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RESONANT CODIMENSION TWO BIFURCATION IN THE HARMONIC OSCILLATOR WITH DELAYED FORCING

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ABSTRACT. We study a delay differential equation modeling the harmonic oscillator with forcing which depends on the state and the derivative of the state at some time in the past. We perform a linearized stability analysis of the equation and describe the location of Hopf and steady state bifurcations in the parameter space. A complete description of the location of points in parameter space where the characteristic equation possesses two pairs of pure imaginary roots, $\pm i\omega_1$, $\pm i\omega_2$ with $\omega_1 : \omega_2 = m : n$, $m, n \in \mathbf{Z}^+$, is given.

1. Introduction. Consider the harmonic oscillator with forcing which depends on the state and/or derivative of the state. In real physical systems such forcing terms are often time delayed, especially when they represent feedback. This gives rise to the delay differential equation

$$\ddot{x}(t) + b\dot{x}(t) + ax(t) = f(x(t-\tau_1), \dot{x}(t-\tau_2)), \tag{1}$$

where f may be nonlinear. By analogy with a mass spring system, we will refer to $x(t), \dot{x}(t)$ as, respectively, the position and velocity at time t, b as the damping constant and a as the spring constant. In most physical systems $a > 0$ and $b \geq 0$; however, we won't make such restrictions. τ_1 and τ_2 are the time delays, assumed to be nonnegative. This equation and its variants have been used to model a number of systems, [**1, 4, 12, 23, 24, 25**]; see also [**9** and references therein]. While there is no physical reason to assume the delays in x and $\dot{x}$ are the same, this is usually done to simplify the analysis. We will follow this practice and take $\tau_1 = \tau_2 = \tau$. (For an exception see [**6**].)

Equation (1) has a fixed point (equilibrium) solution $x(t) = x^*$, found by solving $ax^* = f(x^*, 0)$. In a similar manner to that for ordinary differential equations, the equation may be linearized around this fixed point leading to

$$\ddot{x}(t) + b\dot{x}(t) + ax(t) = dx(t-\tau) + g\dot{x}(t-\tau), \tag{2}$$

where $d = D_1 f(x^*, 0)$, $g = D_2 f(x^*, 0)$. To study the stability of the fixed point, we look for solutions of (2) of the form $x(t) = \exp(\lambda t)$, which leads to the characteristic equation

$$\lambda^2 + b\lambda + a = de^{-\lambda\tau} + g\lambda e^{-\lambda\tau}. \tag{3}$$

For clarity we review some terminology. The roots of the characteristic equation are commonly called the eigenvalues of the fixed point. It can be shown [**15, 17**] that the fixed point is (linearly) stable if all of its eigenvalues have zero real parts, and (linearly) unstable if at least one of its eigenvalues has positive real part. Thus changes of stability of the fixed point may occur when an eigenvalue has zero real part, i.e., when there is real, zero eigenvalue or a pure imaginary pair. The former indicates the presence of a steady state bifurcation which, depending on the nonlinearity f, may be a pitchfork, transcritical or saddle node bifurcation. Under suitable conditions on f, see, e.g., [**15**, Chapter 10] the latter indicates the presence of a Hopf bifurcation.

The characteristic equation (3) has been studied by many authors [**2, 3, 4, 5, 7, 9, 11, 13, 18, 23, 27**], who have shown the presence of both steady state and Hopf bifurcations under various conditions on the parameters. In certain cases, [**9**] it is also possible to show the presence of points where the characteristic equation has *two* pairs of pure imaginary roots, $\pm i\omega_1$, $\pm i\omega_2$. As such points commonly occur where two curves of Hopf bifurcation cross, we refer to them as points of double Hopf bifurcation.

Here we are primarily concerned with the situation when the forcing depends only on the position or the velocity, but not both (this corresponds to taking $g = 0$ or $d = 0$ in (2)). These two cases possess points of double Hopf bifurcation that are resonant, that is, the imaginary parts (frequencies of the Hopf bifurcations) obey $\omega_1 : \omega_2 = m : n$ for some $m, n \in \mathbf{Z}$.

The outline of the paper is as follows. In Section 2 we consider the stability analysis of (2) under the restrictions mentioned. In Section 3 we show explicitly where the points of resonant double Hopf bifurcation for the equation are located in parameter space. In Section 4 we discuss the implications of these results for the full equation (1) and the physical systems it models.

2. Linearized stability analysis. Some aspects of the stability analysis of (2) under the restrictions

Case 1. Position feedback ($b = g = 0$),

Case 2. Velocity feedback ($b = d = 0$),

have been studied by Bhatt and Hsu [**3, 18**]. Here we present a different proof of stability, following the method of Cooke and Grossman [**13**], and consider the parameter space of the delay, τ, and the gain, d or g. This parameter space is relevant to physical systems where the forcing represents feedback and the delay and/or gain may be adjustable by the experimenter.

Our stability results will rely upon the following facts which may be found in many books on delay differential equations, for example, [**21**].

F1. The roots of the characteristic equation (3) are continuous functions of the parameters d, g.

F2. The number of roots of the characteristic equation (3) with positive real parts may change with the variation of the parameters d, g only by the passage of a root through the imaginary axis.

