bifurcation curves are determined via continuation in MATCONT. The points $\bar{g}$ and g^* at $\tilde{I} = 0$ correspond to non-smooth co-dimension 2 bifurcation points.



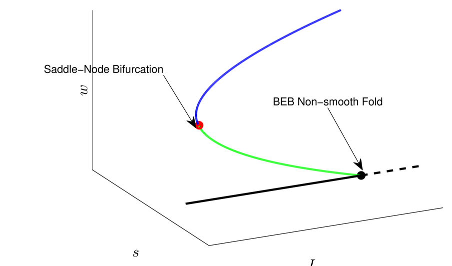
a: $g < \bar{g}$ Persistence



b: $\bar{g} < g < g^*$ Persistence



c: $g > g^*$ Non-Smooth Fold



d: $g \gg g^*$ Non-Smooth Fold

Figure 5: Shown above are the four branches of boundary equilibrium bifurcations. In all figures, the equilibria are $(0,0)$ (in black), $(s_+, \eta s_+)$ (blue) and $(s_-, \eta s_-)$ (green). The limit cycle is shown in magenta and is determined through direct integration of the reduced mean-field system.

- These co-dimension 2 non-smooth bifurcations appear to be non-generic versions of those that appear for regular PWSC systems [3].

Non-Smooth Continued

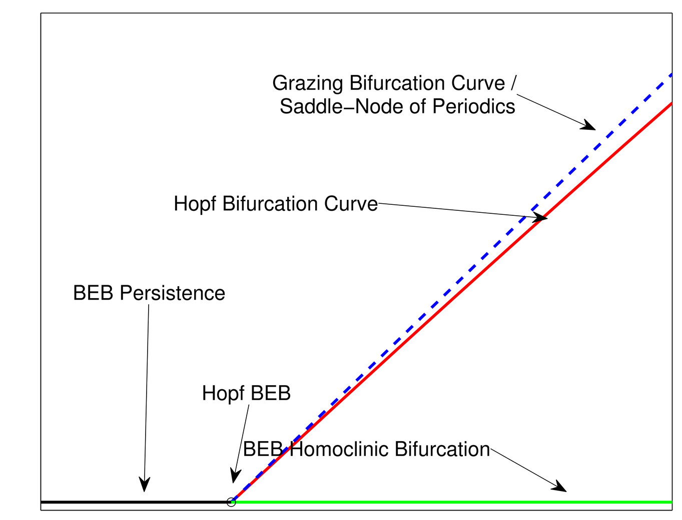


Figure 6: Hopf-BEB Point

Figure 7: Shown above is the collision of the Hopf bifurcation curve with the the non-smooth bifurcation branches.

Conclusions

- A system of PWSC ODE's is derived in the limit that $\tau_s, \tau_w \gg 1$, $H(s,w) \ll 1$ from the moment closure reduced population density equation for a network of Type-I neurons.
- The system of ODE's has saddle-node and hopf bifurcation branches, and two non-smooth co-dimension 2 bifurcation points. The bursting behavior of a full network fo neurons is organized by these points.

References

[1] W. Nicola, and S.A. Campbell. Bifurcations of large networks of two-dimensional integrate and fire neurons *Journal of Computational Neuroscience*, 35: 87–108, 2013.

[2] L.F. Abbott and C. van Vreeswijk Asynchronous states in networks of pulse-coupled oscillators *Phys. Rev. E.*, 48: 1483–1490, 1993.

[3] D.J.W. Simpson Bifurcations in piece-wise smooth continuous systems World Scientific. 2010.

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