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TIME DELAY AND FEEDBACK CONTROL OF AN INVERTED PENDULUM WITH STICK SLIP FRICTION

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ABSTRACT

We consider an experimental system consisting of a pendulum, which is free to rotate 360 degrees, attached to a cart which can move in one dimension. There is stick slip friction between the cart and the track on which it moves. Using two different models for this friction we design feedback controllers to stabilize the pendulum in the upright position. We show that controllers based on either friction model give better performance than one based on a simple viscous friction model. We then study the effect of time delay in this controller, by calculating the critical time delay where the system loses stability and comparing the calculated value with experimental data. Both models lead to controllers with similar robustness with respect to delay. Using numerical simulations, we show that the effective critical time delay of the experiment is much less than the calculated theoretical value because the basin of attraction of the stable equilibrium point is very small.

1 Introduction

We study the experimental system depicted schematically in Figure 1. In this system, a pendulum is attached to the side of a cart by means of a pivot which allows the pendulum to swing in the xy-plane. A force $F(t)$ is applied to the cart in the x direction, with the purpose of keeping the pendulum balanced upright.

The equations of motion of the cart and pendulum from Figure 1

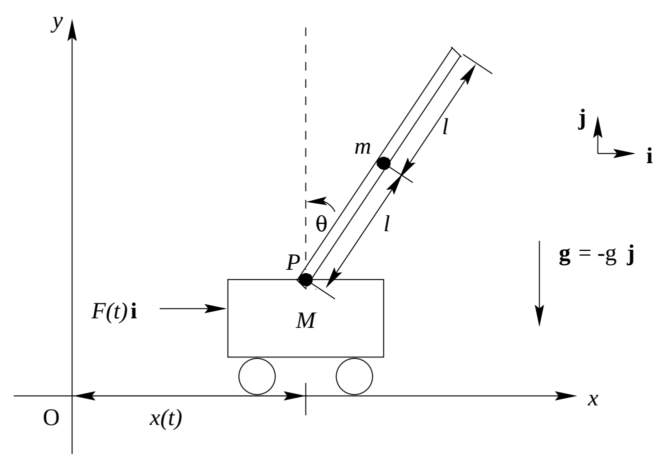

Figure 1. Inverted Pendulum System

can be found using Hamilton's Principle, e.g. [1]. They are:

$$\begin{aligned}(M+m)\ddot{x}(t) - ml\sin\theta(t)\dot{\theta}^2(t) + ml\cos\theta(t)\ddot{\theta}(t) &= F(t) + F_{\text{fric}}\\ ml\cos\theta(t)\ddot{x}(t) - mgl\sin\theta(t) + \tfrac{4}{3}ml^2\ddot{\theta}(t) &= 0\end{aligned} \quad (1)$$

where x is the position of the cart, θ is the pendulum angle, measured in degrees away from the upright position, F is the force applied to the cart and F_{fric} is the force of friction. The definitions of the parameters are given in Table 1.

In our experimental system, supplied by Quanser Limited, the

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Table 1. Parameter Values

Param.	Description	Value
M	mass of the cart	0.8150 Kg
m	mass of the pendulum	0.210 Kg
l	pivot to pendulum c.o.m. distance	0.3050 m
g	gravity constant	9.8 m/s
α	voltage/force conversion	1.7189
β	electrical resistance/force conversion	7.682

applied force is due to a motor in the cart and is given by

$$F(t) = \alpha V(t) - \beta \dot{x}(t), \tag{2}$$

where V is the voltage supplied to the engine, and the second term represents electrical resistance in the cart motor. The values of the constants α and β for the motor used in our experimental apparatus are given in Table 1.

If there is no applied force, $F(t) = 0$, then the system has an (orbitally) asymptotically stable steady state with the pendulum hanging straight down, and the cart in any position on the track. A classical control problem is to design a feedback law which will stabilize the pendulum in the upright position. A standard approach to this problem is to design a law of the form

$$V(t) = K \cdot [x, \theta, \dot{x}, \dot{\theta}]^T, \tag{3}$$

where the feedback gain, $K = [k_1, k_2, k_3, k_4]$, is chosen so that the linearization of the system about the equilibrium point has all eigenvalues with negative real parts. To determine K for our system, we use an optimal linear quadratic controller, e.g. [2], with weights $Q = \mathrm{diag}(5000, 3000, 20, 20)$ and $r = 1$.