2.1. *Case* 1. *Position feedback.* This situation can be modeled by the following restriction of (1)

$$\ddot{x}(t) + ax(t) = f_1(x(t-\tau)), \tag{4}$$

which has a fixed point, $x(t) = x^*$, determined by solving $ax^* = f_1(x^*)$. The linearization about this fixed point is

$$\ddot{x}(t) + ax(t) = dx(t-\tau), \tag{5}$$

leading to the characteristic equation

$$\lambda^2 + a = de^{-\lambda\tau}, \quad \text{where } d = f_1'(x^*). \tag{6}$$

As previously mentioned, changes in stability may occur when the system has a zero eigenvalue or a pure imaginary pair. The former occurs when $\lambda = 0$ in (6) or

$$a = d. \tag{7}$$

The latter occurs when $\lambda = \pm i\omega$ in (6) or (separating real and imaginary parts)

$$a - \omega^2 = d\cos\omega\tau \tag{8a}$$

$$0 = -d\sin\omega\tau. \tag{8b}$$

Solving these equations for τ and ω as functions of the other parameters, we find that there are two families of surfaces (or curves if we think of the physical parameter a as fixed) in the parameter space along which the fixed point has a pair of purely imaginary eigenvalues:

$$\omega = \sqrt{a+d}, \qquad \tau = \frac{(2j+1)\pi}{\sqrt{a+d}}, \quad -a < d \tag{9a}$$

$$\omega = \sqrt{a-d}, \qquad \tau = \frac{2j\pi}{\sqrt{a-d}}, \quad d < a, \tag{9b}$$

for $j = 0, 1, 2, \ldots$. This leads to the following

Theorem 2.1. *For $a > 0$ the fixed point is linearly stable only in the following regions*:

(i) $d < 0$,

$$\frac{(2j-1)\pi}{\sqrt{a+d}} < \tau < \frac{2j\pi}{\sqrt{a-d}}, \quad j = 1, 2, \ldots$$

(ii) $0 < d < a$,

$$\frac{2j\pi}{\sqrt{a-d}} < \tau < \frac{(2j+1)\pi}{\sqrt{a+d}}, \quad j = 0, 1, 2, \ldots .$$

Proof. Let $S(d)$ be the supremum of the real parts of the roots of the characteristic equation (6). We begin by noting that for $d = 0$ the roots are $\lambda = \pm\sqrt{a}\, i$ so $S(0) = 0$. Consideration of the rate of change of the real part of the eigenvalues with respect to d along $d = 0$ gives

$$\begin{aligned}
\left.\frac{d\mathrm{Re}\,(\lambda)}{d\,d}\right|_{d=0} &= -\frac{\sin\sqrt{a}\tau}{2\sqrt{a}} \\
&> 0 \quad \text{for } \frac{(2j-1)\pi}{\sqrt{a}} < \tau < \frac{2j\pi}{\sqrt{a}}, \quad j = 1, 2, \ldots \\
&< 0 \quad \text{for } \frac{2j\pi}{\sqrt{a}} < \tau < \frac{(2j+1)\pi}{\sqrt{a}}, \quad j = 0, 1, 2, \ldots .
\end{aligned}$$

Thus by F1 we have

$$\lim_{d \to 0^-} S(d) = 0^- \quad \text{for } \frac{(2j-1)\pi}{\sqrt{a}} < \tau < \frac{2j\pi}{\sqrt{a}}, \quad j = 1, 2, \dots,$$

and

$$\lim_{d \to 0^+} S(d) = 0^- \quad \text{for } \frac{2j\pi}{\sqrt{a}} < \tau < \frac{(2j+1)\pi}{\sqrt{a}}, \quad j = 0, 1, 2, \dots.$$

By F2 all the real parts will remain negative until the parameters cross one of the curves given by (7) or (9a)–(9b). Noting that the $\lambda = i\omega$ curves (9a)–(9b) intersect the $d = 0$ axis at

$$\tau = \frac{(2j-1)\pi}{\sqrt{a}} \quad \text{or} \quad \tau = \frac{2j\pi}{\sqrt{a}},$$

for $j = 1, 2, \dots$, we see that the fixed point is linearly stable in regions (i) and (ii) above. Consideration of the rate of change with respect to d of the real part of the eigenvalues along the curves (7)

$$\left. \frac{d\mathrm{Re}\,(\lambda)}{d\,d} \right|_{d=a} = \frac{1}{a\tau} > 0$$

or (9a)–(9b)

$$\begin{aligned} \left. \frac{d\mathrm{Re}\,(\lambda)}{d\,d} \right|_{\lambda = \pm i\omega} &= \frac{d\tau}{d^2\tau^2 + 4\omega^2} \\ &> 0 \quad \text{for } d > 0 \\ &< 0 \quad \text{for } d < 0, \end{aligned}$$

shows that the fixed point cannot restabilize outside this region. □

This region is illustrated (for $a = 1$) with shading in Figure 1. As can be seen in the figure, for fixed a (9a)–(9b) define curves in τ, d space. It is straightforward to show that for f sufficiently smooth, (4) satisfies the conditions of the Hopf bifurcation theorem for delay differential equations, [**15**, Chapter 10] almost everywhere[1] along these curves; we

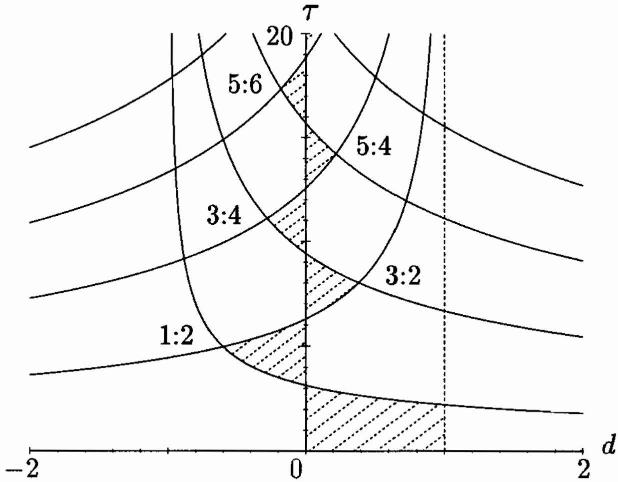

FIGURE 1. Region of linearized stability (shading) of the fixed point of (4), illustrated in the parameter space of the delay (τ) and gain ($d = f_1'(x^*)$). Here $a = 1$, but the picture remains qualitatively the same for any $a > 0$. Ratios of frequencies ($\omega_1 : \omega_2$) at double Hopf points are as indicated for any $a > 0$.

will thus refer to them as Hopf bifurcation curves. Note that for a fixed value of d increasing the delay, τ, causes the fixed point to destabilize and restabilize a finite number of times, ultimately remaining unstable. A similar phenomenon was observed for the damped system by Cooke and Grossman [**13**]. Also, for $d < 0$, the fixed point in the undelayed system is unstable but introduction of a large enough delay *restabilizes* it.

