Calculating Centre Manifolds for Delay Differential Equations Using Maple™

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1 Introduction

To begin we will briefly review some results and terminology from the theory of ordinary differential equations (ODEs). Consider an autonomous ODE

$$\mathbf{x}' = \mathbf{f}(\mathbf{x}), \tag{1}$$

which admits an equilibrium point, $\mathbf{x}^*$. The linearization of (1) about $\mathbf{x}^*$ is given by

$$\mathbf{x}' = A\mathbf{x} \tag{2}$$

where $A = D\mathbf{f}(\mathbf{x}^*)$. Recall that $\mathbf{x}^*$ is called *nonhyperbolic* if at least one of the eigenvalues of A has zero real part. Given a complete set of generalized eigenvectors for the eigenvalues of A with zero real part, one can construct a basis for the subspace solutions of (2) corresponding to these eigenvalues. This subspace is called the *centre eigenspace* of (2). Nonhyperbolic equilibrium points are important as they often occur at bifurcation points of a differential equation. The *centre manifold* is a powerful tool for studying the behaviour of solutions (and hence the nature of the bifurcation) of (1) in a neighbourhood of a nonhyperbolic equilibrium point. It is a nonlinear manifold which is tangent to the centre eigenspace at $\mathbf{x}^*$. For a more detailed review of the theory and construction of centre manifolds for ODES see (Guckenheimer and Holmes, 1983, Section 3.2), (Wiggins, 1990, Section 2.1) or (Perko, 1996, Section 2.12).

In this chapter we will study centre manifolds for nonhyperbolic equilibrium points of delay differential equations (DDEs). In general, one cannot find the centre manifold exactly, thus one must construct an approximation. Some authors have performed the construction by hand, e.g., Faria and Magalhães (1995a,b); Gilsinn (2002); Guo and Huang (2003); Guo et al. (2004); Guo (2005); Jiang et al. (2006); Kalmár-Nagy et al. (1999, 2001); Liu and Yuan (2005); Nayfeh (2008); Orosz and Stépán (2004, 2006); Sri Namachchivaya

and van Roessel (2003); Wischert et al. (1994) and Wu et al. (1999). However, the construction generally involves a lot of computation and is most easily accomplished either numerically or with the aid of a symbolic algebra package. Here we focus on the symbolic algebra approach, which has been used by several authors, e.g., Bélair and Campbell (1994); Campbell et al. (2005); Qesmi et al. (2006b,a, 2007); Stone and Campbell (2004); Yuan et al. (2004); Yuan and Campbell (2004); Yuan and Wei (2005) and Wei and Yuan (2005). Unfortunately, there is rarely space in journal articles to give details of the implementation of such computations. Thus, my purpose here is to give these details, for a particular example DDE and for the symbolic algebra package Maple™, so that other authors may reproduce them in other contexts.

In the following section we will outline the theoretical setting for calculating centre manifolds. In the second section, we will show how the computations may be implemented in the symbolic algebra package Maple™by applying the theory to a model due to Stone and Askari (2002). In the final section we will discuss extensions of this approach as well as alternate approaches.

2 Theory

In this section we briefly outline the theoretical setting for calculating centre manifolds. More detail on the theory can be found in Ait Babram et al. (1997), Gilsinn (2008) and Hale and Verduyn Lunel (1993).

Consider the general delay differential equation

$$\mathbf{x}'(t) = \mathbf{g}(\mathbf{x}(t), \mathbf{x}(t - \tau_1), \ldots, \mathbf{x}(t - \tau_p); \mu) \tag{3}$$

where $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{g} : \mathbb{R}^n \times \mathbb{R}^n \times \ldots \times \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$, p is a positive integer and $\mu \in \mathbb{R}^k$ and $\tau_j > 0$, $j = 1, \ldots, p$ are parameters of the model. We shall assume that $\mathbf{g}$ is as smooth as necessary for our subsequent computations (i.e., $\mathbf{g} \in C^r$ for r large enough) and the equation admits an equilibrium solution $\mathbf{x}(t) = \mathbf{x}^*$. In general $\mathbf{x}^*$ may depend on μ, but not on the τ_j. Shifting the equilibrium to zero and separating the linear and nonlinear terms gives

