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Change in Criticality of Synchronous Hopf Bifurcation in a Multiple-delayed Neural System

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Abstract. We consider a network of three identical neurons with multiple signal transmission delays. The model for such a network is a system of delay differential equations. With the aid of the symbolic computation language MAPLE, we derive the corresponding system of ordinary differential equations describing the semiflow on the centre manifold. It is shown that two cases of a single Hopf bifurcation may occur at the trivial fixed point of the full nonlinear system of delay equations, primarily as a consequence of the structure of the associated characteristic equation. These are (i) the simple root Hopf, and (ii) the double root Hopf. This paper focusses on the first case, paying particular attention to possible change of the criticality of the bifurcations.

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1 Introduction

It is well-known (see [1, 2, 10, 11, 15] and references therein) that a coupled system of multiple-delayed differential equations exhibits some interesting bifurcation phenomena as linear stability is lost. For example, in [1], a scalar multiple-delayed differential equation of the form

$$\dot{x}(t) = f_1(x(t - T_1)) + f_2(x(t - T_2)) \ ,$$

where $f_i(u) = -A_i \tanh(u)$, $i = 1, 2$ (A_i are positive constants) is considered. The bifurcations occurring as linear stability is lost are studied via the construction of a centre manifold. In particular, the nature of Hopf and more degenerate, higher codimension bifurcations are explicitly determined.

In [15], Wu *et al.* studied a coupled system of three multiple-delayed identical neurons of the general form

$$\dot{x}_i(t) = -x_i(t) + \alpha f(x_i(t - \tau)) + \beta[f(x_{i-1}(t - \tau)) + f(x_{i+1}(t - \tau))] \ , \qquad (1.1)$$

where i (mod 3), $f : \mathbb{R} \to \mathbb{R}$ is a sufficiently smooth sigmoid amplification function, normalised so that $f(0) = 0$ and $f'(0) = 1$, α and β measure respectively the normalised synaptic strength of self-connection and nearest-neighbour connection. It was assumed that the nearest-neighbour and self connection transmission delays are identical. Wu *et al.* [15] showed that, in a certain region of the space (α, β), each solution of the network is convergent to the set of synchronous states in the phase space. Furthermore, it was shown that this synchronisation is independent of the size of the delay. Also, a ***bifurcation surface*** was obtained, as the graph of a continuous function of $\tau = \tau(\alpha, \beta)$ in some region of (α, β), where Hopf bifurcation of periodic solutions occurs.

In this paper we study a generalisation of (1.1) in which there are different time delays in the self connections and the nearest-neighbour interactions. This network is shown schematically in Fig. 1, and modelled by the system of nonlinear delay

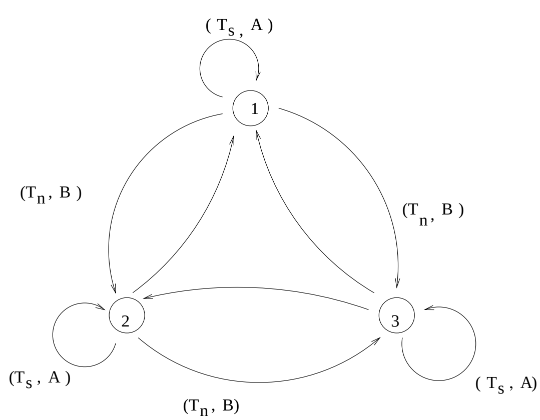

Figure 1 *Architecture of a network of three identical neurons with multiple time delays. The parameters T_s and T_n denote, respectively, the self-connection and nearest-neighbour connection signal transmission delays, while A and B are, respectively, the synaptic strengths of self- and nearest-neighbour connections.*

differential equations

$$\dot{x}_i(t) = -\mu x_i(t) + Af(x_i(t - T_s)) + B[f(x_{i-1}(t - T_n)) + f(x_{i+1}(t - T_n))] \ , \quad (1.2)$$

where i $(mod\, 3)$, $f : \mathbb{R} \to \mathbb{R}$ and T_s and T_n are, respectively, the signal transmission delays for self- and nearest-neighbour connections. Throughout this paper, we assume an activation function of the form $f(x) = \tanh(\gamma x)$. Note that equation (1.2) has D_3 symmetry since the vector field is invariant under permutation of the coordinates, i.e. the transformation $(x_1, x_2, x_3) \to (x_2, x_3, x_1)$ and under reflections such as $(x_1, x_2, x_3) \to (x_1, x_3, x_2)$ and its permutations. The symmetries of the equation lead to various invariant subspaces for the flow of the delay differential equation (DDE). The permutation leads to the invariant lines $x_1 = x_2 = x_3$, and the reflections to the invariant planes $x_j = x_{j+1}$, j mod 3.

We shall rescale (1.2) using the following change of variables: $\tilde{x}_i = \gamma x_i$, $\tilde{t} = \mu t$. Introducing the parameters $\alpha = \frac{\gamma}{\mu} A$, $\beta = \frac{\gamma}{\mu} B$, $\tau_s = \mu T_s$, $\tau_n = \mu T_n$, equation (1.2) becomes

$$\dot{x}_i(t) = -x_i(t) + \alpha \tanh(x_i(t-\tau_s)) + \beta[\tanh(x_{i-1}(t-\tau_n)) + \tanh(x_{i+1}(t-\tau_n))] \ , \quad (1.3)$$

where i mod 3 and we have dropped the tilde's on t and x_i for simplicity. Our goal is to describe all possible spatio-temporal patterns in system (1.3) with four parameters $(\alpha,\ \beta,\ \tau_s,\ \tau_n)$ and we hope, eventually, we may exhibit *all* reported stable patterns in a dynamical system with D_3 symmetry through a single model by choosing suitable parameters. Our focus in this preliminary work is to show, with the aid of the symbolic computation language MAPLE [13], that when $\tau_s \neq \tau_n$, the criticality of one Hopf bifurcation of periodic solutions may be changed as $(\alpha,\ \beta)$ are varied. This change of criticality may then lead to a secondary and more complicated bifurcations. The rest of this paper is organised as follows. In section 2, we discuss some aspects of the stability of fixed points of (1.3). Section 3 gives a brief review of centre manifold theory for delay differential equations and section 4 applies this theory to determine the criticality of the Hopf bifurcation of synchronous periodic solutions of (1.3). Section 5 describes and illustrates the surfaces in the (α, β, τ_n) parameter space along which this bifurcation occurs. In section 6 we make some concluding remarks.