Theorem 2.2. *For $a \leq 0$, the fixed point is linearly unstable for all values of d and τ, except at $a = d = 0$ where it is linearly neutrally stable.*

Proof. $\mathbf{a \leq 0}$, $\mathbf{a < d}$. Consider the function $h_1 : \mathbf{R} \to \mathbf{R}$, defined by

$$h_1(\alpha) = \alpha^2 + a - de^{-\alpha\tau}.$$

Note that $\lim_{\alpha\to\infty} h_1(\alpha) = +\infty$; thus, for each fixed a, d, τ and M there is a K such that $h_1(\alpha) > M$ for $\alpha > K$. Pick $M > 0$; then

$h_1(K+1) > 0$. Further, $h_1(0) = a - d < 0$. Since h_1 is a continuous function of α there must be $0 < \alpha_0 < K+1$ such that $h_1(\alpha_0) = 0$. Thus the characteristic equation (6) has a real, positive root and the fixed point is unstable.

$\mathbf{d = a < 0}$. Consider again the real-valued function $h_1(\alpha)$. In this case for $0 < \alpha \ll 1$ we have

$$\begin{aligned} h_1(\alpha) &= \alpha^2 + a - a\left[1 - \alpha\tau + \frac{1}{2}\alpha^2\tau^2 - \frac{1}{6}\alpha^3\tau^3 + \cdots\right] \\ &\approx a\alpha\tau. \end{aligned}$$

Thus for α small enough, $h_1(\alpha) < 0$. As before, $\lim_{\alpha\to\infty} h_1(\alpha) = +\infty$, so continuing as in the previous case, it is easily shown that there is $0 < \alpha_0 < K+1$ such that $h_1(\alpha_0) = 0$. As before, we conclude that the fixed point is linearly unstable.

$\mathbf{d = a = 0}$. In this case the characteristic equation (6) becomes $\lambda^2 = 0$, which obviously has the zero as a double root. Hence the fixed point is linearly neutrally stable.

$\mathbf{d < a \le 0}$. For $\tau = 0$ characteristic equation has roots $\lambda = \pm i\sqrt{a-d}$. Consideration of the rate of change of the real parts of the eigenvalues as τ increases across this line yields

$$\left.\frac{d\,\mathrm{Re}\,(\lambda)}{d\tau}\right|_{\tau=0} = -\frac{d}{2} > 0.$$

Since the roots of the characteristic equation (6) are continuous functions of τ [**14**] the real parts of these roots of the characteristic equation must be positive as τ becomes positive. We conclude that the fixed point is linearly unstable for small, positive τ. Now as τ increases, the fixed point can only regain stability if the eigenvalues pass through the imaginary axis [**13**], that is, if the parameters cross one of the curves given by (7) or (9a)–(9b). The curves described by (7) and (9a) do not lie in the region we are considering. Consideration of the rate of change with respect to τ of the real parts of the eigenvalues along (9b) yields

$$\left.\frac{d}{d\tau}\mathrm{Re}\,(\lambda)\right|_{\tau=(2j\pi/\sqrt{a-d})} = -\frac{2d\omega^2}{d^2\tau^2 + 4\omega^2} > 0.$$

Since this quantity is positive, the fixed points cannot restabilize along these curves. □

2.2. *Case* 2. *Velocity feedback.* This situation can be modeled by the following restriction of (1)

$$\ddot{x}(t) + ax(t) = f_2(\dot{x}(t-\tau)), \tag{10}$$

which has a fixed point, $x(t) = x^*$, given by $x^* = f_2(0)/a$. The linearization about this fixed point is

$$\ddot{x}(t) + ax(t) = g\dot{x}(t-\tau), \tag{11}$$

leading to the characteristic equation

$$\lambda^2 + a = g\lambda e^{-\lambda\tau}, \quad \text{where } g = f_2'(0). \tag{12}$$

Again, changes of stability will occur at points in parameter space where the real part(s) of the eigenvalue(s) are zero. Substituting $\lambda = 0$ in (12) gives

$$a = 0. \tag{13}$$

Substituting $\lambda = i\omega$ in (12) gives

$$a - \omega^2 = g\omega \sin \omega\tau \tag{14a}$$
$$0 = g \cos \omega\tau. \tag{14b}$$

Solving for τ and ω in the case $a > 0$ shows that there exist two families of surfaces (or curves for a fixed) along which the fixed point has a pair of pure imaginary eigenvalues:

$$\omega = \frac{1}{2}[\sqrt{g^2+4a} - g], \qquad \tau = \frac{(4j-3)\pi}{\sqrt{g^2+4a} - g} \tag{15a}$$
$$\omega = \frac{1}{2}[\sqrt{g^2+4a} + g], \qquad \tau = \frac{(4j-1)\pi}{\sqrt{g^2+4a} + g}, \tag{15b}$$

where $j = 1, 2, \ldots$. This leads to

Theorem 2.3. *For $a > 0$, the fixed point is linearly stable in the following regions:*