The control of the inverted pendulum is a well-studied problem which has application to both biological and mechanical balancing tasks. As such there have been many papers written on the subject. Here we briefly review the ones most directly related to our work, i.e., those that involve time-delayed feedback. Other references can be found within the papers cited. In contrast with model (1) and choice of feedback (3), all other papers we are aware of eliminate the cart dynamics from the problem by neglecting the friction (and resistance in the cart motor) and using feedback which only depends on θ and $\dot{\theta}$. Stability analysis of the resulting second-order equation can be found in the work of Stépán and collaborators [3, 4]. Stépán and Kollár [5] formulate conditions on the delay such that stabilizing controllers can and cannot exist. Atay [6] does a similar analysis employing position feedback only, but with multiple delays. Sieber and Krauskopf [7] show there is a codimension three bifurcation point in the model and use centre manifold and normal form analysis to show that this point acts as an organizing centre for the dynamics of the system. The emphasis of these studies is on theoretical analysis of the model for arbitrary parameters whereas we focus on understanding the model with the parameters dictated by our experimental setup. Finally, we note the work of Cabrera and Milton [8, 9] who study, theoretically and experimentally, an inverted pendulum where the control is provided by a person (the "stick balancing problem"). The emphasis of this work is on the interplay between the time delay and the noise in the system.

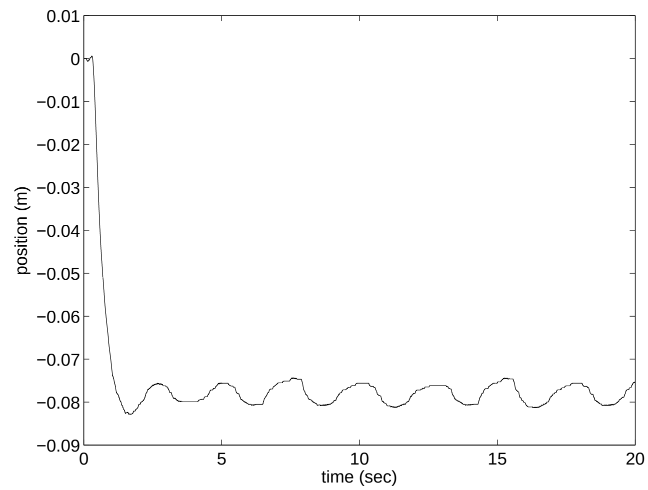

Figure 2. Experiment with controller based on a simple viscous friction model.

In our previous work [10] we studied system (1), when only viscous friction is included in the model, i.e., $F_{\mathrm{fric}} = -\varepsilon \dot{x}$. Using this model, we designed a feedback controller for the system in the manner described above. However, when the controller was implemented in the experimental system, small amplitude oscillations resulted, an example is shown in Figure 2. These oscillation do not appear in simulations of the model. We also studied the effect of time delay on this feedback, showing that, for sufficiently large delay oscillations will occur due to a delay-induced Hopf bifurcation.

In this paper, we focus on including a stick slip friction in the model (1). In [11] a detailed study of two stick slip friction models for system (1) is carried out, including estimation of the relevant parameters. The important aspects of these models are summarized in Section 2. We then show that controllers designed using the two friction models both stabilize the pendulum in the upright position when implemented in the experimental system. In

Section 3, we consider the effect of time delays on these two controllers. In Section 4 we summarize our results and draw some conclusions.

2 Friction Models and Controller Design

An accurate model of the friction between the cart and the track in our experimental system must include static and Coulomb (sliding) friction as well as viscous friction. Static friction is the friction that must be overcome to start an object moving; it is only present when the object is not moving. Coulomb and viscous friction are both present only when the object is moving. In the following subsections, we consider two models which take into account these effects and allow for smooth transitions between the resting and moving states. We then show that the controllers designed using the two friction models both stabilize the pendulum in the upright position.