2 Fixed points and bifurcations

Fixed points of equation (1.3) may be symmetric or non-symmetric. Those that are permutation symmetric will have the form (x^*, x^*, x^*), where

$$x^* = \alpha f(x^*) + 2\beta f(x^*) \ .$$

Reflection symmetric fixed points will have the form (y^*, x^*, x^*) (or a permutation thereof) with

$$\begin{aligned} x^* &= (\alpha+\beta) f(x^*) + \beta f(y^*) \\ y^* &= \alpha f(y^*) + 2\beta f(x^*) \ . \end{aligned}$$

Non symmetric fixed points will have the form (x_1^*, x_2^*, x_3^*), where

$$x_j^* = \alpha f(x_j^*) + \beta f(x_{j-1}^*) + \beta f(x_{j+1}^*) \ , \quad j \text{ mod } 3 \ .$$

The linearisation of (1.3) about $(0,0,0)$ is given by

$$\dot{u}_j(t) = -u_j(t) + \alpha u_j(t-\tau_s) + \beta[u_{j-1}(t-\tau_n) + u_{j+1}(t-\tau_n)] \ , \qquad (2.1)$$

where j $mod\, 3$. To study the linearised stability of $(0,0,0)$ we consider solutions of (2.1) of the form

$$u_j(t) = c_j e^{\lambda t} \quad j = 1, \ldots, 3, \qquad (2.2)$$

where $\lambda \in \mathbb{C}$, $c_j \in \mathbb{R}$. The c_j will be nontrivial if and only if

$$S(\lambda) \stackrel{def}{=} \Delta_1(\lambda)\Delta_2^2(\lambda) = 0 , \tag{2.3}$$

where

$$\begin{aligned} \Delta_1(\lambda) &= \lambda + 1 - \alpha e^{-\lambda\tau_s} - 2\beta e^{-\lambda\tau_n} , \\ \Delta_2(\lambda) &= \lambda + 1 - \alpha e^{-\lambda\tau_s} + \beta e^{-\lambda\tau_n} . \end{aligned}$$

Equation (2.3) is the characteristic equation for (2.1). It follows from standard results [9, 12] that the trivial fixed point of (1.3) will be locally asymptotically stable if all the roots, λ, of the characteristic equation (2.3) have negative real parts and unstable if at least one root has positive real part. A complete description of the stability region of the trivial fixed point of (1.3) is beyond the scope of this paper, however, we give the following two delay independent results.

Theorem 2.1 *If the parameters satisfy $|\beta| < \frac{1}{2}(1 - |\alpha|)$ the trivial solution of (1.3) is locally asymptotically stable for all $\tau_s \geq 0$ and $\tau_n \geq 0$.*

Proof Let $\lambda = \nu + i\omega$, $\nu, \omega \in \mathbb{R}$ in the two factors of the characteristic equation and separate into real and imaginary parts to obtain $\Delta_j(\lambda) = R_j(\nu, \omega) + iI_j(\nu, \omega)$, where

$$\begin{aligned} R_1(\nu, \omega) &= \nu + 1 - \alpha e^{-\nu\tau_s}\cos(\omega\tau_s) - 2\beta e^{-\nu\tau_n}\cos(\omega\tau_n), \\ I_1(\nu, \omega) &= \omega + \alpha e^{-\nu\tau_s}\sin(\omega\tau_s) + 2\beta e^{-\nu\tau_n}\sin(\omega\tau_n). \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} R_2(\nu, \omega) &= \nu + 1 - \alpha e^{-\nu\tau_s}\cos(\omega\tau_s) + \beta e^{-\nu\tau_n}\cos(\omega\tau_n), \\ I_2(\nu, \omega) &= \omega + \alpha e^{-\nu\tau_s}\sin(\omega\tau_s) - \beta e^{-\nu\tau_n}\sin(\omega\tau_n), \end{aligned} \tag{2.5}$$

From equation (2.4), we see that

$$R_1(\nu, \omega) \geq \nu + 1 - |\alpha| e^{-\nu\tau_s} - |\beta| e^{-\nu\tau_n}. \tag{2.6}$$

Denote the right-hand side of (2.6) by $\mathcal{R}_1(\nu)$. Clearly,

$$\mathcal{R}_1(0) = 1 - |\alpha| - |\beta| > 0,$$

under the assumptions of the theorem. Furthermore,

$$\mathcal{R}_1'(\nu) = 1 + |\alpha|\tau_s e^{-\nu\tau_s} + \tau_n|\beta| e^{-\nu\tau_n} .$$

Hence, $\mathcal{R}_1(\nu) > 0$ for all $\nu \geq 0$ and $R_1(\nu, \omega) > 0$ for all $\nu \geq 0, \omega \in \mathbb{R}$. In a similar manner, it may be shown that $R_2(\nu, \omega) > 0$ for all $\nu \geq 0$.

Now let $\lambda = \nu + i\omega$ be an arbitrary root of the characteristic equation. Then ν and ω must satisfy $R_1(\nu, \omega) = 0$ and $I_1(\nu, \omega) = 0$ or $R_2(\nu, \omega) = 0$ and $I_2(\nu, \omega) = 0$. But from the discussion above it follows that $\nu < 0$. Thus all the roots of the characteristic equation have negative real parts. □

Theorem 2.2 *If $1 < \alpha$, then the trivial fixed point of (1.3) is unstable for all values of β, $\tau_s \geq 0$ and $\tau_n \geq 0$.*

Proof Recall from the characteristic equation, that $\Delta_2(\lambda) = (\lambda + 1 - \alpha e^{-\lambda\tau_s} + \beta e^{-\lambda\tau_n})$. Then, under the assumption of the theorem, with $\beta \leq 0$

$$\Delta_2(0) = (1 - \alpha + \beta) < 0$$

and, for $\lambda \in \mathbb{R}$,

$$\lim_{\lambda \to +\infty} \Delta_2(\lambda) = \lim_{\lambda \to +\infty} [\lambda + 1 - \alpha e^{-\lambda \tau_s} + \beta e^{-\lambda \tau_n}] = +\infty.$$

for all $\beta \leq 0$, $\tau_s \geq 0$, and $\tau_n \geq 0$. Hence, as $\Delta_2 : \mathbb{R} \to \mathbb{R}$ is a continuous function, there exists a $\lambda^* > 0$ such that $\Delta_2(\lambda^*) = 0$ for any fixed values of $\tau_s \geq 0$, $\tau_n \geq 0$, $\beta \leq 0$ and $1 < \alpha$.

