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STABILITY AND BIFURCATION IN THE HARMONIC OSCILLATOR WITH MULTIPLE, DELAYED FEEDBACK LOOPS.

Sue Ann Campbell[1]

[1]Department of Applied Mathematics
University of Waterloo, Waterloo, Ontario N2L 3G1
and
Centre for Nonlinear Dynamics in Physiology and Medicine
McGill University, Montréal, Québec

Abstract. We analyze the second order differential equation describing a damped harmonic oscillator with nonlinear feedback depending on both the state and the derivative of the state at some time in the past. The characteristic equation for the linear stability of the equilibrium is completely solved, and the stability region is illustrated in a parameter space consisting of the time delay and the strengths of the two feedback loops. The bifurcations which occur when stability is lost are described and the location of Hopf-Hopf and Hopf-steady state bifurcation interactions are given. Numerical simulations reveal the presence of quasiperiodic solutions and multistability near such codimension two bifurcation points.

AMS (MOS) subject classification: 34K20, 70K10, 93D15

1. Introduction

Second-order delay differential equations arise in a variety of mechanical, or neuromechanical systems in which inertia plays an important role [1, 2, 4, 10, 11, 16, 17] (see also [8] and references therein). Many of these systems are regulated by feedback which depends on the state and/or the derivative of the state. In this case the model equations take the form

$$\ddot{u}(t) + b\dot{u}(t) + au(t) = f(u(t-\tau), \dot{u}(t-\tau)), \tag{1}$$

where a, b are positive constants representing physical attributes of the system, τ is the time delay, $u, u(t-\tau)$ are the values of the regulated variable evaluated at, respectively, times t and $t-\tau$, and the function, $f(x, y)$, describes the feedback. The time delay arises because of, for example, finite conduction and production times in physiological systems, and finite measurement and processing times in mechanical systems. By analogy with the mechanical systems we will refer to u and $\dot{u}$ as the position and velocity, respectively.

Much of the work cited above has been done on systems of the form (1) where the feedback depends either on the position or on the velocity of the system. It is our purpose to consider here the case when *both* feedback loops are present in order to study the effect that the interaction of such loops has on the dynamics.

2. Linear Stability Analysis

Before we begin the local (linear) stability analysis of (1), we note that the case where f depends on either u or $\dot{u}$ has been discussed by several investigators [2, 3, 4, 5, 6, 8, 9, 11, 13, 15].

The fixed point(s) of (1), $u(t) = const = u^*$, are found by solving $au^* = f(u^*, 0)$. Assuming that this is satisfied for some u^* and f is sufficiently smooth, one can linearize equation (1) about the fixed point (using $u(t) = u^* + v(t)$) yielding the equation

$$v''(t) + bv'(t) + av(t) = dv(t-\tau) + gv'(t-\tau), \tag{2}$$

where $d = f_{,1}(u^*, 0),\ g = f_{,2}(u^*, 0)$. Physically, d and g represent strengths or gains of the position and velocity feedback loops, respectively. It is well known [12] that the local stability of the fixed point $u(t) = u^*$ of (1) is given by the stability of the trivial fixed point of (2).

To study the stability of this fixed point, we use the usual ansatz ($v(t) = e^{\lambda t}$) which yields the characteristic equation

$$\lambda^2 + b\lambda + a = de^{-\lambda\tau} + g\lambda e^{-\lambda\tau}. \tag{3}$$

From standard results [14, 17] the trivial fixed point of (2) will be stable if all roots, λ, of the characteristic equation have negative real parts. The fixed point may change stability when $Re(\lambda) = 0$ for some λ; this can occur in the following two ways. First, a real eigenvalue passes through zero ($\lambda = 0$). This occurs when $d = a$; note that this can occur only if $d > 0$, i.e. the position feedback loop has *positive* feedback. Second, a pair of complex eigenvalues crosses the imaginary axis ($\lambda = \pm i\omega,\ \omega > 0$). This occurs when the following equations are satisfied

$$\begin{aligned} a - \omega^2 &= d\cos\omega\tau + g\omega\sin\omega\tau \\ b\omega &= g\omega\cos\omega\tau - d\sin\omega\tau. \end{aligned} \tag{4}$$