$$\mathbf{x}'(t) = A_0(\mu)\,\mathbf{x}(t) + \sum_{j=1}^{p} A_j(\mu)\,\mathbf{x}(t-\tau_j) + \mathbf{f}(\mathbf{x}(t), \mathbf{x}(t-\tau_1), \ldots, \mathbf{x}(t-\tau_p); \mu), \tag{4}$$

where

$$A_j(\mu) = D_{j+1}\mathbf{g}(\mathbf{x}^*, \ldots, \mathbf{x}^*; \mu) \tag{5}$$

and

$$\begin{aligned}
\mathbf{f}(\mathbf{x}(t), \mathbf{x}(t - \tau_1), \ldots, \mathbf{x}(t - \tau_p); \mu) &= \mathbf{g}(\mathbf{x}(t), \mathbf{x}(t - \tau_1), \ldots, \mathbf{x}(t - \tau_p); \mu) \\
&\quad - A_0(\mu)\,\mathbf{x}(t) - \textstyle\sum_{j=1}^{p} A_j(\mu)\,\mathbf{x}(t - \tau_j).
\end{aligned} \tag{6}$$

Here $D_j\mathbf{g}$ means the Jacobian of $\mathbf{g}$ with respect to its jth argument.

Let $\tau = \max_j \tau_j$. To pose an initial value problem at $t = t_0$ for this DDE one must specify the value of $\mathbf{x}(t)$ not just at $t = t_0$, but on the whole interval $[t_0 - \tau, t_0]$. Thus an appropriate initial condition is

$$\mathbf{x}(t_0 + \theta) = \zeta_0(\theta), \ -\tau \leq \theta \leq 0, \tag{7}$$

where $\zeta_0 : [-\tau, 0) \to \mathbb{R}^n$ is a given function. It can be shown (see e.g. (Hale and Verduyn Lunel, 1993, Section 2.2) that if ζ_0 is continuous and $\mathbf{f}$ is Lipschitz there exists a unique solution to the initial value problem (4)–(7) which is defined and continuous on a (maximal) interval $[t_0 - \tau, \beta), \ \beta > 0$. In the following, we will assume that ζ_0 and $\mathbf{f}$ satisfy these conditions.

To define an appropriate phase space for the solutions of the DDE, make the following definition

$$\mathbf{x}_t(\theta) \stackrel{def}{=} \mathbf{x}(t + \theta), \ -\tau \leq \theta \leq 0.$$

Note that the initial condition can now be expressed as $\mathbf{x}_{t_0} = \zeta_0$ and that $\mathbf{x}_t$ will be a continuous mapping from $[-\tau, 0] \to \mathbb{R}^n$ for each $t \in [t_0, \beta)$.

With this in mind, it is usual (see, e.g., Hale and Verduyn Lunel (1993)) to take the phase space for (4) to be the Banach space $\mathcal{C} \stackrel{def}{=} C([-\tau, 0], \mathbb{R}^n)$ of continuous mappings from $[-\tau, 0]$ into $\mathbb{R}^n$, equipped with the norm

$$\|\zeta\|_\tau = \sup_{\theta \in [-\tau, 0]} \|\zeta(\theta)\|,$$

where $\| \cdot \|$ is the usual Euclidean norm on $\mathbb{R}^n$. We can then define the flow for the DDE as a mapping on $\mathcal{C}$ which takes the initial function ζ_0 into the function $\mathbf{x}_t$.

The equation (4) for $\mathbf{x}(t)$ can be expressed as a *functional differential equation* (FDE)

$$\mathbf{x}'(t) = L(\mathbf{x}_t; \mu) + \mathbf{F}(\mathbf{x}_t; \mu), \tag{8}$$

where $L : \mathcal{C} \times \mathbb{R}^k \to \mathbb{R}^n$ is a linear mapping defined by

$$L(\phi; \mu) = A_0(\mu)\,\phi(0) + \sum_{j=1}^p A_j(\mu)\,\phi(-\tau_j), \tag{9}$$

and $\mathbf{F} : \mathcal{C} \times \mathbb{R}^k \to \mathbb{R}^n$ is a nonlinear functional defined by

$$\mathbf{F}(\phi; \mu) = \mathbf{f}(\phi(0), \phi(-\tau_1), \ldots, \phi(-\tau_p); \mu). \tag{10}$$