2.1 Exponential Friction Model

This model is due to Hauschild [12] and combines standard models for static, viscous and coulomb friction with an exponential term to smooth the transition between the resting and moving states. Recall the standard model for static friction

$$F_{\text{static}} = \begin{cases} -F_{\text{applied}} & \text{if } |F_{\text{applied}}| < \mu_s F_N \\ -\mu_s F_N \operatorname{sgn}(F_{\text{applied}}) & \text{if } |F_{\text{applied}}| \geq \mu_s F_N \end{cases}, \tag{4}$$

where μ_s is the coefficient of static friction and F_N is the magnitude of the normal force. In our model, the object is moving horizontally, so $F_N = (M+m)g$. The exponential friction model of Hauschild [12] is given by

$$F_{\text{fric}} = \begin{cases} F_{\text{static}} & \text{if } \dot{x} = 0 \\ -(\mu_c + (\mu_s - \mu_c)e^{-(\frac{\dot{x}}{v_s})^\gamma})F_N \operatorname{sgn}(\dot{x}) - \varepsilon\dot{x} & \text{if } \dot{x} \neq 0 \end{cases} \tag{5}$$

where and μ_c, ε are the coefficients Coulomb (sliding) and viscous friction, respectively, v_s is called the Stribeck velocity and γ the form factor.

The appropriate parameters for our system were estimated by performing various tests on the experimental apparatus. Specifically, μ_s was estimated by slowing ramping up the applied voltage in the motor (and hence the applied force) and measuring the minimum applied force needed to start the cart moving. To estimate μ_c and ε we applied two different constant forces and measured the associated steady state constant velocity. For ν_s we used the value given in [13], while γ was chosen so to give the best fit of simulations to experimental data showing the transition from $\dot{x} = 0$ to $\dot{x} \neq 0$. Details of these estimations can be found in [11]. The resulting values are given in Table 2. With these parameter values, the dependence of the friction force on $\dot{x}$ is illustrated in Figure 3.

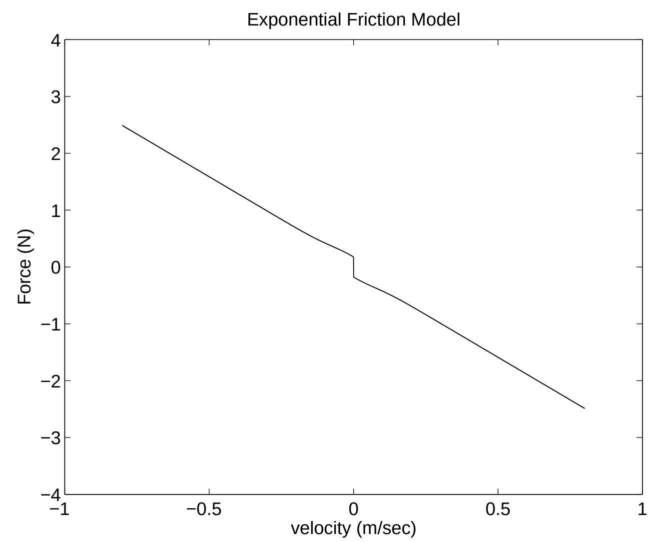

Figure 3. Exponential friction model.

Table 2. Parameters for Exponential Friction Model

Param.	Description	Value
μ_s	coefficient of static friction	0.08610
μ_c	coefficient of Coulomb friction	0.04287
ε	coefficient of viscous friction	3
γ	form factor	2
v_s	Stribeck velocity	0.105

To design a feedback control for a system as described in the introduction, one needs to linearize the system about the trivial solution. However, our friction model (5) is not differentiable. Thus to calculate the linearization of our system, we approximated the nondifferentiable functions in (5) by smooth functions. Specifically, rewriting (5) as

$$F_{\text{fric}} = F_{\text{static}}(1 - \operatorname{sgn}^2(\dot{x})) - (\mu_c + (\mu_s - \mu_c)e^{-(\frac{\dot{x}}{v_s})^\gamma})F_N \operatorname{sgn}(\dot{x}) - \varepsilon\dot{x}, \tag{6}$$