(i) $g < 0$,

$$0 < \tau < \frac{\pi}{\sqrt{g^2+4a}-g}$$

$$\frac{(4j-1)\pi}{\sqrt{g^2+4a}+g} < \tau < \frac{(4j+1)\pi}{\sqrt{g^2+4a}-g}, \quad j = 1, 2, \ldots .$$

(ii) $0 < g$,

$$\frac{(4j-3)\pi}{\sqrt{g^2+4a}-g} < \tau < \frac{(4j-1)\pi}{\sqrt{g^2+4a}+g}, \quad j = 1, 2, \ldots .$$

Proof. Proceeding analogously to Theorem 2.1 we define $S(g)$ to be the supremum of the real parts of the roots of (12), and note that for $g = 0$ the roots are $\lambda = \pm\sqrt{a}\, i$ so $S(0) = 0$. Consideration of the rate of change of the real part of the eigenvalues with respect to g along the line $g = 0$ gives

$$\begin{aligned} \left.\frac{d\mathrm{Re}\,(\lambda)}{d\,g}\right|_{g=0} &= \frac{1}{2}\cos\sqrt{a}\tau \\ &> 0 \quad \text{for } 0 < \tau < \frac{\pi}{2\sqrt{a}} \\ &> 0 \quad \text{for } \frac{(4j-1)\pi}{2\sqrt{a}} < \tau < \frac{(4j+1)\pi}{2\sqrt{a}}, \\ &< 0 \quad \text{for } \frac{(4j-3)\pi}{2\sqrt{a}} < \tau < \frac{(4j-1)\pi}{2\sqrt{a}}, \end{aligned}$$

$j = 1, 2, \ldots .$ Thus, by F1, we have

$$\begin{aligned} \lim_{g\to 0^-} S(g) = 0^- \quad &\text{for } 0 < \tau < \frac{\pi}{2\sqrt{a}}, \\ &\text{or } \quad \frac{(4j-1)\pi}{2\sqrt{a}} < \tau < \frac{(4j+1)\pi}{2\sqrt{a}}, \end{aligned}$$

and

$$\lim_{g\to 0^+} S(g) = 0^- \quad \text{for } \frac{(4j-3)\pi}{2\sqrt{a}} < \tau < \frac{(4j-1)\pi}{2\sqrt{a}},$$

$j = 1, 2, \ldots$. By F2 all the real parts will remain negative until the parameters cross one of the curves given by (15a)–(15b). Noting that the intersections of these curves with the $g = 0$ axis are given by

$$\tau = \frac{(4j-1)\pi}{2\sqrt{a}} \quad \text{or} \quad \tau = \frac{(4j-3)\pi}{2\sqrt{a}},$$

for $j = 1, 2, \ldots$, we see that the fixed point is linearly stable in regions (i) and (ii) above. Consideration of the rate of change with respect to g of the real part of the eigenvalues along the curves (15a) and (15b),

$$\begin{aligned} \left.\frac{d\mathrm{Re}\,(\lambda)}{d\,g}\right|_{\lambda=\pm i\omega} &= \frac{g\omega^2\tau}{4a + g^2(1+\omega^2\tau^2)} \\ &> 0 \quad \text{for } g > 0 \\ &< 0 \quad \text{for } g < 0, \end{aligned}$$

shows that the fixed point cannot restabilize along these curves. □

This region is illustrated with shading in Figure 2. As in the case of position feedback, it can be shown that for f in (10) sufficiently smooth, the curves defined by (15a)–(15b) and illustrated in the figure are indeed curves of Hopf bifurcation. In this case for fixed g and increasing τ the fixed point alternates between stability and instability a finite number of times, finally becoming unstable.

Theorem 2.4. *For $a < 0$, the fixed point is linearly unstable for all g and τ.*

Proof. Consider the real valued function $h_2 : \mathbf{R} \to \mathbf{R}$ defined by

$$h_2(\alpha) = \alpha^2 + a - g\alpha e^{-\alpha\tau}.$$

Note that $\lim_{\alpha\to\infty} h_2(\alpha) = +\infty$; thus, for each fixed a, g, τ and M there is a K such that $h_2(\alpha) > M$ for $\alpha > K$. Pick $M > 0$; then $h_2(K+1) > 0$. Further, $h_2(0) = a < 0$. Since h_2 is a continuous function of α, there must be a $0 < \alpha_0 < K+1$ such that $h_2(\alpha_0) = 0$. Thus the characteristic equation (12) has a real, positive root and the fixed point is unstable. □

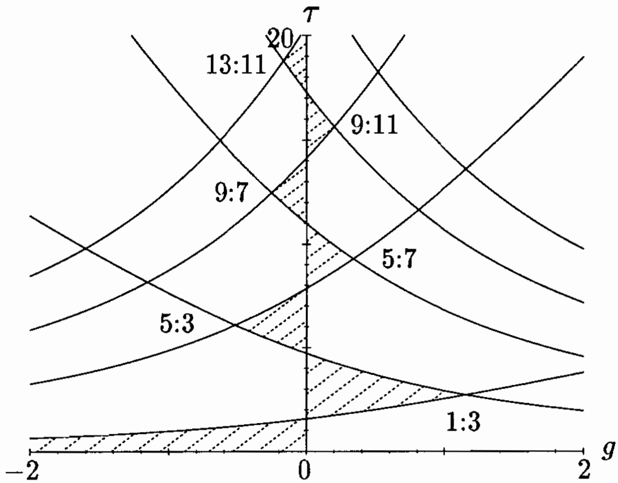

FIGURE 2. Region of linearized stability (shading) of the fixed point of (10), illustrated in the parameter space of the delay (τ) and gain ($g = f_2'(0)$). Here $a = 1$, but the picture remains qualitatively the same for any $a > 0$. Ratios of frequencies ($\omega_1 : \omega_2$) at double Hopf points are as indicated for any $a > 0$.