Squaring and adding these equations yields a quartic for ω:

$$\omega^4 + (b^2 - g^2 - 2a)\omega^2 + a^2 - d^2 = 0, \tag{5}$$

with roots

$$\begin{aligned} \omega_\pm &= \sqrt{a + \frac{1}{2}(g^2 - b^2) \pm \sqrt{(a + \frac{1}{2}(g^2 - b^2))^2 - a^2 + d^2}} \\ &= \sqrt{a + \frac{1}{2}(g^2 - b^2) \pm \sqrt{a(g^2 - b^2) + \frac{1}{4}(g^2 - b^2)^2 + d^2}}. \end{aligned} \tag{6}$$

Since $\omega_\pm$ must be *real* and *positive*, the existence of these roots is as described in Table 1.

Region in parameter space	Roots	Figure
$g \in \mathbb{R}$ and $a^2 < d^2$ or $b^2 - 2a \le g^2$ and $a^2 = d^2$	1	1, 2
$g^2 \le b^2 - 2a$ and $d^2 \le a^2$	0	1(a)
$b^2 - 2a < g^2 < b^2$ and $(b^2 - g^2)(a - (b^2 - g^2)/4) < d^2 < a^2$	2	1(b)
$b^2 - 2a < g^2 < b^2$ and $(0 < d^2 < (b^2 - g^2)(a - (b^2 - g^2)/4))$	0	1(b)
$b^2 \le g^2$ and $d^2 < a^2$	2	2

Table 1: Existence of roots for equation (6)

Now (4) may be rearranged to yield

$$\cos \omega_\pm \tau = \frac{ad + \omega_\pm^2 (bg - d)}{d^2 + g^2 \omega_\pm^2} \tag{7}$$

$$\sin \omega_\pm \tau = \frac{\omega \left[g(a - \omega_\pm^2) - db \right]}{d^2 + g^2 \omega_\pm^2}, \tag{8}$$

which together with (6) and Table 1 define the delay, τ, as a function of the other parameters. Note that there will be two sets of functions corresponding to two possible roots for ω. These are given by

$$\tau = \begin{cases} \tau_{1\pm}^j \stackrel{def}{=} \dfrac{1}{\omega_\pm} \left[2j\pi + \text{Arccos} \left[\dfrac{ad + \omega_\pm^2 (bg - d)}{d^2 + g^2 \omega_\pm^2} \right] \right], & g(a - \omega_\pm^2) - db > 0 \\ \tau_{2\pm}^j \stackrel{def}{=} \dfrac{1}{\omega_\pm} \left[(2j+2)\pi - \text{Arccos} \left[\dfrac{ad + \omega_\pm^2 (bg - d)}{d^2 + g^2 \omega_\pm^2} \right] \right], & g(a - \omega_\pm^2) - db < 0 \end{cases} \tag{9}$$

where $j = 0, 1, \ldots$, and Arccos refers to the branch of the inverse cosine function with range $[0, \pi]$.

For fixed a, b and g, (9) defines curves in d, τ parameter space. These curves are illustrated by the solid lines in Figs. 1 and 2, where it can be seen that they, along with the line $d = a$, divide the d, τ plane in to various regions. By consideration of the sign of the real parts of the roots of the characteristic equation (3) in these regions we will establish some theorems on the stability of the trivial fixed point of (2) and hence on the linearized stability of the fixed point u^* of (1). Our proofs will rely on the following facts which may be found in many books on delay differential equations, for example [14].

F1 The roots of the characteristic equation (3) are continuous functions of the parameters d, g.