As shown in e.g., Faria and Magalhães (1995a,b) and Wischert et al. (1994) one may extend (8) to a differential equation for $\mathbf{x}_t(\theta)$ as follows

$$\frac{d}{dt}\mathbf{x}_t(\theta) = \begin{cases} \frac{d}{d\theta}(\mathbf{x}_t(\theta)) & , \ -\tau \leq \theta < 0 \\ L(\mathbf{x}_t; \mu) + \mathbf{F}(\mathbf{x}_t; \mu) , & \theta = 0 , \end{cases} \tag{11}$$

This equation will be important for the centre manifold construction.

2.1 Linearization

Clearly (4) and (8) admit the trivial solution $\mathbf{x}(t) = 0$, $\forall t$, which corresponds to the equilibrium solution $\mathbf{x}(t) = \mathbf{x}^*$ of (3). The stability of this equilibrium solution can be studied via the linearization of (8) about the trivial solution:

$$\mathbf{x}'(t) = L(\mathbf{x}_t; \mu) \tag{12}$$

or, in the DDE form,

$$\mathbf{x}'(t) = A_0(\mu)\,\mathbf{x}(t) + \sum_{j=1}^{p} A_j(\mu)\,\mathbf{x}(t - \tau_j). \tag{13}$$

Substituting the ansatz $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$, $\mathbf{v} \in \mathbb{R}^n$ into (13) yields the matrix vector equation

$$\left[\lambda I - A_0(\mu) - \sum_{j=1}^{p} A_j(\mu) e^{-\lambda \tau_j} \right] \mathbf{v} = 0, \tag{14}$$

which we will sometimes write in the compact form $\Delta(\lambda; \mu)\mathbf{v} = 0$. Requiring nontrivial solutions ($\mathbf{v} \neq \mathbf{0}$) yields the constraint $\det(\Delta(\lambda); \mu) = 0$, i.e., that λ is a root of the characteristic equation

$$\det \left[\lambda I - A_0(\mu) - \sum_{j=1}^{p} A_j(\mu) e^{-\lambda \tau_j} \right] = 0. \tag{15}$$

It can be shown (Hale and Verduyn Lunel, 1993, Corollary 7.6.1) that the trivial solution of (12) (or (13)) will be asymptotically stable (and hence the equilibrium solution of (33) will be locally asymptotically stable) if all the roots of (15) negative real parts. We will call these roots the eigenvalues of the equilibrium point.

Consider a point, $\mu = \mu_c$, in the parameter space where the characteristic equation (15) has m roots with zero real parts and the rest of the roots have negative real parts. The following results are shown in Hale and Verduyn Lunel (1993). At such a point there exists a decomposition of the solution space for the linear FDE (12) as $\mathcal{C} = N \oplus S$, where N is an m-dimensional subspace spanned by the solutions to (12) corresponding to the eigenvalues with zero real part, S is infinite dimensional and N and S are invariant under the flow associated with (12). N and S are analogous to the centre and stable eigenspaces for ODEs.

For simplicity, we will assume that all the eigenvalues with zero real part have multiplicity one. This includes the most common cases studied: single Hopf bifurcation, double Hopf bifurcation (with nonidentical frequencies) and zero-Hopf bifurcation. For a discussion of DDEs with a zero eigenvalue of

multiplicity two (Bogdanov Takens bifurcation) or three, see Campbell and Yuan (2008) or Qesmi et al. (2007). For a discussion of DDEs where complex conjugate eigenvalues with higher multiplicity arise due to symmetry, see Campbell et al. (2005); Guo and Huang (2003); Guo et al. (2004); Guo (2005); Krawcewicz and Wu (1999); Orosz and Stépán (2004); Wu (1998); Wu et al. (1999) or Yuan and Campbell (2004). For a general discussion of eigenspaces associated with eigenvalues of higher multiplicity in DDEs see (Hale and Verduyn Lunel, 1993, Section 7.4).