Theorem 2.5. *For $a = 0$, the fixed point is linearly unstable for $g > 0$, $\tau > 0$ and $g < -(\pi/2\tau) < 0$ and linearly neutrally stable for $g = 0$, $\tau > 0$ and $-(\pi/2\tau) \le g < 0$.*

Proof. We begin by noting that when $a = 0$ the characteristic equation (12) reduces to $\lambda^2 = \lambda g e^{-\lambda\tau}$. Thus one root is $\lambda = 0$ for all values of g, τ and the other roots must satisfy $\lambda = g e^{-\lambda\tau}$.

$g = 0$. In this case the characteristic equation (12) becomes $\lambda^2 = 0$, which obviously has zero as a double root. Hence the fixed point is linearly neutrally stable.

$g > 0$. Consideration of the real valued function

$$h_3(\alpha) = \alpha - g e^{-\alpha\tau}, \quad \alpha \in \mathbf{R}$$

in a similar manner to that of Theorem 2.4 shows that h_3 (and hence the characteristic equation) possesses a real, positive root, so the fixed point is unstable.

$g < 0$. Note that if $\tau = 0$ the fixed point is neutrally stable (it has eigenvalues $\lambda = 0$, $g < 0$). Now, as stated in Theorem 2.2, the stability of the fixed point can only change, as τ is varied, if an eigenvalue passes through the imaginary axis. Thus, as τ increases, the fixed point must remain neutrally stable until reaching a point where $\lambda = 0$ or $\lambda = \pm i\omega$. Only the latter exist in this region and are given by $\tau = -((4j-3)\pi/2g)$, $j = 1, 2, \dots$. Consideration of the rate of change of the real part of the eigenvalues with respect to τ along these curves yields

$$\left.\frac{d\mathrm{Re}\,(\lambda)}{d\tau}\right|_{\tau=-((4j-3)\pi/2g)} = \frac{4g^2}{4+(4j-3)^2\pi^2} > 0.$$

Hence the fixed point is linearly neutrally stable for $-(\pi/2\tau) \le g < 0$ and linearly unstable for $g < -(\pi/2\tau) < 0$. □

3. Double Hopf bifurcation. Recall that a double Hopf bifurcation point occurs where two curves of Hopf bifurcation intersect. It can be readily seen in Figures 1 and 2 that there are many such points in the system being studied. What is noteworthy about the particular parameter restrictions we are considering is that *all* the double Hopf points that occur are resonant, that is, the imaginary parts (frequencies of the Hopf bifurcations) obey $\omega_1 : \omega_2 = n : m$ for some $n, m \in \mathbf{Z}$.

3.1. *Case* 1. *Position feedback.*

Theorem 3.1. *For $a > 0$, every double Hopf point of* (6) *is resonant. More specifically, when $d = d_{\mathrm{res}}$ as defined by* (17), (6) *possesses two pairs of pure imaginary roots $\pm i\omega_1, \pm i\omega_2$ with frequencies in the ratio $\omega_1 : \omega_2 = (2k-1) : 2l$ for all $k, l \in \mathbf{Z}^+$.*

Proof. The curves of Hopf bifurcation in this case are given by (9a)–(9b). Points of intersection of these curves (double Hopf points) occur when τ has the same value on the two curves, i.e.,

$$\frac{(2k-1)}{\sqrt{a+d}} = \frac{2l}{\sqrt{a-d}}, \tag{16}$$

for some $k, l \in \mathbf{Z}^+$. Rearranging this equation shows that the frequencies are in the ratio $\omega_1 : \omega_2 = \sqrt{a+d} : \sqrt{a-d} = (2k-1) : 2l$. Squaring both sides of (16) and solving for d shows that the intersections occur when

$$d = \frac{(2k-1)^2 - (2l)^2}{(2k-1)^2 + (2l)^2} a \stackrel{\text{def}}{=} d_{\text{res}}. \qquad \square \tag{17}$$

The first few resonant double Hopf points are illustrated in Figure 1. Note that some of these points occur on the boundary of the region of stability of the fixed point, hence they may influence the observed behavior of the system. We discuss this further in Section 4.

For $a \leq 0$, there is only a single value of ω for each value of d, τ, namely,

$$\omega = \sqrt{a+d}, \quad 0 \leq -a \leq d \tag{18a}$$

$$\omega = \sqrt{a-d}, \quad d \leq a \leq 0. \tag{18b}$$

Thus there can be no double Hopf points.

3.2. *Case* 2. *Velocity feedback.*

Theorem 3.2. *For $a > 0$ every double Hopf point of* (12) *is resonant. More specifically, when $g = g_{\text{res}}$ as defined by* (20), (12) *possesses two pairs of pure imaginary roots $\pm i\omega_1, \pm i\omega_2$ with frequencies in the ratio $\omega_1 : \omega_2 = 4k-3 : 4l-1$, for all $k, l \in \mathbf{Z}^+$.*

Proof. The curves of Hopf bifurcation in this case are given by (15a)–(15b). Points of intersection of these curves (double Hopf points) occur when τ has the same value on two curves, i.e.,

$$\frac{(4k-3)\pi}{(\sqrt{4a+g^2}-g)} = \frac{(4l-1)\pi}{(\sqrt{4a+g^2}+g)}, \tag{19}$$

for some $k, l \in \mathbf{Z}^+$. Rearranging this equation shows that the frequencies are in the ratio $\omega_1 : \omega_2 = \sqrt{a+(g^2/4)} - (g/2) : \sqrt{a+(g^2/4)} + (g/2) = (4k-3) : (4l-1)$. Rearranging, and squaring both sides of (19) and solving for g shows that these intersections occur when

$$g = \frac{2(2l-2k+1)}{\sqrt{16kl-4k-12l+3}} \sqrt{a} \stackrel{\text{def}}{=} g_{\text{res}}. \qquad \square \tag{20}$$

The first few resonant double Hopf points are illustrated in Figure 2. Note that some of these points occur on the boundary of the region of stability of the fixed point, hence they may influence the observed behavior of the system. We discuss this further in Section 4.