F2 The number of roots of the characteristic equation (3) with positive real parts may change with the variation of the parameters d, g only by the passage of a root through the imaginary axis.

To begin, we establish the following lemma on the stability when the feedback depends only on the velocity. A proof of this Lemma can be found in [3], we include a different proof here for completeness.

Lemma 1.1 *For fixed a, b and $d = 0$ the trivial fixed point of (2) is stable in the following regions*

$$
\begin{aligned}
-b < g < b, \quad & 0 < \tau \\
g < -b, \quad & 0 < \tau < \frac{1}{\omega_+} \operatorname{Arccos} \frac{b}{g} \\
& \frac{1}{\omega_-}\left[(2j+2)\pi - \operatorname{Arccos}\frac{b}{g}\right] < \tau < \frac{1}{\omega_+}\left[(2j+2)\pi + \operatorname{Arccos}\frac{b}{g}\right] \\
b < g, \quad & \frac{1}{\omega_-}\left[2j\pi + \operatorname{Arccos}\frac{b}{g}\right] < \tau < \frac{1}{\omega_+}\left[(2j+2)\pi - \operatorname{Arccos}\frac{b}{g}\right]
\end{aligned}
\tag{10}
$$

where $j = 0, 1, \dots$ and

$$\omega_\pm \stackrel{def}{=} \sqrt{a + \frac{g^2 - b^2}{4}} \pm \frac{\sqrt{g^2 - b^2}}{2}. \tag{11}$$

Proof. If $d = 0$ the characteristic equation (3) becomes

$$\lambda^2 + b\lambda + a = g\lambda e^{-\lambda\tau}.$$

Note that if $g = 0$ then this equation has the roots $\lambda = (-b \pm \sqrt{b^2 - 4a})/2$ and hence the trivial fixed point of (2) is stable. From the results mentioned above, for any fixed $\tau > 0$, the fixed point will remain stable for $g \neq 0$ until g reaches a value for which the characteristic equation has a root with zero real part. Now there are no zero roots of this equation, but pure imaginary roots, $\lambda = i\omega$, will occur when the following equations (which come from (4) with $d = 0$) are satisfied

$$a - \omega^2 = g\omega \sin \omega\tau \tag{12}$$

$$b\omega = g\omega \cos \omega\tau. \tag{13}$$

From the second of these equations it is easy to see that no such roots will exist if $-b < g < b$, and thus the fixed point must remain stable for all values of g, τ in this region. For $b < |g|$ there are two such roots as given by (11), thus solving for τ as in terms of the other parameters yields two sets of functions:

$$\tau_+ = \begin{cases} \frac{1}{\omega_+}\left[(2j+2)\pi - \operatorname{Arccos}\frac{b}{g}\right] & g < 0 \\ \frac{1}{\omega_+}\left[2j\pi + \operatorname{Arccos}\frac{b}{g}\right] & g > 0 \end{cases}, \tag{14}$$

$$\tau_- = \begin{cases} \frac{1}{\omega_-}\left[2j\pi + \operatorname{Arccos}\frac{b}{g}\right] & g < 0 \\ \frac{1}{\omega_-}\left[(2j+2)\pi - \operatorname{Arccos}\frac{b}{g}\right] & g > 0 \end{cases}, \tag{15}$$

where $j = 0, 1, 2, \ldots$. Consideration of the relative positions of these curves in the g, τ plane yields the result.□

We now present our main theorems. These will describe how the region of stability of the trivial fixed point of (2) in the d, τ plane varies with the parameters a, b and g.