a smooth approximation may be made by replacing $\operatorname{sgn}(\dot{x})$ by

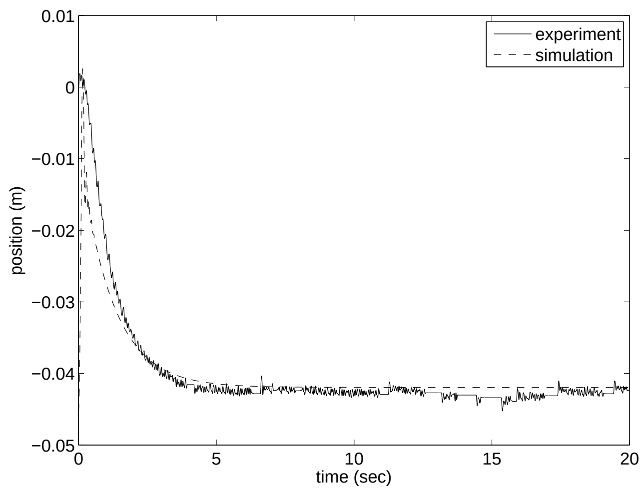

Figure 4. Implementation of the controller based on the exponential friction model in the experiment (solid line) and in a numerical simulation of the system with initial condition $(x,\theta,\dot{x},\dot{\theta}) = (0, 0.06\,\text{rad}, 0, 0)$ (dashed line).

$\tanh(\delta\dot{x})$ where $\delta > 0$ is large. Using $\delta = 100$ leads to the feedback gain [11]

$$K = [70.7107, 304.9273, 143.3152, 61.0833]. \tag{7}$$

We implemented this controller in our experimental system and found that it did stabilize the system. An example experimental run is shown, together with a numerical simulation, in Figure 4. The performance of the controlled system is improved considerably over that obtained with the controller designed used the simple viscous friction model. Note that there remains some noise in the system. This can been shown to be due to quantization error in the angle sensor [11].

2.2 Dynamic Friction Model

This model is due to Canudas de Wit and Lipschinsky [13]. It attempts to represent friction more physically by thinking of two surfaces making contact through elastic bristles. The main idea is that when a force is applied, the bristles will deflect like springs, which gives rise to the friction force. The friction force generated by the bristles is

$$F_{\text{fric}} = -\sigma_0 z - \sigma_1 \frac{dz}{dt} - \varepsilon\dot{x},$$

where z is the average deflection of the bristles, and σ_0, σ_1 are the stiffness and damping coefficient. Our expression appears to have the opposite sign for the friction force from that presented in [13] as they incorporate the sign of the force into their equation of motion (see equation (7) of [13]), whereas we incorporate it into the expression for the force. In either case, the variation of z is modelled via:

$$\frac{dz}{dt} = \dot{x} - \sigma_0 \frac{|\dot{x}|}{g(\dot{x})} z \tag{8}$$

where

$$g(\dot{x}) = (\mu_c + (\mu_s - \mu_c) e^{-(\frac{\dot{x}}{v_S})^{\gamma}}) F_N.$$

This leads to a dynamic friction force

$$F_{\text{fric}} = -(\sigma_1 + \varepsilon)\dot{x} - \sigma_0 z \left(1 - \sigma_1 \frac{|\dot{x}|}{g(\dot{x})}\right), \tag{9}$$

where z satisfies (8). Note that when this model is used, the dimension of the system will be increased by one due to the introduction of the variable z. The standard friction parameters are as in the exponential model and are given in Table 2. The remaining parameters, as determined in [11], are given in Table 3. These were chosen so that simulations of the transition from $\dot{x} = 0$ to $\dot{x} \neq 0$ gave the best fit of the experimental data.

Table 3. Parameters for Dynamic Friction Model

Parameter	Description	Value
σ_0	stiffness	121
σ_1	damping coefficient	70

When $\dot{x}$ is constant, the bristle state z, and hence F_{fric}, approach constant values:

$$\begin{aligned} z &= \frac{1}{\sigma_0} g(\dot{x})\,\text{sgn}(\dot{x}), \\ F_{\text{fric}} &= -g(\dot{x})\,\text{sgn}(\dot{x}) - \varepsilon\dot{x} \\ &= -(\mu_c + (\mu_s - \mu_c) e^{-(\frac{\dot{x}}{v_S})^{\gamma}}) F_N\,\text{sgn}(\dot{x}) - \varepsilon\dot{x}. \end{aligned}$$

Thus, the steady state friction force is the same as that given by the exponential friction model (5) and illustrated in Figure 3. The dynamic nature of this friction model can be seen when $\dot{x}$ is a periodic function of time. This is illustrated in Figure 5.