For $a < 0$, $g < 0$, the curves of pure imaginary eigenvalues are given by

$$\omega = \frac{1}{2}[-\sqrt{g^2+4a}-g], \qquad \tau = \frac{(4j-3)\pi}{-\sqrt{g^2+4a}-g} \tag{21a}$$

$$\omega = \frac{1}{2}[\sqrt{g^2+4a}-g], \qquad \tau = \frac{(4j-3)\pi}{\sqrt{g^2+4a}-g}, \tag{21b}$$

where $j = 1, 2, \dots$. This leads to

Theorem 3.3. *For $a < 0$, $g < 0$, every double Hopf point of* (12) *is resonant. More specifically, when $g = g_{\text{res}-}$ as defined by* (22), (12) *possesses two pairs of pure imaginary roots $\pm i\omega_1$, $\pm i\omega_2$ with frequencies in the ratio $\omega_1 : \omega_2 = 4k-3 : 4l-3$, for all $k \neq\in \mathbf{Z}^+$.*

Proof. The proof is analogous to that of Theorem 3.2 with

$$g_{\text{res}-} \stackrel{\text{def}}{=} -\frac{2(2k+2l-3)}{\sqrt{16kl-12k-12l+9}}\sqrt{-a}. \qquad \square \tag{22}$$

For $a < 0$, $g > 0$, the curves of pure imaginary eigenvalues are given by

$$\omega = \frac{1}{2}[-\sqrt{g^2+4a}+g], \qquad \tau = \frac{(4j-1)\pi}{-\sqrt{g^2+4a}+g} \tag{23a}$$

$$\omega = \frac{1}{2}[\sqrt{g^2+4a}+g], \qquad \tau = \frac{(4j-1)\pi}{\sqrt{g^2+4a}+g}, \tag{23b}$$

where $j = 1, 2, \dots$. This leads to

Theorem 3.4. *For $a < 0$, $g > 0$, every double Hopf point of* (12) *is resonant. More specifically, when $g = g_{\text{res}+}$ as defined by* (24), (12) *possesses two pairs of pure imaginary roots* $\pm i\omega_1, \pm i\omega_2$ *with frequencies in the ratio* $\omega_1 : \omega_2 = 4k-1 : 4l-1$, *for all* $k \neq\in \mathbf{Z}^+$.

Proof. The proof is analogous to that of Theorem 3.2 with

$$g_{\mathrm{res}+} \stackrel{\mathrm{def}}{=} \frac{2(2k+2l-1)}{\sqrt{16kl-4k-4l+1}}\sqrt{-a}. \qquad \square \tag{24}$$

For $a = 0$, there is a single value of ω for each value of g, τ, namely,

$$\omega = -g, \quad g < 0, \tag{25a}$$

$$\omega = g, \quad g > 0. \tag{25b}$$

Thus there can be no double Hopf points.

The resonances double Hopf points described in Theorems 3.3–3.4 can be illustrated as in Figures 1–2. These points are less interesting as the fixed point is linearly unstable in the regions where they exist; hence, they will have little effect on the observable dynamics of the system.

3.3. Nonzero damping. So far we have shown that when there is no damping and the feedback depends only on the position or the velocity, all double Hopf points are resonant. Now let us consider the general equation (1). By consideration of the existence of pure imaginary eigenvalues for the fixed point of this equation, one can derive the following necessary condition for the occurrence of a resonant double Hopf point with frequencies in the ratio $m : n$:

$$g^2 = b^2 - 2a + \frac{m^2+n^2}{mn}\sqrt{a^2-d^2}. \tag{26}$$

Further, one can show that the frequencies are given by the simple expressions:

$$\omega_1 = \sqrt{\frac{m}{n}\sqrt{a^2-d^2}}, \qquad \omega_2 = \sqrt{\frac{n}{m}\sqrt{a^2-d^2}}. \tag{27}$$

Sufficient conditions are obtained by appending to (26) an equation defining the intersection points of the Hopf curves:

$$\text{(28)}\quad \arccos \frac{ad + (m/n)\sqrt{a^2 - d^2}(bg - d)}{d^2 + g^2(m/n)\sqrt{a^2 - d^2}} = \frac{m}{n} \arccos \frac{ad + (n/m)\sqrt{a^2 - d^2}(bg - d)}{d^2 + g^2(n/m)\sqrt{a^2 - d^2}},$$

g defined by (26). The value of the delay at the resonant double Hopf point is then given by:

$$\text{(29)}\qquad \tau = \sqrt{\frac{n}{m\sqrt{a^2 - d^2}}} \arccos \frac{ad + (m/n)\sqrt{a^2 - d^2}(bg - d)}{d^2 + g^2(m/n)\sqrt{a^2 - d^2}}.$$

Clearly, (28) cannot be solved closed in form, except when there are certain restrictions on the parameter values. We have seen examples of such restrictions in Theorems 3.1–3.4. One final example is given below.