Now consider $\Delta_1(\lambda) = (\lambda + 1 - \alpha e^{-\lambda \tau_s} - 2\beta e^{-\lambda \tau_n})$. For $\beta \geq 0$, under the conditions of the theorem

$$\Delta_1(0) = 1 - \alpha - 2\beta < 0$$

and, for $\lambda \in \mathbb{R}$,

$$\lim_{\lambda \to +\infty} \Delta_1(\lambda) = \lim_{\lambda \to +\infty} [\lambda + 1 - \alpha e^{-\lambda \tau_s} - 2\beta e^{-\lambda \tau_n}] = +\infty.$$

Hence, as $\Delta_1 : \mathbb{R} \to \mathbb{R}$ is a continuous function, there exists a $\lambda^* > 0$ such that $\Delta_1(\lambda^*) = 0$ for any fixed values of $\tau_s \geq 0$, $\tau_n \geq 0$, $\beta \geq 0$ and $1 < \alpha$.
Thus, the characteristic equation has a positive real root for all β and all $\tau_s \geq 0$, $\tau_n \geq 0$ and $1 < \alpha$. □

As the parameters are varied, stability may be lost by a real root of the characteristic equation passing through zero, or by a pair of complex conjugate roots passing through the imaginary axis. The former occurs when $\beta = \frac{1}{2}(1 - \alpha)$, where the characteristic equation has a simple zero root, and when $\beta = \alpha - 1$, where the characteristic equation has a double zero root. The latter happens if at least one of the following two situations occurs:

1. The characteristic equation has a simple pair of pure imaginary roots $\pm i\omega$ for parameter values such that $\Delta_1(\pm i\omega) = 0$.
2. The characteristic equation has a repeated pair of pure imaginary roots $\pm i\omega$ for parameter values such that $\Delta_2(\pm i\omega) = 0$.

In this paper, we focus on the first case, showing that a Hopf bifurcation occurs under generic conditions and analysing the criticality of this bifurcation to determine the stability of the resulting periodic solutions.

Consider the factor (of the characteristic equation)

$$\Delta_1(\lambda) = \lambda + 1 - \alpha e^{-\lambda \tau_s} - 2\beta e^{-\lambda \tau_n} = 0 ,$$

and let $\lambda = i\omega$. This gives the two relationships between parameters α, β, τ_s, and τ_n

$$\begin{aligned} \omega + \alpha \sin(\omega \tau_s) &= -2\beta \sin(\omega \tau_n) , \\ 1 - \alpha \cos(\omega \tau_s) &= 2\beta \cos(\omega \tau_n) , \end{aligned} \tag{2.7}$$

which determine parameter values for which (2.3) has a pair of pure imaginary roots. We will consider two cases, namely:

(i) $\tau_n = \tau_s = \tau$, for a constant $\tau \geq 0$;
(ii) $\tau_s = \tau$, $\tau_n = \tau + \epsilon$, where $\tau \geq 0$ and ϵ are given constants.

Case (i):
Applying Theorem 1.1 on page 332 of [8], we proceed as follows. In (2.3), suppose that $\lambda = \lambda(\tau)$. Then

$$\frac{dS}{d\tau} = \frac{\partial S}{\partial \tau} + \frac{\partial S}{\partial \lambda}\frac{d\lambda}{d\tau} = 0 ,$$

which gives

$$\frac{d\lambda}{d\tau} = -\left.\frac{\partial S}{\partial \tau}\right/\frac{\partial S}{\partial \lambda}\ , \tag{2.8}$$

where

$$\frac{\partial S}{\partial \tau} = \Delta_2^2(\lambda)\frac{\partial \Delta_1}{\partial \tau} + 2\Delta_1(\lambda)\Delta_2(\lambda)\frac{\partial \Delta_2}{\partial \tau}\ .$$

Recalling that, for the simple root case, $\Delta_1(i\omega) = 0$, we then have that

$$\left.\frac{\partial S}{\partial \tau}\right|_{\lambda=i\omega} = \Delta_2^2(i\omega)\left.\frac{\partial \Delta_1}{\partial \tau}\right|_{\lambda=i\omega}\ .$$

From the above, we get that

$$Re\left(\left.\frac{\partial \lambda}{\partial \tau}\right|_{\lambda=i\omega}\right) = \frac{-\omega(\alpha+2\beta)[K_1\sin(\omega\tau) - K_2\cos(\omega\tau)]}{K_1^2+K_2^2}\ ,$$

where $K_1 = 1 + \tau(\alpha+2\beta)\cos(\omega\tau)$ and $K_2 = \tau(\alpha+2\beta)\sin(\omega\tau)$. Hence, the usual transversality condition is met if and only if

$$\omega(\alpha+2\beta)[K_1\sin(\omega\tau) - K_2\cos(\omega\tau)] \neq 0\ . \tag{2.9}$$

Case(ii):
Similarly, it may be shown that

$$Re\left(\left.\frac{\partial \lambda}{\partial \tau}\right|_{\lambda=i\omega}\right)$$
$$= \frac{\omega}{D_*}[-\{\alpha\sin(\omega\tau) + 2\beta\sin(\omega(\tau+\epsilon))\}Q_1 + \{\alpha\cos(\omega\tau) + 2\beta\cos(\omega(\tau+\epsilon))\}Q_2]\ ,$$

where $Q_1 = 1 + \alpha\tau\cos(\omega\tau) + 2\beta(\tau+\epsilon)\cos(\omega(\tau+\epsilon))$, $Q_2 = \alpha\tau\sin(\omega\tau) + 2\beta(\tau+\epsilon)\sin(\omega(\tau+\epsilon))$, and $D_* = Q_1^2 + Q_2^2$. Thus, the usual transversality condition is met if and only if

$$\omega[\{\alpha\sin(\omega\tau) + 2\beta\sin(\omega(\tau+\epsilon))\}Q_1 - \{\alpha\cos(\omega\tau) + 2\beta\cos(\omega(\tau+\epsilon))\}Q_2] \neq 0 \tag{2.10}$$