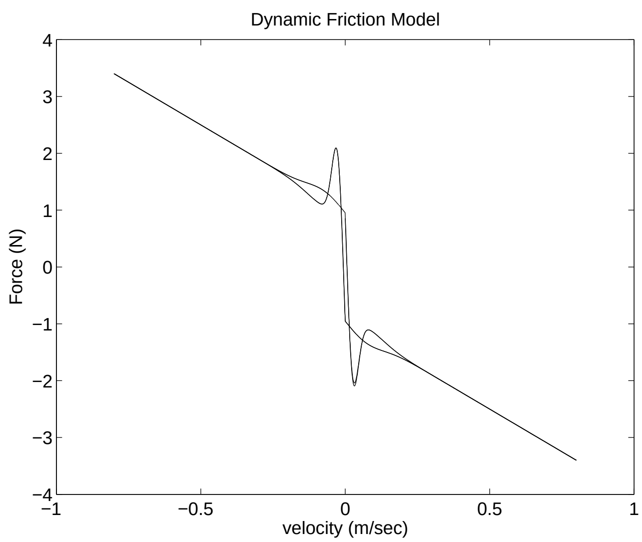

Figure 5. Dynamic friction model when $\dot{x}(t) = 0.8\sin(0.5t)$.

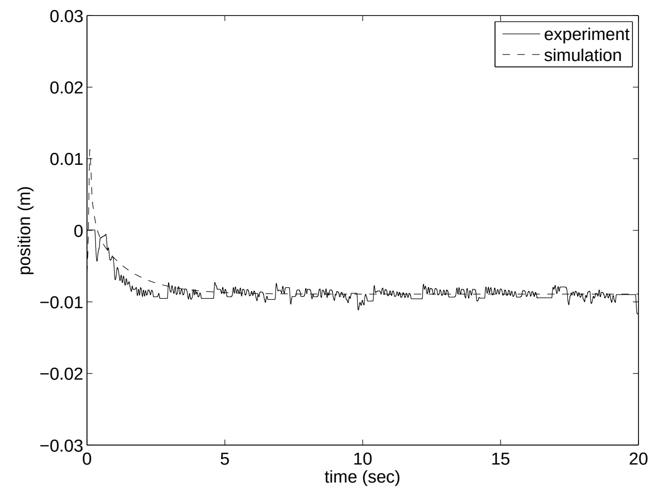

Figure 6. Implementation of the controller based on the dynamic friction model in the experiment (solid line) and in a numerical simulation of the system with initial condition $(x, \theta, \dot{x}, \dot{\theta}) = (0, 0.02 \text{ rad}, 0, 0)$ (dashed line).

Once again, to design a feedback control for our system with the friction model (9) requires that we make a smooth approximation to the nondifferentiable absolute value function. However, it can be shown that any smooth approximation of the absolute value function will have zero derivative at zero, so the approximation plays no role in the linearization or the controller design.

A further complication in this case is that the linearized system is not stabilizable, due to the introduction of the new state z. This may be dealt with by an appropriate transformation of the system [11]. Finally, one obtains the feedback gain

$$K = [85.0828, 290.4957, 136.0180, 58.1005]. \tag{10}$$

We implemented this controller in our experimental system and found that it did stabilize the system. An example experimental run is shown, together with a numerical simulation, in Figure 6. The performance of the controlled system is improved considerably over that obtained with the controller designed used the simple viscous friction model (Figure 2). Note that there remains some noise in the system. This can been shown to be due to quantization error in the angle sensor [11].

3 Effect of Time Delay

As illustrated in the last section, the controllers designed with either friction model achieve stability of the upright equilibrium point. We would now like to compare their robustness with respect to time delay in the feedback loop.

We assume that there is a time delay, $\tau > 0$, between when the variables used in the feedback law (3) are measured and when the voltage $V(t)$ is applied. Adding this delay to the model, the force applied by the cart ($F(t)$ in (1)) now becomes

$$F(t) = d_1 x(t-\tau) + d_2\theta(t-\tau) + d_3\dot{x}(t-\tau) + d_4\dot{\theta}(t-\tau) - \beta\dot{x}(t), \tag{11}$$

where $d_i = \alpha k_i$.