Consider (1) with nonzero damping, but forcing which depends only on the velocity:

$$\text{(30)}\qquad \ddot{x}(t) + b\dot{x}(t) + ax(t) = f_2(\dot{x}(t - \tau)).$$

Proceeding as before, we can show that there is a fixed point and find the characteristic equation of the linearization about this fixed point:

$$\text{(31)}\qquad \lambda^2 + b\lambda + a = g\lambda e^{-\lambda\tau}, \quad \text{where} \quad g = f_2'(0).$$

As this characteristic equation has been studied in detail by several authors, e.g., [**3, 13, 18**], we only repeat here what is necessary for our analysis. Putting $\lambda = i\omega$ in (31), we find possible Hopf bifurcations will occur when

$$\text{(32a)}\qquad a - \omega^2 = g\omega \sin \omega\tau$$

$$\text{(32b)}\qquad b\omega = g\omega \cos \omega\tau.$$

If $a > 0$ and $g^2 > b^2 - 2a > 0$, there exist two families of surfaces (or curves if we think of the physical parameters a and b as fixed) satisfying these equations

(33a)

$$\omega = \sqrt{a + \frac{g^2 - b^2}{2} + \sqrt{\frac{(g^2 - b^2)^2}{4} + a(g^2 - b^2)}}, \qquad \tau = \frac{1}{\omega} \arccos\left(\frac{b}{g}\right),$$

(33b)

$$\omega = \sqrt{a + \frac{g^2 - b^2}{2} - \sqrt{\frac{(g^2 - b^2)^2}{4} + a(g^2 - b^2)}}, \qquad \tau = \frac{1}{\omega} \arccos\left(\frac{b}{g}\right).$$

See Cooke and Grossman [**13**] for details of the derivation. This leads to the following

Theorem 3.5. *For $a > 0$, (31) possesses double Hopf points with frequencies having all possible ratios $\omega_1 : \omega_2 = m : n$, $m < n \in \mathbf{Z}^+$.*

Proof. Consideration of $n\omega_1 = m\omega_2$ where ω_1, ω_2 are defined by (33a) and (33b), respectively, yields

$$g^2 - b^2 = \frac{(n-m)^2}{mn} a.$$

Substituting this relation back into the equations for ω_1 and ω_2 in turn yields

(34) $$\omega_1 = \sqrt{\frac{m}{n} a}, \qquad \omega_2 = \sqrt{\frac{n}{m} a}.$$

Now at the double Hopf point (32b) must be satisfied for *both* ω_1 and ω_2; thus, $\cos(\omega_1\tau) = \cos(\omega_2\tau)$ and we have

$$\omega_2\tau = \pm\omega_1\tau + 2l\pi$$

for some $l \in \mathbf{Z}$. This in turn implies that $\sin\omega_2\tau = \pm\sin\omega_1\tau$. However, consideration of (32a) and (34) shows that $\sin\omega_2\tau$ and $\sin\omega_1\tau$ must have opposite signs, eliminating the + sign in the expression above. Thus, we find

$$\tau = \frac{2l\pi}{\omega_1 + \omega_2} = \frac{\sqrt{mn}}{m+n} \frac{2l\pi}{\sqrt{a}},$$

TABLE 1.

$\omega_1 : \omega_2$	Parameter Values		$\omega_1 : \omega_2$	Parameter Values	
	b, d, g	ω_1, τ		b, d, g	ω_1, τ
1:2	$0,\ -\frac{3}{5},\ 0$ $\frac{1}{\sqrt{6}},\ 0,\ -\sqrt{\frac{2}{3}}$	$\sqrt{0.4},\ \frac{\pi}{\sqrt{0.4}}$ $\frac{1}{\sqrt{2}},\ \frac{4}{3}\sqrt{2}\pi$			
1:3	$0,\ 0,\ \frac{2}{\sqrt{3}}$ $-\frac{4}{5},\ 1.03,\ 1.03$	$\frac{1}{\sqrt{3}},\ \frac{\sqrt{3}}{2}\pi$ $\sqrt{0.2},\ \frac{2\pi}{3\sqrt{0.2}}$	2:3	$\frac{1}{\sqrt{6}}\cot\frac{2}{5}\pi,\ 0,\ \frac{1}{\sqrt{6}}\csc\frac{2}{5}\pi$ $0.297,\ 0.471,\ 0$	$\frac{2}{\sqrt{3}},\ \frac{6}{5}\sqrt{6}\pi$ $0.767,\ 8.19$
1:4	$0,\ -0.882,\ 0$ $0,\ 0.311,\ 1.43$	$\sqrt{0.118},\ \frac{\pi}{\sqrt{0.118}}$ $0.487,\ 2.36$			
1:5	$0,\ 0.507,\ 1.584$ $\frac{4}{\sqrt{15}},\ 0,\ \frac{8}{\sqrt{15}}$	$0.415,\ 2.20$ $\frac{1}{\sqrt{5}},\ \frac{\sqrt{5}}{3}\pi$	2:5	$0,\ -0.225,\ 0.909$ $-0.468,\ 0.789,\ 0$	$0.624,\ 3.12$ $0.496,\ 12.7$

for some $l \in \mathbf{Z}$. It is a simple matter to derive from (32b)–(32a) expressions giving b and g at the double Hopf point

$$b = (n-m)\sqrt{\frac{a}{mn}}\cot\left(2l\pi\frac{m}{m+n}\right),$$
$$g = (n-m)\sqrt{\frac{a}{mn}}\csc\left(2l\pi\frac{m}{m+n}\right). \qquad \square$$

A similar theorem may be stated for $a < 0$. Note that, although all resonances are possible, unlike the zero damping cases, not all double Hopf points are resonant.

We conclude this section with the results of a numerical investigation to find some low order resonant double Hopf points of (3), as shown in Table 1. Note that some of these points could be predicted by Theorems 3.1, 3.2 and 3.5.

4. Discussion. We have shown that resonant double Hopf bifurcation points with frequencies in all ratios $m : n$, $m < n \in \mathbf{Z}$ occur in the damped harmonic oscillator with delayed forcing. That such points may be important sources of interesting behavior is clear from studies of the normal forms for certain "strong" ($\omega_2/\omega_1 = 1, 2, 3$) resonances. Studies of both the 1:2 and 1:3 resonances [**19, 20, 22, 26**] have shown the presence of three or more families of periodic orbits. Further, near 1:2 resonant double Hopf points, other dynamics such as period doubling bifurcations [**22**], phase locking and quasiperiodic dynamics [**20**] have been shown to occur. The behavior of the "weak" resonances has been shown to be similar to that of the nonresonant case by Iooss [**19**] and Schmidt [**26**], i.e., there exist two families of periodic orbits and quasiperiodic solutions. See [**28**] and [**16**] for thorough studies of the nonresonant case.