Theorem 1.1 *For fixed a, b and g with $0 \leq g^2 < b^2 - 2a$ the trivial fixed point of (2) is stable in the following regions*

$$-a < d < a, \qquad \tau > 0$$

$$d < -a, \qquad \tau < \frac{1}{\omega_+} \operatorname{Arccos}\left[\frac{ad + \omega_+^2(bg - d)}{d^2 + g^2\omega_+^2}\right]$$

Proof. To begin, note from Lemma 1.1 that when $d = 0$ the fixed point is stable. As before, for any fixed $\tau > 0$, the the fixed point will remain stable for $d \neq 0$ until d reaches a value for which the characteristic equation has a root with zero real part. This will occur when it reaches the innermost curve of the set consisting of $d = a$ (corresponding to the zero root) and (9) (corresponding to pure imaginary roots). From Table 1 we see that of the latter, only those with $\omega = \omega_+$ are relevant under the conditions of this theorem. Further, rewriting (5) as $d^2 = \omega^4 + (b^2 - g^2 - 2a)\omega^2 + a^2$ we can consider the curves (9) to be parametrically defined in terms of ω. For $g^2 < b^2 - 2a$, d^2 is clearly monotone increasing in ω thus the curves must be nested and can have no intersection points. Finally, $\lim_{\omega \to 0} d^2 = a^2$, thus the innermost curve for $d > 0$ is the line $d = a$ and for $d < 0$ is the curve $\tau = \tau_{1+}^0$ from (9). The result follows. □

This region is illustrated by the dashed lines in Fig. 1(a) for $a = 1,\ b = 1.5$ and $g = -0.5$. The picture is qualitatively the same for any a, b, g satisfying $0 \leq g^2 < b^2 - 2a$.

Theorem 1.2 *For fixed a, b and g with $b^2 - 2a < g^2 < b^2$ the trivial fixed point of (2) is stable in the following regions*

$$\begin{array}{rclrcl}
-d_{min} < & d & < d_{min}, & 0 < & \tau & \\
 & d & < -d_{min}, & 0 < & \tau & < \dfrac{1}{\omega_+} \operatorname{Arccos}\left[\dfrac{d + \omega_+^2(bg - d)}{d^2 + g^2\omega_+^2}\right] \\
 & & & \tau_{1-}^j < & \tau & < \tau_{1+}^{j+1} \\
d_{min} < & d & < a, & 0 < & \tau & < \dfrac{1}{\omega_+}\left\{2\pi - \operatorname{Arccos}\left[\dfrac{d + \omega_+^2(bg - d)}{d^2 + g^2\omega_+^2}\right]\right\} \\
 & & & \tau_{2-}^j < & \tau & < \tau_{2+}^{j+1}
\end{array}$$

where $j = 0, 1, 2, \ldots$, $\tau_{1\pm}^j$, $\tau_{2\pm}^j$ are defined as in (9) and

$$d_{min} \stackrel{def}{=} \sqrt{(b^2 - g^2)\left(a - \frac{b^2 - g^2}{4}\right)} \tag{16}$$