Summarising the above discussions and applying Theorem 1.1 of [8, page 332], we obtain the following:

Theorem 2.3 *System (1.3) undergoes a Hopf bifurcation on the surfaces defined by (2.7) if condition (2.9) holds in case (i), and if condition (2.10) holds in case (ii).*

3 Centre manifold analysis

We consider the extended DDE expressed, in standard form (see [6, page 167]), as

$$\dot{\mathbf{x}}_t(\theta) = \begin{cases} \frac{d}{d\theta}[\mathbf{x}_t(\theta)] & , \quad -r \le \theta < 0 \\ \int_{-r}^{0}[d\eta(\theta)]\mathbf{x}_t(\theta) + \mathbf{f}[\mathbf{x}_t(\theta)] & , \quad \theta = 0\ , \end{cases} \tag{3.1}$$

with $\mathbf{x}_t = \mathbf{x}(t+\theta)$, $-r \le \theta \le 0$, $C = C([-r,0],\mathbb{R}^n)$, and $\mathbf{f} \in C^k(C,\mathbb{R}^n)$, $k \ge 1$. In addition, it is usual to assume that $\eta : [-r,0] \to \mathbb{R}^n$ is a function of bounded variation. It is assumed that any parameters in the model are such that the linear part of the equation

$$\dot{\mathbf{x}}_t(\theta) = Lx_t = \begin{cases} \frac{d}{d\theta}[\mathbf{x}_t(\theta)] & , \quad -r \le \theta < 0 \\ \int_{-r}^{0}[d\eta(\theta)]\mathbf{x}_t(\theta) & , \quad \theta = 0\ , \end{cases} \tag{3.2}$$

where $L : C \to \mathbb{R}^n$ is a linear operator, has m eigenvalues with zero real parts, all other eigenvalues having nonzero real parts. In such a situation, Hale and Lunel [8, Chapter 10] have shown that there exists, in the state space C, an m-dimensional centre manifold. Below, we outline the steps involved in computing this manifold, and, in Section 4, we apply it to the system (1.3). Note that if all the eigenvalues of L with nonzero real parts in fact have negative real parts, then this manifold is attracting and that long term behaviour of solutions to the nonlinear equation is well approximated by the flow on this manifold.

At a point in parameter space where (3.2) possesses m eigenvalues with zero real parts, there exists a splitting of the space $C = N \oplus U \oplus S$, where N is an m-dimensional subspace spanned by the solutions to (3.2) corresponding to the m zero real part eigenvalues, U is a finite dimensional subspace spanned by the solutions to (3.2) corresponding to the eigenvalues with positive real part (if any exist), and N, U and S are invariant under the flow associated with (3.2). Furthermore, the centre manifold introduced above is given by

$$M_f = \{\phi \in C \mid \phi = \mathbf{\Phi z} + \mathbf{h}(\mathbf{z}, f)\ ,\ \mathbf{z} \text{ in a neighbourhood of zero in } \mathbb{R}^m\}\ .$$

The flow on this centre manifold is

$$\mathbf{x}_t(\theta) = \Phi(\theta)\mathbf{z}(t) + \mathbf{h}(\theta, \mathbf{z}(t))\ , \tag{3.3}$$

where $\Phi(\theta),\ \theta \in [-r, 0]$ is the basis for the invariant subspace, N, of the linear problem. Substituting (3.3) into (3.1) gives

$$[\Phi(\theta)+D_z\mathbf{h}(\theta,\mathbf{z}(t))]\dot{\mathbf{z}}(t) = \begin{cases} \Phi(\theta)B\mathbf{z}(t) + \frac{\partial \mathbf{h}}{\partial \theta} & , \quad -\tau \le \theta < 0 \\ \Phi(0)B\mathbf{z}(t) + \mathbf{f}[\Phi(\theta)\mathbf{z}(t) + \mathbf{h}(\theta,\mathbf{z}(t))] & \\ \quad + \int_{-\tau}^{0}[d\eta(\theta)]\mathbf{h}(\theta,\mathbf{z}(t)) & , \quad \theta = 0\ , \end{cases} \tag{3.4}$$

where B is an $m \times m$ matrix, whose eigenvalues have null real part. This coupled system must be solved for $\mathbf{z}(t)$ and $\mathbf{h}(\theta, \mathbf{z}(t))$. Let

$$\langle \psi, \phi \rangle = \psi(0)\phi(0) - \int_{-r}^{0}\int_{0}^{\theta} \psi(\xi - \theta)[d\eta(\theta)]\phi(\xi)d\xi \tag{3.5}$$

be the bilinear form associated with (3.2). Then the basis for the adjoint linear problem, $\mathbf{\Psi}(s),\ s \in [0, r]$, satisfies

$$\langle \mathbf{\Psi}(s), \mathbf{\Phi}(\theta) \rangle = \mathbf{I}\ , \tag{3.6}$$

where $\mathbf{I}$ is the $m \times m$ identity matrix.

Taking the scalar product of (3.4) with $\mathbf{\Psi}(s)$, and using the fact that $\langle \mathbf{\Psi}(s), \mathbf{h}(\theta, \mathbf{z}(t)) \rangle = 0$, gives a system of ODE's for $\mathbf{z}(t)$, namely:

$$\dot{\mathbf{z}}(t) = B\mathbf{z}(t) + \mathbf{\Psi}(0)\mathbf{f}[\Phi(\theta)\mathbf{z}(t) + \mathbf{h}(\theta, \mathbf{z}(t))]\ . \tag{3.7}$$

Using (3.7) in (3.4) then yields a system of partial differential equations (PDE's) for $\mathbf{h}(\theta, \mathbf{z})$, viz.