We wish to find the smallest time $\tau_c > 0$ that makes the upright equilibrium point of this system unstable. Rewriting the model as a first order system and linearizing about the upright equilibrium point yields the system [11]

$$\dot{\mathbf{y}} = A\mathbf{y}(t) + E\mathbf{y}(t-\tau) \tag{12}$$

where $\mathbf{y}$ and A and E depend on which friction model is used. We find the critical time delay, τ_c, by assuming that equation (12) has a solution of the form $\mathbf{y}(t) = \mathbf{v}e^{\lambda t}$, substituting $\mathbf{y}(t)$ back into the equation and solving for the smallest value of τ such that the system has an eigenvalue λ with $Re(\lambda) = 0$.

3.1 Time Delay with the Exponential Model

For the exponential model, $\mathbf{y} = [x, \theta, \dot{x}, \dot{\theta}]$, and A and E in (12) are:

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{-3mg}{4M+m} & \frac{-(\beta+\varepsilon+\delta\mu_s(M+m)g)}{4M+m} & 0 \\ 0 & \frac{3(M+m)g}{(4M+m)l} & \frac{3(\beta+\varepsilon+\delta\mu_s(M+m)g)}{(4M+m)l} & 0 \end{pmatrix}, \tag{13}$$

$$E = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{4d_1}{4M+m} & \frac{4d_2}{4M+m} & \frac{4d_3}{4M+m} & \frac{4d_4}{4M+m} \\ \frac{-3d_1}{(4M+m)l} & \frac{-3d_2}{4M+m} & \frac{-3d_3}{4M+m} & \frac{-3d_4}{4M+m} \end{pmatrix}. \tag{14}$$

Using the parameters in Tables 1–3 and $\delta = 100$, we find that the critical time delay is $\tau_c \approx 0.1069$ seconds.

To study this experimentally, we implemented the feedback control based on the exponential friction model, i.e., (11) with gain given by (7), in our experimental system. The actual time delay of the experimental system is insignificant, however, we can artificially vary it using the computer system which implements the feedback control. We did this and observed that the upright equilibrium point was asymptotically stable until $\tau = 0.01$. For $0.01 < \tau < 0.025$ the experimental system exhibited oscillations about the upright equilibrium point, with amplitude that increased with τ. For $\tau \geq 0.025$, the pendulum fell down. Our initial conditions for the experiment are set manually, by holding the pendulum “close” to the upright position. Thus they are given by

$$x(t) = 0, \theta(t) = \theta_0, \dot{x}(t) = 0, \dot{\theta}(t) = 0, \ -\tau \leq t \leq 0. \tag{15}$$

The range of initial angles that we used was approximately $1 \leq \theta_0 \leq 5$ degrees.

To understand the discrepancy between our theoretical prediction and experimental results, we performed numerical simulations of the full model (1) with the delayed force (11) and exponential friction model (5). The simulations were performed in the Matlab Simulink tool box using the solver `ode23tb`, which is a variable step size solver for stiff differential equations, using initial conditions (15) with $\theta_0 \leq 10$ deg. The results are summarized in the top graph of Figure 7 (dashed lines) with a zoom in shown in the bottom graph. For all initial conditions tested, we found that the equilibrium point was stable for $\tau < 0.005$. With $0.06 < \theta_0 < 6$ degrees, the system exhibited oscillations for $0.005 \leq \tau < 0.041$ and the pendulum fell over for $\tau \geq 0.041$. For $\theta_0 \geq 6$ degrees, the delay value at which the transition from oscillations to instability occurred decreased with increasing θ_0. The

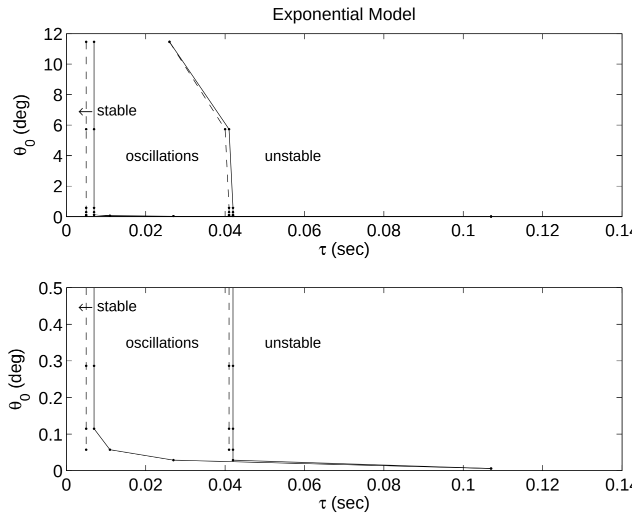

Figure 7. Results of numerical simulations of (1) with the exponential friction model and corresponding feedback gain (7). Dashed lines correspond to using the friction model (5). Solid lines correspond to replacing $\text{sgn}(\dot{x})$ with $\tanh(100\dot{x})$. Initial conditions used are as in (15) for various values θ_0 of the time delay, τ. The bottom graph shows a zoom in of the top graph.