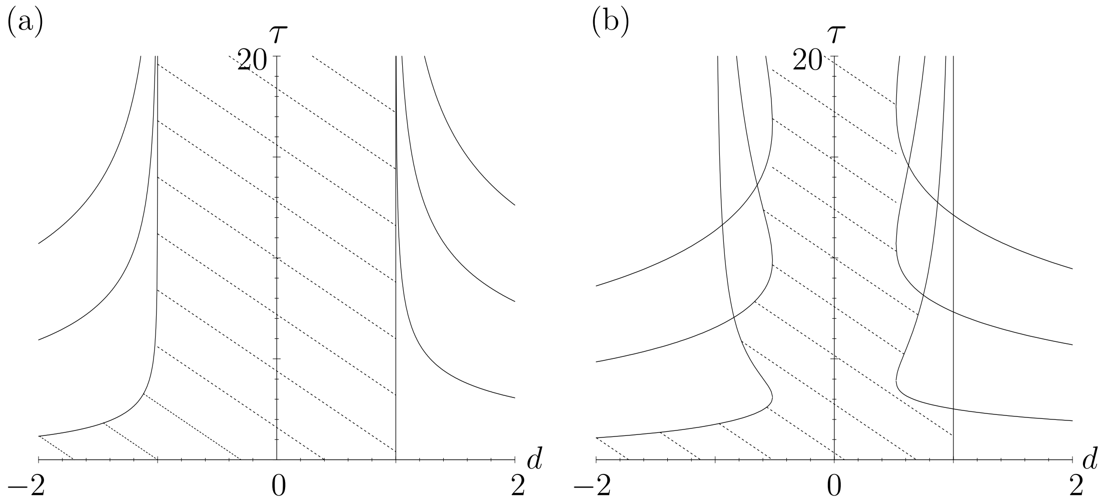

Figure 1: Stability diagrams in d, τ plane for $a = 1,\ b = 1.5$ and (a) $g = -0.5$, (b) $g = -1.4$. The region of stability of the trivial fixed point of (2) is indicated by the dashed lines.

Proof. To begin, note from Lemma 1.1 that when $d = 0$ the fixed point is stable. As before, any fixed $\tau > 0$, the the fixed point will remain stable for $d \neq 0$ until d reaches a value for which the characteristic equation has a root with zero real part. This will occur when it reaches the innermost curve of the set consisting of $d = a$ (corresponding to the zero root) and (9) (corresponding to pure imaginary roots). From Table 1 we see the latter curves are not defined for $-d_{min} < d < d_{min}$, hence there are no imaginary roots in this region. Noting further $d_{min} < a$ we see that there are also no zero roots, and thus the trivial fixed point of (2) is stable, in this region.

For $|d| \geq d_{min}$ the fixed point will be stable until $d = a$ or it takes on a value corresponding to one of curves (9), whichever comes first. Consideration of the relative positions of these curves in the d, τ plane leads to the result.□

This region is illustrated by the dashed lines in Fig. 1(b) for $a = 1,\ b = 1.5$ and $g = -1.4$. The picture is qualitatively the same for any a, b, g satisfying $b^2 - 2a < g^2 < b^2$.

Theorem 1.3 *For fixed a, b and g with $b^2 \leq g^2$ the trivial fixed point of (2) is stable in the following regions*

$$g < 0, \qquad 0 < \tau < \frac{1}{\omega_+} \text{Arccos}\left[\frac{ad + \omega_+^2(bg - d)}{d^2 + g^2\omega_+^2}\right] \qquad (d < a)$$

$$\tau_{2-}^{j} < \tau < \tau_{1+}^{j+1}$$

$$g > 0, \qquad \tau_{1-}^{j} < \tau < \tau_{2+}^{j} \qquad (d < a)$$

where $j = 0, 1, 2, \ldots,\ \tau_{1\pm}^{j},\ \tau_{2\pm}^{j}$ are defined as in (9).

Proof. To begin, we note from Lemma 1.1 that when $d = 0$ the fixed point is stable only for the values of τ as defined by (10). Fixing τ at a value for which the fixed point is stable, it will remain stable for $d \neq 0$ until d reaches a value for which the characteristic equation has a root with zero real part. As in the previous Theorems, this will occur when $d = a$ or d reaches one of the curves defined by (9), whichever comes first. Fixing τ at a value for which the fixed point is unstable with $d = 0$ and consideration of the rate of change of the real part of the roots of the characteristic equation along these curves shows that the fixed point cannot restabilize along these curves. The result follows.□

This region is illustrated by the dashed lines in Fig. 2 for $a = 1,\ b = 1.5$ and (a) $g = -1.5$, (b) $g = 1.5$, (c) $g = -1.6$, and (d) $g = 1.6$. The picture is qualitatively the same as 2(a), respectively 2(b), for any $b > 0, g < 0$, respectively $b, g > 0$, satisfying $b^2 = g^2$. The picture is qualitatively the same as 2(c), respectively 2(d), for any $b > 0, g < 0$, respectively $b, g > 0$, satisfying $b^2 < g^2$.