$$D_z\mathbf{h}(\theta,\mathbf{z})\,[B\mathbf{z} + \mathbf{\Psi}(0)\mathbf{f}[\Phi(\theta)\mathbf{z} + \mathbf{h}(\theta,\mathbf{z})]] + \mathbf{\Phi}(\theta)\mathbf{\Psi}(0)\mathbf{f}[\mathbf{\Phi}(\theta)\mathbf{z} + \mathbf{h}(\theta,\mathbf{z})]$$

$$= \begin{cases} \frac{\partial \mathbf{h}}{\partial \theta} & , \quad -\tau \le \theta < 0 \\ \int_{-\tau}^{0}[d\eta(\theta)]\mathbf{h}(\theta,\mathbf{z}) + \mathbf{f}[\Phi(\theta)\mathbf{z} + \mathbf{h}(\theta,\mathbf{z})] & , \quad \theta = 0\ . \end{cases} \tag{3.8}$$

Thus, the evolution of solutions on the centre manifold is determined by solving (3.8) for $\mathbf{h}(\theta, \mathbf{z})$ and then (3.7) for $\mathbf{z}(t)$. To solve (3.8), one uses a standard approach in centre manifold theory, namely, one assumes that $\mathbf{h}(\theta, \mathbf{z}(t))$ may be expanded in power series in $\mathbf{z}$. Equation (3.8) can then be reduced to a system of ODE's and boundary conditions for the coefficients of the power series. Note that the first term in the power series for $\mathbf{h}(\theta, \mathbf{z})$ will correspond to the lowest order terms in the power series of the nonlinearity, $\mathbf{f}$, of (3.1). Thus, if only the lowest order terms of the nonlinearity are needed to determine the evolution of $\mathbf{z}(t)$, in (3.7), and hence the solutions on the centre manifold (3.3), then it is unnecessary to solve for $\mathbf{h}(\theta, \mathbf{z})$.

More detail on this approach to the calculation of centre manifolds for delay differential equations can be found in Hale [7] and Wischert *et al.* [14] for the scalar ($n = 1$) case and in Faria and Magalhães [3, 4] for the vector case.

4 The single Hopf simple root case

We now return to the particular nonlinear system, (1.3). In the neighbourhood of the equilibrium $(0, 0, 0)$, the hyperbolic tangent may be expanded in a Taylor series (to third order), giving

$$\begin{aligned} \dot{x}_i(t) &= -x_i(t) + \alpha x_i(t-\tau_s) + \beta[x_{i-1}(t-\tau_n) + x_{i+1}(t-\tau_n)] \\ &\quad -\frac{1}{3}[\alpha x_i^3(t-\tau_s) + \beta x_{i-1}^3(t-\tau_n) + \beta x_{i+1}^3(t-\tau_n)]\ ,\ i \ mod\ 3 \\ &= -x_{i\,t}(0) + \alpha x_{i\,t}(-\tau_s) + \beta[x_{i-1\,t}(-\tau_n) + x_{i+1\,t}(-\tau_n)] \\ &\quad -\frac{1}{3}[\alpha x_{i\,t}^3(-\tau_s) + \beta x_{i-1\,t}^3(-\tau_n) + \beta x_{i+1\,t}^3(-\tau_n)]\ ,\ i \ mod\ 3 \\ &\stackrel{def}{=} L_i x_t + f_i(x_t)\ ,\ i \ mod\ 3\ , \end{aligned} \tag{4.1}$$

which defines our DDE. This can be written in the form (3.1) with $n = 3$, $\mathbf{f} = [f_1, f_2, f_3]^T$, $r = max\{\tau_s, \tau_n\}$ and

$$\eta(\theta) = \begin{pmatrix} -\delta(\theta) + \alpha\delta(\theta+\tau_s) & \beta\delta(\theta+\tau_n) & \beta\delta(\theta+\tau_n) \\ \beta\delta(\theta+\tau_n) & -\delta(\theta) + \alpha\delta(\theta+\tau_s) & \beta\delta(\theta+\tau_n) \\ \beta\delta(\theta+\tau_n) & \beta\delta(\theta+\tau_n) & -\delta(\theta) + \alpha\delta(\theta+\tau_s) \end{pmatrix}\ ,$$

where $-r \le \theta \le 0$ and $\delta(x)$ is the Dirac distribution at the point $x = 0$.

Note that, since the lowest order nonlinear terms in f_i are $O(3)$, the lowest order terms in $\mathbf{h}$ of equation (3.3) will also be $O(3)$. Hence they will only affect the terms of $O(4)$ in (3.7). Since these are not necessary for determining the criticality of the Hopf, *we do not need to calculate* $\mathbf{h}$.

In the single Hopf simple root case, only two purely imaginary eigenvalues exist. In this situation, the calculations are somewhat simplified by working with a complex coordinate, $z = x + iy$, in which case the elements needed to write (3.7) are

$$\mathbf{\Phi}(\theta) = (\phi_1(\theta), \phi_2(\theta))\ ,\ B = \begin{pmatrix} i\omega & 0 \\ 0 & -i\omega \end{pmatrix}\ ,\ \text{and}\ \mathbf{z} = (z, \bar{z})^T\ , \tag{4.2}$$

where

$$\phi_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{i\omega\theta}\ ,\ \text{and}\ \phi_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} e^{-i\omega\theta}\ .$$

Therefore, the solutions (which are synchronous, in this case) to the linear system (2.1) corresponding to the above eigenfunctions are given by

$$u_j(t) = \sqrt{k_1^2 + k_2^2} \cos\left[\omega t + \tan^{-1}\left(\frac{k_2}{k_1}\right)\right] \ , \ j = 1, \dots 3,$$

where $k_1, \ k_2 \in \mathbb{R}$ are arbitrary constants.