numerical integration routine had difficulty dealing with initial conditions smaller than 0.06 degree due to the signum function. Clearly, the simulation results match the experimental observations better than the theoretical prediction based on the linearization of the model with the smooth approximation to the signum function. However, we still have no explanation for the discrepancy between the destabilizing delay predicted by linearization and that observed in the experiment.

To investigate this discrepancy further, we performed simulations using the smooth approximation $\tanh(100\dot{x})$ for $\text{sgn}(\dot{x})$ in (5). The results of these simulations are summarized in the top graph of Figure 7 (solid lines) with a zoom in shown in the bottom graph. For initial conditions with $\theta_0 > 0.1$ degree, these simulations give very similar results to the model with the signum function. For smaller initial conditions, however, the two transition values of the delay increase and finally coalesce. For small enough initial conditions the delay where stability is lost matches the theoretical prediction from the linearization, as it should.

These results lead to two possible explanations for the discrepancy between the theoretical prediction and the experimental observation. The simplest is that while the smooth approximation is good enough to design a controller which will stabilize the system, it is not good enough to use for quantitative predictions of delay induced instability. Alternatively, we note that simulations with the smooth approximation show that when $0.007 < \tau \leq \tau_c$

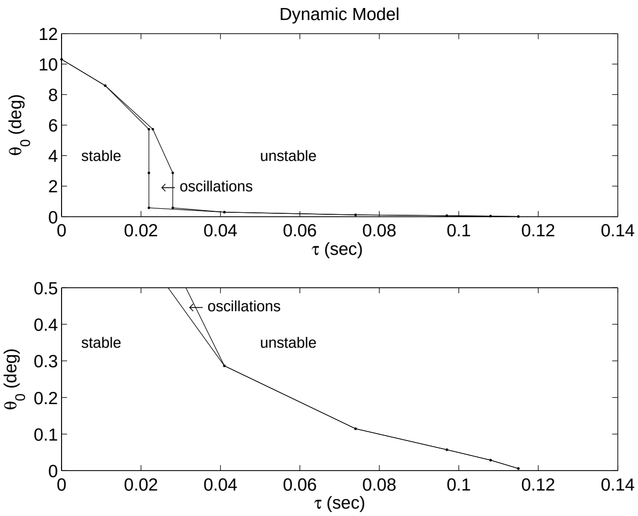

Figure 8. Results of numerical simulations of (1) with the dynamic friction model and corresponding feedback gain. Initial conditions used are as in (15) for various values θ_0 of the time delay, τ. The bottom graph shows a zoom in of the top graph.

only for very small initial conditions will the system return to the stable equilibrium point. This situation, where the equilibrium point is asymptotically stable, but only for initial conditions sufficiently close to the point is called *local* asymptotic stability. Of course, local stability is all that is guaranteed by the linear stability analysis. We cannot verify this idea experimentally, since it is impossible to get the pendulum sufficiently close to the equilibrium point. However, we can conclude from both sets of simulations that the *effective* critical delay (i.e. the delay where stability will be lost with experimentally achievable initial conditions) is considerably less than that predicted by the linear stability analysis.

3.2 Time Delay with the Dynamic Model

For the dynamic friction model, $\mathbf{y} = [x, \theta, \dot{x}, \dot{\theta}, z]$, and A and E in (12) are:

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & \frac{-3mg}{4M+m} & \frac{-4(b+\sigma_1+\sigma_2)}{4M+m} & 0 & \frac{-4\sigma_0}{4M+m} \\ 0 & \frac{3(M+m)g}{(4M+m)l} & \frac{3(b+\sigma_1+\sigma_2)}{(4M+m)l} & 0 & \frac{3\sigma_0}{(4M+m)l} \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \tag{16}$$

$$E = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ \frac{4d_1}{4M+m} & \frac{4d_2}{4M+m} & \frac{4d_3}{4M+m} & \frac{4d_4}{4M+m} & 0 \\ \frac{-3d_1}{(4M+m)l} & \frac{-3d_2}{4M+m} & \frac{-3d_3}{4M+m} & \frac{-3d_4}{4M+m} & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{17}$$

Using the parameters in Tables 1–3, we find that the critical time delay is $\tau_c \approx 0.1169$ seconds.