Let $\{\phi_1(t), \phi_2(t), \ldots, \phi_m(t)\}$ be a basis for N and $\{\lambda_1, \lambda_2, \ldots, \lambda_m\}$ the corresponding eigenvalues. This basis can be constructed in a similar manner to that for ODEs. Since $\mathrm{Re}(\lambda_k) = 0$ for each k, either $\lambda_k = 0$ or $\lambda_k = i\omega_k$. In the latter case, it is easy to check that $-i\omega_k$ is also a root, and we will order the eigenvalues so that $\lambda_{k+1} = -i\omega_k$. With the restriction of simple eigenvalues, the construction of the basis functions is straight forward. If $\lambda_k = 0$, then $\phi_k = \mathbf{v}_k$ where $\mathbf{v}_k$ a solution of $\Delta(0; \mu_c)\mathbf{v}_k = 0$. If $\lambda_k = i\omega$, then $\phi_k = \mathrm{Re}(e^{i\omega_k t}\mathbf{v}_k)$ and $\phi_{k+1} = \mathrm{Im}(e^{i\omega_k t}\mathbf{v}_k)$, where $\mathbf{v}_k$ a solution of $\Delta(i\omega_k; \mu_c)\mathbf{v}_k = 0$. In the following, we will usually write the basis as an $n \times m$ matrix, with the kth column given by ϕ_k, viz.:

$$\mathbf{\Phi}(t) = [\, \phi_1(t) \,|\, \phi_2(t) \,|\, \ldots \,|\, \phi_m(t) \,]. \tag{16}$$

A simple calculation then shows that $\mathbf{\Phi}$ satisfies the following matrix ordinary differential equation:

$$\mathbf{\Phi}' = \mathbf{\Phi}B \tag{17}$$

where B is a block diagonal matrix, with block $[\, 0 \,]$ for the zero eigenvalue, if present, and block

$$\begin{bmatrix} 0 & \omega_k \\ -\omega_k & 0 \end{bmatrix}$$

for each pair of complex conjugate eigenvalues, $\pm i\omega_k$.

Note that the basis functions may also be treated as functions on $\mathcal{C}$, by changing their argument to $\theta \in [-\tau, 0]$. Now consider

$$L(e^{\lambda_k \theta}\mathbf{v}_k; \mu_c) = A_0(\mu_c)\mathbf{v}_k + \sum_{j=1}^{p} A_j(\mu_c)e^{-\lambda_k \tau_j}\mathbf{v}_k$$

$$= \lambda_k \mathbf{v}_k,$$

which follows from (14). This implies that $L(\phi_k; \mu_c) = 0$ when $\lambda_k = 0$, and when $\lambda_k = i\omega_k$, $L(\phi_k; \mu_c) = -\omega_k\phi_{k+1}(0)$ and $L(\phi_{k+1}; \mu_c) = \omega_k\phi_k(0)$. We then have the following result:

$$L(\mathbf{\Phi}; \mu_c) = \mathbf{\Phi}(0)B. \tag{18}$$

As for ODEs, the decomposition of the solution space may be accomplished via the introduction of the adjoint equation for (12). However, a different equation, which is closely related to the adjoint equation, may also be used to

decompose the solution space. It turns out that this latter equation is useful for the centre manifold construction, so we focus on it.

Let R^{n*} be the n-dimensional row vectors and $\mathcal{C}^* = C([0,r], \mathbb{R}^{n*})$. For $\psi \in \mathcal{C}^*$ and $\phi \in \mathcal{C}$, define the following bilinear form

$$\langle \psi, \phi \rangle = \sum_{i=1}^{n} \psi_j(0)\phi_j(0) + \sum_{j=1}^{p} \int_{-\tau_j}^{0} \psi(\sigma + \tau_j) \, A_j \, \phi(\sigma) d\sigma. \qquad (19)$$

As shown in (Hale and Verduyn Lunel, 1993, Section 7.5), this can be used to define a system dual to (12) given by

$$\mathbf{y}'(t) = L^T(\mathbf{y}^s; \mu), \qquad s \leq 0, \qquad (20)$$

where $y^s = y(s+\xi)$, $0 \leq \xi \leq \tau$ and L^T is a linear mapping on $\mathcal{C}^* \times \mathbb{R}^k$ given by

$$L^T(\psi; \mu) = -\psi(0) \, A_0(\mu) - \sum_{j=1}^{p} \psi(\tau_j) \, A_j(\mu). \qquad (21)$$