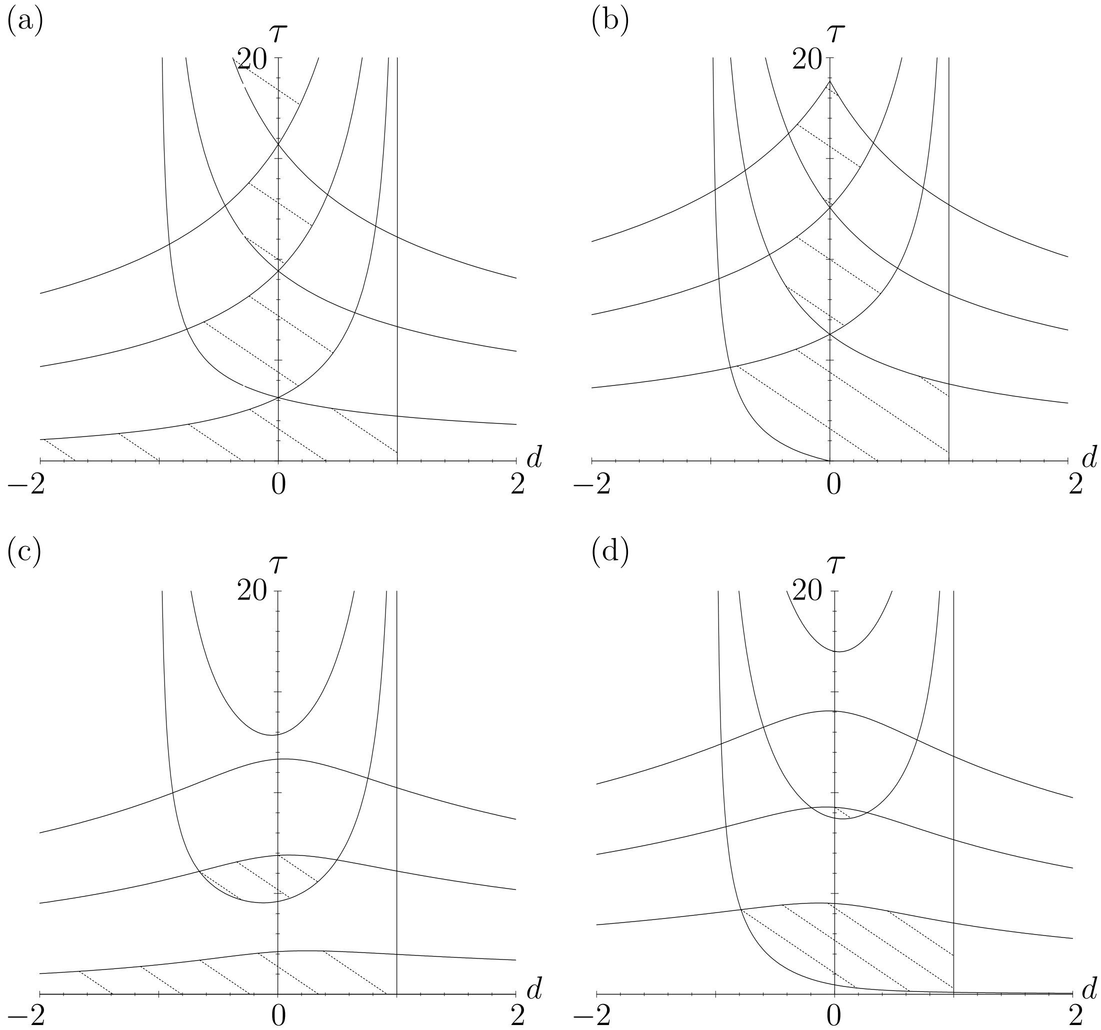

Figure 2: Stability diagrams in d, τ plane for $a = 1,\ b = 1.5$ and (a) $g = -1.5$, (b) $g = 1.5$, (c) $g = -1.6$, (d) $g = 1.6$. The region of stability of the trivial fixed point of (2) is indicated by the dashed lines.