To study this experimentally, we implemented the feedback control based on the dynamic friction model, i.e., (11) with gain given by (10), in our experimental system and varied the time delay as described above. We observed that the upright equilibrium point was asymptotically stable until $\tau = 0.009$. For $0.009 < \tau < 0.023$ the experimental system exhibited oscillations about the upright equilibrium point, with amplitude that increases with τ. For $\tau > 0.023$, the pendulum fell down.

To understand the discrepancy between our theoretical prediction and experimental results, we performed numerical simulations of the full model (1) with the delayed force (11) and the dynamic friction model (9). The simulations were performed using the same package as for the exponential friction model. The results of these simulations are summarized in top graph of Figure 8 with a zoom in shown in the bottom graph. We find a similar situation to that with the numerical simulations of the exponential model with the smooth approximation to the signum function. For initial conditions with $1 < \theta_0 < 5$ degrees, the *effective* critical delay is 0.022 and for larger initial conditions, the effective critical delay is even smaller. The simulations also show that, for $1 < \theta_0 < 5$ degree, the system exhibits oscillations for $0.022 \leq \tau < 0.028$ and the pendulum falls down for $\tau > 0.028$. These predictions are in reasonable agreement with the experimental observations. The prediction of the destabilizing delay is not as good as that using the exponential friction model, however the prediction of the delay when the pendulum falls over is better. The simulations for initial conditions with $\theta_0 < 1$ show that the effective critical delay increases as θ_0 decreases, limiting close to the theoretically predicted critical delay as θ_0 goes to zero. This gives an explanation for the discrepancy between the experimental observations and the theoretical prediction of the critical delay. Namely, the equilibrium point is only locally asymptotically stable, and for large delays can only be observed for initial angles smaller than can be achieved in the experiment.

4 Conclusions

We studied the feedback control of an experimental inverted pendulum system which has stick slip friction. We showed that controllers designed using either the exponential friction model of [12] or the dynamic friction model of [13] achieve stability of the pendulum in the upright position. We compared the robustness to time delay of the two controllers. Linear stability analysis predicts that the controller based on the dynamic model is slightly better than that using the exponential model (critical delay of 0.1169 sec vs 0.1069 sec). Implementing the controllers based on the two friction models in our experimental system we found that the one based on the exponential model was marginally better than the one based on the dynamic model (critical delay of 0.01 sec vs 0.009 sec).

The large discrepancy between the critical delays predicted by the linear stability analysis and those observed in experiment was explained by numerical simulations of the full nonlinear model. These simulations indicate that the equilibrium point is asymptotically stable for $\tau < \tau_c$, where τ_c is the critical delay value predicted by the linear stability analysis. However, for a range of delay values, $\hat{\tau}_c \leq \tau < \tau_c$, the equilibrium point is only *locally* asymptotically stable and the initial position of the pendulum must be very close to the upright position in order for the pendulum to asymptotically tend to the upright position. In the language of dynamical systems, the *basin of attraction* of the equilibrium point corresponding to the upright position is very small. Since the initial position needed to see the asymptotic stability of the upright position is smaller than what is experimentally obtainable, the *effective* critical delay is $\hat{\tau}_c$. These results highlight the fact that linear stability analysis may only be marginally useful for predicting the behaviour of nonlinear systems.

The main conclusion of this work is that when using a model to design a feedback controller, it is important to include an accurate friction model. For systems with stick slip friction, we found that both the models of [12] and [13] are adequate. Controllers based on either model achieve stability and both have similar robustness to delay. Both models have drawbacks. It is more difficult to calculate the controller for the model of [13]. However, numerical simulations of using the model of [12] are difficult, due to the discontinuities in the model. The value of $\hat{\tau}_c$ predicted by the simulations of the system with the exponential friction model is closer to that observed in the corresponding experiments than when the dynamic model is used. However, the dynamical friction model gave a better prediction of the delay value when the pendulum fell over. Simulations of the nonlinear system with either friction model yield better prediction of the stability than the linear analysis.

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