Equation (20) is called the *transposed system* by Hale and Verduyn Lunel (1993). In the literature, it is sometimes called the formal adjoint. The corresponding differential equation is

$$\mathbf{y}'(s) = -\mathbf{y}(s) \, A_0(\mu) - \sum_{j=1}^{p} \mathbf{y}(s+\tau_j) \, A_j(\mu), \qquad s \leq 0. \qquad (22)$$

Using the ansatz $\mathbf{y}(s) = \mathbf{w}e^{-\lambda s}$, $\mathbf{w} \in \mathbb{R}^{n*}$ and proceeding as for (12), shows that $\mathbf{w}$ must satisfy $\mathbf{w}\Delta(\lambda; \mu) = 0$. Thus the characteristic equation of (22) is just (15). It follows that the trivial solutions of (22) and (13) have the same eigenvalues.

Let

$$\mathbf{\Psi}(s) = \begin{bmatrix} \psi_1(s) \\ \vdots \\ \psi_m(s) \end{bmatrix},$$

be a basis for the solutions of (20) (or, equivalently, (22)) corresponding to the m eigenvalues with zero real part (i.e. the "centre eigenspace" of (22)). Note that the ψ_j are row vectors and that they can be considered as functions on $\mathcal{C}^*$ if we change their argument to $\xi \in [0, \tau]$. The fundamental result used in the centre manifold construction is that $\mathbf{\Psi}$ may be used to decompose the solution space. See (Hale and Verduyn Lunel, 1993, Section 7.5) for details and proofs. In particular, for any $\zeta \in S$,

$$\langle \psi_j, \zeta \rangle = 0, \ j = 1, \ldots, m.$$

Further, we can choose a basis so that $\langle \mathbf{\Psi}, \mathbf{\Phi} \rangle = \mathbf{I}$, where $\langle \mathbf{\Psi}, \mathbf{\Phi} \rangle$ is the $m \times m$ matrix with i, j elements $\langle \psi_i, \phi_j \rangle$ and $\mathbf{I}$ is the $m \times m$ identity matrix. Thus

for any $\zeta \in N$ we have $\zeta = \mathbf{\Phi u}$ where $\mathbf{u} = \langle \mathbf{\Psi}, \zeta \rangle \in \mathbb{R}^m$. Finally, one can show that

$$\mathbf{\Psi}' = -B\mathbf{\Psi} \qquad \text{and} \qquad L^T(\mathbf{\Psi}; \mu_c) = -B\mathbf{\Psi}(0). \tag{23}$$

where B is the same block diagonal matrix as in (17).

2.2 Nonlinear Equation

Now let us return to the nonlinear equation (8). For the rest of this section we will assume that $\mu = \mu_c$ and hence that the characteristic equation (15) has m eigenvalues with zero real parts and all other eigenvalues have negative real parts. In this situation Hale and Verduyn Lunel (1993) (Chapter 10) have shown that there exists, in the solution space $\mathcal{C}$ for the nonlinear FDE (8), an m dimensional centre manifold. Since all the other eigenvalues have negative real parts, this manifold is attracting and the long term behaviour of solutions to the nonlinear equation is well approximated by the flow on this manifold. In particular, studying the flow on this manifold will enable us to characterize the bifurcation which occurs as a μ passes μ_c. Below, we outline the steps involved in computing this manifold. The approach we take follows the work of Hale (1985) and Wischert et al. (1994) (scalar case) and of Ait Babram et al. (1997) (vector case). Since all our computations will be done for $\mu = \mu_c$, we not write the dependence on μ explicitly.