These results for the dynamics of the normal form can be applied to delay differential equations by using the center manifold projection [**15, 17**]. This has been done for the 1:2 resonance in (4) by Campbell and LeBlanc [**10**].

Although many of the resonances occur only if one or more of the parameters is zero, it is important to note that the results concerning the solutions of a normal form are valid for parameters in some neighborhood of the double Hopf point. Thus it is conceivable that systems with small nonzero parameters would be "close" to a resonant double Hopf case and thus exhibit some of the same dynamics. Indeed, in a model for the pupil light reflex Campbell et al. [**8**] observed period doubling bifurcations near a point of nonresonant double Hopf bifurcation. The frequencies in this case were close to 1:2 resonance as the parameters were close to a 1:2 double Hopf point of (5), $a = 1$, $b = .07$, $d = -0.63$.

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ENDNOTES

1. The exceptional points are the points of resonant double Hopf bifurcation discussed in Section 3.

REFERENCES

1. U. an der Heiden, *Delays in physiological systems*, J. Math. Biol. **8** (1979), 345–364.

2. ———, *Periodic solutions of a nonlinear second order differential equation with delay*, J. Math. Anal. Appl. **70** (1979), 599–609.

3. S. Bhatt and C. Hsu, *Stability criteria for second order dynamical systems with time lag*, J. Appl. Mech. **33** (1966), 113–118.

4. E. Boe and H.-C. Chang, *Dynamics of delayed systems under feedback control*, Chemical Engrg. Sci. **44** (1989), 1281–1294.

5. F. Boese, *Stability conditions for the general linear difference-differential equation with constant coefficients and one constant delay*, J. Math. Anal. Appl. **140** (1989), 136–176.

6. ———, *Stability criteria for second-order dynamical systems involving several time delays*, SIAM J. Math. Anal. **26** (1995), 1306–1330.

7. F. Boese and P. van den Driessche, *Stability with respect to the delay in a class of differential-delay equations*, Canad. Appl. Math. Quart. **2** (1994), 151–175.

8. S. Campbell, J. Bélair, T. Ohira and J. Milton, *Complex dynamics and multistability in a damped harmonic oscillator with delayed negative feedback*, Chaos **5** (1995), 640–645.

9. ———, *Limit cycles, tori, and complex dynamics in a second-order differential equations with delayed negative feedback*, J. Dynamics and Differential Equations **7** (1995), 213–236.

10. S. Campbell and V.G. LeBlanc, *Resonant Hopf-Hopf interactions in delay differential equations*, J. Dynamics and Differential Equations **10** (1998), 327–346.

11. J. Chuma and P. van den Driessche, *A general second-order transcendental equation*, Appl. Math. Notes **5** (1980), 85–96.

12. L. Chung, A. Reinhorn and T. Soong, *Experiments on active control of seismic structures*, J. Engrg. Mechanics **114** (1988), 241–235.

13. K. Cooke and Z. Grossman, *Discrete delay, distributed delay and stability switches*, J. Math. Anal. Appl. **86** (1982), 592–627.

14. R. Datko, *A procedure for the determination of the exponential stability of certain differential-difference equations*, Quart. Appl. Math. **36** (1978), 279–292.

15. O. Diekmann, S. van Gils, S.V. Lunel and H.-O. Walther, *Delay equations*, Springer-Verlag, New York, 1995.

16. J. Guckenheimer and P. Holmes, *Nonlinear oscillations, dynamical systems and bifurcations of vector fields*, Springer-Verlag, New York, 1983.

17. J. Hale and S.V. Lunel, *Introduction to functional differential equations*, Springer-Verlag, New York, 1993.

18. C. Hsu and S. Bhatt, *Stability charts for second-order dynamical systems with time lag*, J. Applied Mech. **33** (1966), 119–124.

19. G. Iooss, *Direct bifurcation of a steady solution of the Navier-Stokes equations into an invariant torus*, in *Turbulence and the Navier-Stokes equations*, Springer-Verlag, New York, 1975, 69–84.

20. E. Knobloch and M. Proctor, *The double Hopf bifurcation with* 2 : 1 *resonance*, Proc. Roy. Soc. London **415** (1988), 61–90.

21. V. Kolmanovskii and V. Nosov, *Stability of functional differential equations*, in *Mathematics in Science and Engineering*, Academic Press, 1986.

22. V. LeBlanc and W. Langford, *On the* 1 : 2 *resonant Hopf bifurcation*, Arch. Rational Mech. Anal. **136** (1996), 305–354.

23. N. MacDonald, *Biological delay systems: Linear stability theory*, Cambridge University Press, Cambridge, 1989.

24. J. Milton and A. Longtin, *Evaluation of pupil constriction and dilation from cycling measurements*, Vision Research **30** (1990), 515–525.

25. N. Minorsky, *Self-excited oscillations in dynamical systems possessing retarded actions*, J. Appl. Mech. **9** (1942), A65–A71.

26. D.S. Schmidt, *Hopf's bifurcation theorem and the center theorem of Liapunov with resonance cases*, J. Math. Anal. Appl. **63** (1978), 354–370.

27. G. Stépán, *Retarded dynamical systems*, Pitman Res. Notes Math. Ser. **210**, Longman Group, Essex, 1989.

28. F. Takens, *Singularities of vector fields*, Publ. Math. de l'IHES **43** (1974), 47–100.

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