3. Bifurcation

One can verify that the roots of the characteristic equation as described above are simple and that they cross the imaginary axis with non zero speed. Thus for the original nonlinear equation (1) with the nonlinearities satisfying appropriate nondegeneracy conditions, the curves (9) define Hopf bifurcation curves in the d, τ plane, and the vertical line $d = a$ defines a steady state bifurcation. The exact nature of these bifurcations, i.e. whether the Hopf bifurcations are supercritical or subcritical, and what type of steady state bifurcation occurs, would depend on the nonlinearity f.

Codimension two bifurcation points can occur where these different bifurcations interact. Points of Hopf-Hopf interaction occur where the characteristic equation has two pairs of pure imaginary roots $\pm i\omega_1$, $\pm i\omega_2$. For this system these points can occur for any set of values of the parameters a, b, d, g such that $\tau^j_{1,2+} = \tau^k_{1,2-}$, as defined in (9), for some $j, k \in \mathbb{Z}$. The roots in this case will be $\omega_1 = \omega_+$, $\omega_2 = \omega_-$, with $\omega_\pm$ defined by (6). These points cannot, in general, be solved for in closed form. They can, however, be easily computed numerically and are readily seen in Figs. 1 and 2, where they appear as intersection points of two of the Hopf bifurcation curves. From Table 1 it is clear that such points can only occur if $d^2 < a$ and $b^2 - 2a > g^2$.

Hopf-steady state interactions will occur when $d = a$. This necessitates $\omega = \omega_+ = \sqrt{2a + g^2 - b^2}$ and gives the τ values of these points as

$$\tau = \begin{cases} \tau^j_{1+} \stackrel{def}{=} \dfrac{1}{\omega_+}\left[2j\pi + \text{Arccos}\left[\dfrac{a^2 + \omega_+^2(bg - a)}{a^2 + g^2\omega_+^2}\right]\right], & g(a - \omega_+^2) - ab > 0 \\[2ex] \tau^j_{2+} \stackrel{def}{=} \dfrac{1}{\omega_+}\left[(2j+2)\pi - \text{Arccos}\left[\dfrac{a^2 + \omega_+^2(bg - a)}{a^2 + g^2\omega_+^2}\right]\right], & g(a - \omega_+^2) - ab < 0 \end{cases} \tag{17}$$

where $j = 0, 1, \ldots$, and Arccos refers to the branch of the inverse cosine function with range $[0, \pi]$. From the expression for ω, such points can only occur if $b^2 - 2a < g^2$. Further, the only point which can be expected to have significant influence on the dynamics is the one with $j = 0$, i.e. which borders on the region of stability of the fixed point.

Such codimension two points can be the source of more complicated dynamics such as multistability and quasiperiodicity. To illustrate what may occur in this system, we performed numerical simulations near one Hopf-Hopf and the Hopf-steady state interaction points which occur next to the stability region in Fig. 2(b). We chose as our nonlinearity $f(x, y) = d\tanh(x) + g\tanh(y)$, leading to a supercritical pitchfork bifurcation at $d = a$. Plots in the u, $\dot{u}$ plane of two of these simulations are shown in Fig. 3. Fig. 3(a) shows the existence of a stable 2-torus near the Hopf-Hopf interaction at $\tau \approx 4.5$, $d \approx -0.9$. Fig. 3(b) shows the coexistence of two stable limit cycles near the Hopf-pitchfork interaction at $\tau \approx 4$, $d = 1$.