To begin, we note that points on the local centre manifold of $\mathbf{0}$ can be expressed as the sum of a linear part belonging to N and a nonlinear part belonging to S, i.e.,

$$W_{loc}^c(\mathbf{0}) = \{\phi \in \mathcal{C} \mid \phi = \mathbf{\Phi u} + \mathbf{h}(\mathbf{u})\} \,,$$

where $\mathbf{\Phi}(\theta)$, $\theta \in [-\tau, 0]$ is the basis for N introduced above, $\mathbf{u} \in \mathbb{R}^m$, $\mathbf{h}(\mathbf{u}) \in S$ and $\|\mathbf{u}\|$ is sufficiently small. The solutions of (8) on this centre manifold are then given by $\mathbf{x}(t) = \mathbf{x}_t(0)$ where $\mathbf{x}_t(\theta)$ is a solution of (11) satisfying

$$\mathbf{x}_t(\theta) = \mathbf{\Phi}(\theta)\mathbf{u}(t) + \mathbf{h}(\theta, \mathbf{u}(t)) \,. \tag{24}$$

To find the center manifold and the solutions on it, we proceed as follows. Substituting (24) into (11) yields

$$\left[\mathbf{\Phi}(\theta) + \frac{\partial \mathbf{h}}{\partial \mathbf{u}}\right] \dot{\mathbf{u}}(t) = \begin{cases} \mathbf{\Phi}'(\theta)\mathbf{u}(t) + \dfrac{\partial \mathbf{h}}{\partial \theta} & , \ -\tau \leq \theta < 0 \\[2ex] L(\mathbf{\Phi}(\theta))\mathbf{u}(t) + L(\mathbf{h}(\theta, \mathbf{u}(t))) \\ +\mathbf{F}[\mathbf{\Phi}(\theta)\mathbf{u}(t) + \mathbf{h}(\theta, \mathbf{u}(t))] & , \quad \theta = 0 \,. \end{cases} \tag{25}$$

Using (17) and (18) in (25) we obtain

$$\left[\mathbf{\Phi}(\theta) + \frac{\partial \mathbf{h}}{\partial \mathbf{u}}\right] \dot{\mathbf{u}}(t) = \begin{cases} \mathbf{\Phi}(\theta)B\mathbf{u}(t) + \dfrac{\partial \mathbf{h}}{\partial \theta} & , \ -\tau \leq \theta < 0, \\[2ex] \mathbf{\Phi}(0)B\mathbf{u}(t) + L(\mathbf{h}(\theta, \mathbf{u}(t))) \\ +\mathbf{F}[\mathbf{\Phi}(\theta)\mathbf{u}(t) + \mathbf{h}(\theta, \mathbf{u}(t))] & , \quad \theta = 0. \end{cases} \tag{26}$$

This coupled system must be solved for $\mathbf{u}(t)$ and $\mathbf{h}(\theta, \mathbf{u}(t))$.

To derive the equation for $\mathbf{u}(t)$ we will use the bilinear form (19). First we note some useful results. Since $\mathbf{h}(\theta, \mathbf{u}) \in S$ for any $\mathbf{u}$,

$$\langle \boldsymbol{\Psi}(\xi), \mathbf{h}(\theta, \mathbf{u}(t)) \rangle = 0.$$

It then follows from the definition of the partial derivative that

$$\langle \boldsymbol{\Psi}(\xi), \frac{\partial \mathbf{h}}{\partial \mathbf{u}}(\theta, \mathbf{u}(t)) \rangle = 0.$$

Finally, using (23) we have

$$\boldsymbol{\Psi}(0) L(\mathbf{h}(\theta, \mathbf{u})) + \sum_{j=1}^{p} \int_{-\tau_j}^{0} \boldsymbol{\Psi}(\sigma + \tau_j) \, A_j \, \frac{\partial \mathbf{h}}{\partial \sigma} \, d\sigma$$

$$= -L^T(\boldsymbol{\Psi}(\xi)) \mathbf{h}(0, \mathbf{u}(t)) - \sum_{j=1}^{p} \int_{-\tau_j}^{0} \boldsymbol{\Psi}'(\sigma + \tau_j) \, A_j \, \mathbf{h}(\sigma, \mathbf{u}(t)) \, d\sigma$$

$$= B\boldsymbol{\Psi}(0) \mathbf{h}(0, \mathbf{u}(t)) + \sum_{j=1}^{p} \int_{-\tau_j}^{0} B\boldsymbol{\Psi}(\sigma + \tau_j) \, A_j \, \mathbf{h}(\sigma, \mathbf{u}(t)) \, d\sigma$$