Given $\mathbf{\Phi}(\theta)$, the basis for the adjoint linear problem can be calculated using the ansatz $\mathbf{\Psi}(s) = \mathbf{K}\mathbf{\Phi}^*(s)$, where $\mathbf{K}$ is an $m \times m$ matrix of constants and $M^* = \overline{M}^T$. The condition (3.6) then yields $\mathbf{K} = \langle \mathbf{\Phi}^*(s), \mathbf{\Phi}(\theta)\rangle^{-1}$ or $\mathbf{\Psi}(s) = \langle \mathbf{\Phi}^*(s), \mathbf{\Phi}(\theta)\rangle^{-1}\mathbf{\Phi}^*(s)$. Using (3.5) and the relationships (2.7), the elements of $\langle \mathbf{\Phi}^*(s), \mathbf{\Phi}(\theta)\rangle$ are computed as

$$\begin{aligned} \langle \phi_1^*(s), \phi_1(\theta)\rangle &= \overline{\langle \phi_2^*(s), \phi_2(\theta)\rangle} \\ &= 3 + 3\alpha\tau_s \cos(\omega\tau_s) + 6\beta\tau_n \cos(\omega\tau_n) \\ &\quad -i[3\alpha\tau_s \sin(\omega\tau_s) + 6\beta\tau_n \sin(\omega\tau_n)] \ , \end{aligned} \tag{4.3}$$

and

$$\begin{aligned} \langle \phi_1^*(s), \phi_2(\theta)\rangle &= \langle \phi_2^*(s), \phi_1(\theta)\rangle \\ &= 3 + \frac{3\alpha}{\omega}\sin(\omega\tau_s) + \frac{6\beta}{\omega}\sin(\omega\tau_n) \\ &= 0 \ . \end{aligned} \tag{4.4}$$

Thus,

$$\mathbf{\Psi}(0) = \frac{1}{A_1^2 + A_2^2}\begin{pmatrix} A_1 + iA_2 & A_1 + iA_2 & A_1 + iA_2 \\ A_1 - iA_2 & A_1 - iA_2 & A_1 - iA_2 \end{pmatrix} , \tag{4.5}$$

where

$$\begin{aligned} A_1 &= 3 + 3\alpha\tau_s \cos(\omega\tau_s) + 6\beta\tau_n \cos(\omega\tau_n) \\ A_2 &= 3\alpha\tau_s \sin(\omega\tau_s) + 6\beta\tau_n \sin(\omega\tau_n) \ . \end{aligned} \tag{4.6}$$

In the present case, the function $\mathbf{f}$ in (3.7) is a vector, namely:

$$\mathbf{f}(\mathbf{\Phi z}) = [f_1 \ \ f_2 \ \ f_3]^T(\mathbf{\Phi z}) \ ,$$

where

$$\begin{aligned} f_j(\mathbf{\Phi z}) &= -\frac{1}{3}[\alpha(ze^{-i\omega\tau_s} + \bar{z}e^{i\omega\tau_s})^3 + 2\beta(ze^{-i\omega\tau_n} + \bar{z}e^{i\omega\tau_n})^3] \ , \ j = 1, \dots, 3 \\ &= -\frac{8}{3}[\alpha(x\cos(\omega\tau_s) + y\sin(\omega\tau_s))^3 + 2\beta(x\cos(\omega\tau_n) + y\sin(\omega\tau_n))^3]. \end{aligned} \tag{4.7}$$

From (4.5) and (4.7), we have that

$$\mathbf{\Psi}(0)\mathbf{f}(\mathbf{\Phi z}) = 3\begin{pmatrix} C_1 f_1(\mathbf{\Phi z}) \\ C_2 f_1(\mathbf{\Phi z}) \end{pmatrix} , \tag{4.8}$$

where

$$C_1 = \frac{A_1 + iA_2}{A_1^2 + A_2^2} = \overline{C}_2 \ .$$

Substituting (4.2) and (4.8) back in (3.7), and using $z = x + iy$, yields the dynamical system on the centre manifold

$$\begin{aligned} \dot{x} &= -\omega y + F_{111}^1 x^3 + F_{112}^1 x^2 y + F_{122}^1 x y^2 + F_{222}^1 y^3 \\ \dot{y} &= \omega x + F_{111}^2 x^3 + F_{112}^2 x^2 y + F_{122}^2 x y^2 + F_{222}^2 y^3 \ , \end{aligned} \tag{4.9}$$

where the parameter-dependent coefficients of x^3, x^2y, xy^2, and y^3 are computed with the help of MAPLE [13]. However, these coefficients are complicated and, for this reason, we omit them. The system in (4.9) may be simplified, using a near-identity transformation [5], to the normal form (up to third order)

$$\begin{aligned} \dot{x} &= a(x^2+y^2)x - [\omega + b(x^2+y^2)]y \\ \dot{y} &= [\omega + b(x^2+y^2)]x + a(x^2+y^2)y \ . \end{aligned} \tag{4.10}$$

In polar coordinates, this degenerate system reduces to

$$\dot{r} = ar^3 \ , \ \ \dot{\theta} = \omega + br^2 \tag{4.11}$$

and its unfolding is [5]

$$\dot{r} = \zeta r + ar^3 \ , \ \ \dot{\theta} = \omega + br^2 \ , \tag{4.12}$$

where ζ is an unfolding parameter. Analysis of these equations reveals that there are two distinct cases, depending on the sign of a. For $a < 0$, the Hopf bifurcation gives rise to a stable limit cycle (supercritical). If $a > 0$, the bifurcation yields an unstable limit cycle (subcritical). Employing the relationships (2.7), the parameter a is given by

$$\begin{aligned} a &= \frac{1}{8}[3F^1_{111} + F^1_{122} + F^2_{112} + 3F^2_{222}] \\ &= -\frac{\alpha(\tau_n - \tau_s)(\omega\sin(\omega\tau_s) - \cos(\omega\tau_s)) + 1 + \tau_n(1+\omega^2)}{(1+\tau_n - \alpha(\tau_n-\tau_s)\cos(\omega\tau_s))^2 + (\alpha(\tau_n-\tau_s)\sin(\omega\tau_s) + \omega\tau_n)^2} . \end{aligned} \tag{4.13}$$

Therefore, from (4.13), it is clear that the criticality of the bifurcation is determined by considering only the sign of the expression

$$N = -\alpha(\tau_n - \tau_s)(\omega\sin(\omega\tau_s) - \cos(\omega\tau_s)) - 1 - \tau_n(1+\omega^2). \tag{4.14}$$

The criticality of the Hopf bifurcation is studied, with the aid of a combination of analytical and symbolic computation techniques, in Section 5.

5 Hopf bifurcation surfaces

We start off by recalling the relationships between the parameters at the Hopf bifurcation, as given in (2.7):

$$\begin{aligned} w + \alpha\sin(\omega\tau_s) &= -2\beta\sin(\omega\tau_n) \\ 1 - \alpha\cos(\omega\tau_s) &= 2\beta\cos(\omega\tau_n) \ . \end{aligned}$$

In the analysis of (4.14) that follows, we shall primarily concern ourselves with two cases, namely: (i) $\tau_s = \tau_n = \tau$, and (ii) $\tau_s = \tau$, $\tau_n = \tau + \epsilon$. The rest of the parameters shall be considered fixed.