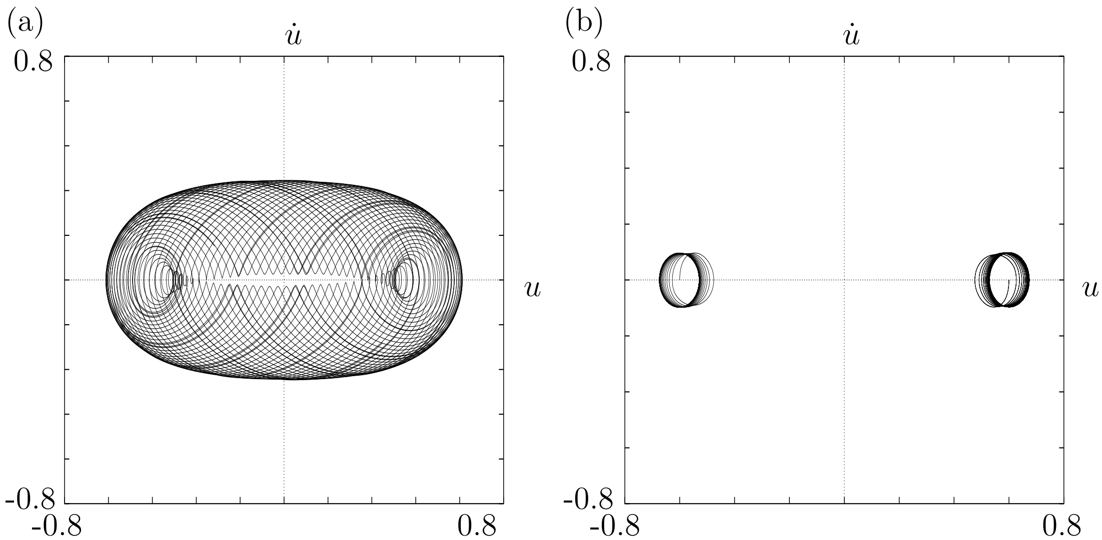

Figure 3: Numerical simulations of (1) with $a = 1$, $b = 1.5$ and $f(x, y) = d\tanh(x) + 1.5\tanh(y)$. Other parameter values are (a) $d = -0.96$, $\tau = 5$ (b) $d = 1.12$, $\tau = 4.12$.

4. Conclusions

We have presented the linear stability analysis of a second order differential equation with delayed feedback depending on both the state and the derivative of the state. The location of Hopf and steady state bifurcations and their points of interaction have been explicitly identified. We now discuss these results in light of the physical systems modelled by equation (1). One (standard) such system is that of an object attached to a spring, in which case the parameters a and b would represent the spring force and damping constants normalized by the the mass of the object. For a particular physical systems these constants are usually fixed, however we have some freedom to vary the parameters associated with the feedback loops, i.e. the nonlinearity f (and hence the gains, d, g) and perhaps the time delay, τ.

A natural goal when regulating a physical system is to simplify the dynamics as much as possible, i.e. force the fixed point to be stable whenever possible and avoid such complex dynamics as discussed in the previous section. With this in mind we make the following remarks.

As a general principle, making the gains as small as possible should stabilize the fixed point. In addition, for nonzero damping (which is generally the case for physical systems) one can always choose the gains of the feedback loops so that the fixed point is stable for all positive values of the delay.

If the damping and and spring force constants are such that $0 < b^2 - 2a$ (which will necessarily be true if the original system is over-damped) one can always choose the gains of the feedback loops so that there are no interactions between the bifurcations, eliminating a source of more complicated dynamics. We note that a similar result holds for a system with position feedback only [7], however, no such result exists for a system with velocity feedback only [3]. That is, the position feedback loop is necessary for this result.

Clearly, having two feedback loops allows for more variability in the stability regions. This may be especially important if we have a limited range for the values of the gains, as the interaction of the two feedback loops may allow us to achieve goals not be possible with just one.

Finally, we note that to be truly realistic, one should incorporate different time delays for the two feedback loops. As this can be expected to greatly complicated the analysis, we leave this problem to future work.

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