5.1 The case $\tau_n = \tau_s = \tau$. For some given (α, β), it is straightforward to solve (2.7) to find the corresponding values of ω and τ:

$$\omega = \sqrt{(\alpha+2\beta)^2 - 1} \stackrel{def}{=} \omega^*(\alpha,\beta) \ , \tag{5.1}$$

and

$$\tau = \frac{1}{\omega(\alpha,\beta)} \arctan\left[-\omega(\alpha,\beta)\right] \stackrel{def}{=} \tau^*(\alpha,\beta) \ . \tag{5.2}$$

Fig. 2 shows plots of the surface $\tau = \tau^*(\alpha, \beta)$ for various values of $(\alpha,\ \beta) \in \mathbb{R}^2$. Note that, for any given 2-tuple (α^*, β^*), the corresponding point on the surface gives a $\tau^*(\alpha^*, \beta^*)$ value for which the factor (of equation (2.3))

$$\Delta_1(\lambda) = (-\lambda - 1 + \alpha e^{-\lambda\tau_s} + 2\beta e^{-\lambda\tau_n}) = 0$$

has purely imaginary roots. Hence from Theorem 2.3, the system undergoes a Hopf bifurcation at this point. We shall thus refer to $\tau^*(\alpha, \beta)$ as the *Hopf bifurcation surface.* Furthermore, for $\tau_s = \tau_n = \tau^*,\ \omega = \omega^*$, (4.13) becomes

$$a = -\frac{1 + \tau^*(1 + (\omega^*)^2)}{(1 + \tau^*)^2 + (\omega^*\tau^*)^2}. \tag{5.3}$$

Clearly $a < 0$ for all parameter values and hence each point on the surface $\tau = \tau^*(\alpha, \beta)$ corresponds to a *supercritical* Hopf bifurcation point. This is consistent with the result of Wu *et al.* [15].

5.2 The case $\tau_s = \tau,\ \tau_n = \tau + \epsilon$. Let $\tau_s = \tau$ and $\tau_n = \tau + \epsilon$ where ϵ is arbitrarily small. Then, (2.7) becomes

$$\begin{aligned} \omega + [\alpha + 2\beta\cos(\omega\epsilon)]\sin(\omega\tau) &= -2\beta\sin(\omega\epsilon)\cos(\omega\tau) \\ 1 - [\alpha + 2\beta\cos(\omega\epsilon)]\cos(\omega\tau) &= -2\beta\sin(\omega\epsilon)\sin(\omega\tau)\ . \end{aligned} \tag{5.4}$$

The two expressions above may be rearranged to yield

$$\tau(\omega, \alpha, \beta, \epsilon) = \frac{1}{\omega}\arctan\left\{\frac{-\omega(\alpha + 2\beta\cos(\omega\epsilon)) - 2\beta\sin(\omega\epsilon)}{\alpha + 2\beta\cos(\omega\epsilon) - 2\omega\beta\sin(\omega\epsilon)}\right\} \tag{5.5}$$

where $\omega = \omega(\alpha,\ \beta,\ \epsilon)$ satisfies

$$\alpha^2 + 4\beta^2 + 4\alpha\beta\cos(\omega\epsilon) - (1 + \omega)^2 = 0. \tag{5.6}$$

Given values of α and β, one can solve (5.6) numerically for ω, and substitute into (5.5) to obtain the corresponding value for τ on the Hopf bifurcation surface. This is a daunting exercise. Instead, we now follow the procedure of [1] to gain some insight into the geometry of the bifurcation surfaces and their criticality.

Using (2.7), it is straightforward to solve for τ_n and β in terms of the Hopf frequency ω and the parameters α and τ_s giving

$$\tau_n = \frac{1}{\omega}\arctan\left\{\frac{-\omega - \alpha\sin(\omega\tau_s)}{1 - \alpha\cos(\omega\tau_s)}\right\} \stackrel{def}{=} \tau_n^*(\omega, \alpha, \tau_s)\ , \tag{5.7}$$

and

$$\beta = \frac{1}{2}\sqrt{1 + \omega^2 + \alpha^2 + 2\alpha\omega\sin(\omega\tau_s) - 2\alpha\cos(\omega\tau_s)} \stackrel{def}{=} \beta^*(\omega, \alpha, \tau_s)\ . \tag{5.8}$$

For fixed τ_s these equations parametrically define the Hopf bifurcation surface in (α, β, τ_n) space. This surface is illustrated in Fig. 3 for the case $\tau_s = 1$.

To determine the criticality of the Hopf bifurcation on this surface, we substitute (5.7) into (4.14), which gives an expression

$$N = N(\alpha, \omega, \tau_s)\ .$$

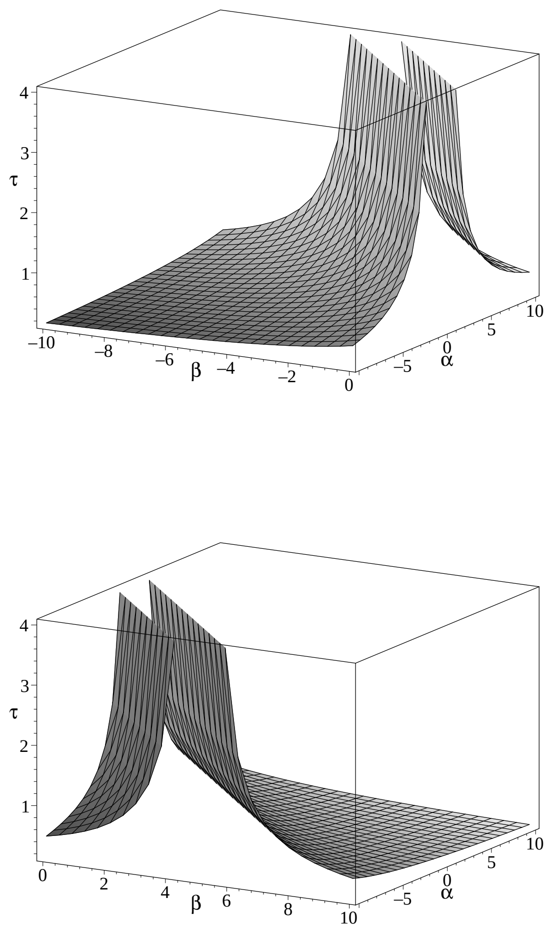

Figure 2 *Geometry of the Hopf bifurcation surface* $\tau = \tau^*(\alpha, \beta)$. Top: $\beta < 0$. Bottom: $\beta > 0$.

For fixed τ_s, it is possible to visualise the surface $N(\alpha, \omega, \tau_s)$ as shown in Fig. 4. This figure illustrates the case $\tau_s = 1$, where the Hopf bifurcation is supercritical if $|\alpha| < 1$, and may be supercritical or subcritical if $|\alpha| > 1$.

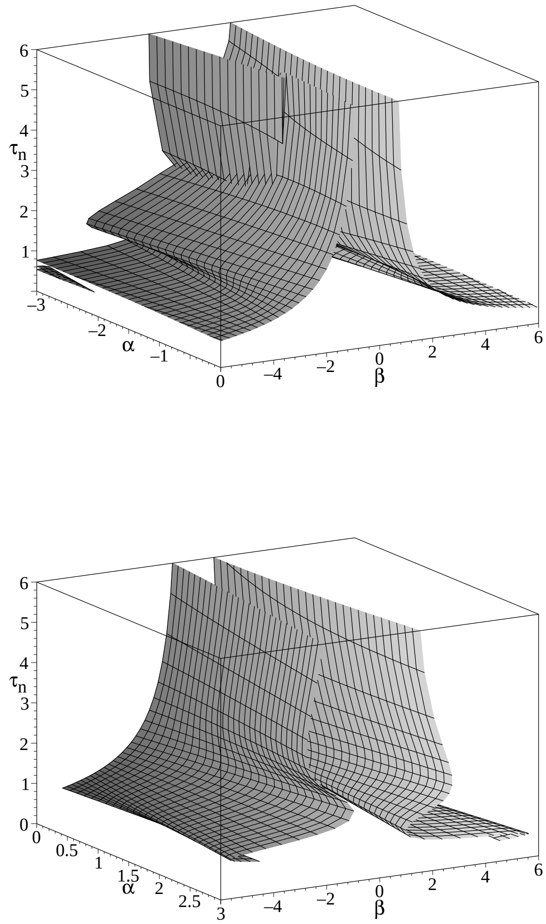

Figure 3 *Geometry of the Hopf bifurcation surface* $\tau_n = \tau_n^*(\omega, \alpha, \tau_s)$, $\beta = \beta^*(\omega, \alpha, \tau_s)$ *for* $\tau_s = 1$. Top: $\alpha < 0$. Bottom: $\alpha > 0$.

6 Conclusions and Remarks

The local stability of a scalar version of (1.1) has been studied in [1]. Following this, Shayer and Campbell [11] considered a system of two coupled neurons with multiple time delays and showed that the trivial fixed point may lose stability via a pitchfork bifurcation, a Hopf bifurcation or one of three types of codimension two bifurcations. Multistability near the latter bifurcations was predicted and confirmed using centre manifold theory and numerical simulations. Furthermore, for the case $\tau_n = \tau_s$, Wu *et al.* in [15] have conducted an insightful study of (1.1). Our approach provides some interesting insight into the nature of simple root single Hopf bifurcations of (1) when $\tau_s \neq \tau_n$. It should be emphasised that this study is a preliminary attempt aimed at grappling with the case $\tau_s \neq \tau_n$, for the simple root single Hopf bifurcation. The double root single Hopf case is even more irksome, vis à vis symbolic computational issues, and remains the subject of ongoing and future

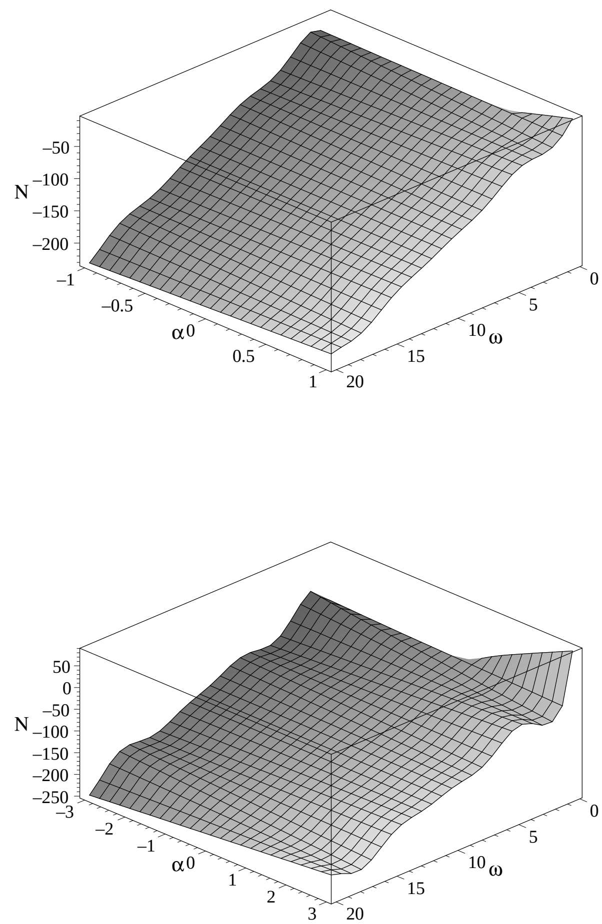

Figure 4 *The surface $N(\alpha, \omega, \tau_s)$ for $\tau_s = 1$.* Top: *For $-1 < \alpha < 1$, N is always negative.* Bottom: *For $-3 < \alpha < 3$, N takes on both negative and positive values.*

investigations. However, as demonstrated in Section 5.2, the case under consideration already presents some serious symbolic computational challenges. This said, we have provided some revealing insights into criticality of the Hopf for this case, as can be seen in Fig. 4. This figure suggests that change of the criticality of the Hopf bifurcation can be easily achieved by varying the parameter α. At the points where such a change takes place, secondary and more complicated bifurcations may occur.

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