$$= B\langle \boldsymbol{\Psi}(\xi), \mathbf{h}(\theta, \mathbf{u}) \rangle$$

$$= 0.$$

Applying the bilinear form to $\boldsymbol{\Psi}$ and (26) and using these results gives the following system of ODEs for $\mathbf{u}(t)$:

$$\dot{\mathbf{u}}(t) = B\mathbf{u}(t) + \boldsymbol{\Psi}(0)\mathbf{F}[\boldsymbol{\Phi}(\theta)\mathbf{u}(t) + \mathbf{h}(\theta, \mathbf{u}(t))] . \tag{27}$$

Using (27) in (26) then yields the following system of partial differential equations for $\mathbf{h}(\theta, \mathbf{u})$:

$$\frac{\partial \mathbf{h}}{\partial \mathbf{u}} \{B\mathbf{u} + \boldsymbol{\Psi}(0)\mathbf{F}[\boldsymbol{\Phi}(\theta)\mathbf{u} + \mathbf{h}(\theta, \mathbf{u})]\} + \boldsymbol{\Phi}(\theta)\boldsymbol{\Psi}(0)\mathbf{F}[\boldsymbol{\Phi}(\theta)\mathbf{u} + \mathbf{h}(\theta, \mathbf{u})]$$

$$= \begin{cases} \dfrac{\partial \mathbf{h}}{\partial \theta} & , \ -\tau \leq \theta < 0 \\ L(\mathbf{h}(\theta, \mathbf{u})) + \mathbf{F}[\boldsymbol{\Phi}(\theta)\mathbf{u} + \mathbf{h}(\theta, \mathbf{u})] \ , & \theta = 0 \ . \end{cases} \tag{28}$$

Thus, the evolution of solutions on the centre manifold is determined by solving (28) for $\mathbf{h}(\theta, \mathbf{u})$ and then (27) for $\mathbf{u}(t)$. To solve (28), one uses a standard approach in centre manifold theory, namely, one assumes that $\mathbf{h}(\theta, \mathbf{u})$ may be expanded in power series in $\mathbf{u}$:

$$\mathbf{h}(\theta, \mathbf{u}) = \mathbf{h}_2(\theta, \mathbf{u}) + \mathbf{h}_3(\theta, \mathbf{u}) + \cdots \tag{29}$$

where

$$\mathbf{h}_2(\theta, \mathbf{u}) = \begin{bmatrix} h_{11}^1(\theta)u_1^2 + \cdots h_{1m}^1(\theta)u_1 u_m + h_{22}^1(\theta)u_2^2 + \cdots + h_{mm}^1(\theta)u_m^2 \\ \vdots \\ h_{11}^n(\theta)u_1^2 + \cdots h_{1m}^n(\theta)u_1 u_m + h_{22}^n(\theta)u_2^2 + \cdots + h_{mm}^n(\theta)u_m^2 \end{bmatrix},$$

and similarly for $\mathbf{h}_3$ and the higher order terms.

Before proceeding to solve for $\mathbf{h}$, we would like to note that to determine the terms of (27) to $O(\|\mathbf{u}(t)\|^l)$, one only needs the terms which are $O(\|\mathbf{u}(t)\|^{l-1})$ in the series for $\mathbf{h}$. To see this, write $\mathbf{F}$ in series form

$$\mathbf{F} = \mathbf{F}_2 + \mathbf{F}_3 + \cdots \tag{30}$$

and hence rewrite (27) as

$$\dot{\mathbf{u}} = B\mathbf{u} + \mathbf{\Psi}(0) \left\{ \mathbf{F}_2[\mathbf{\Phi}(\theta)\mathbf{u} + \mathbf{h}_2(\theta, \mathbf{u}) + \mathbf{h}_3(\theta, \mathbf{u}) + O(\|\mathbf{u}\|^4)] \right.$$
$$\left. + \mathbf{F}_3[\mathbf{\Phi}(\theta)\mathbf{u} + \mathbf{h}_2(\theta, \mathbf{u}) + \mathbf{h}_3(\theta, \mathbf{u}) + \mathrm{O}(\|\mathbf{u}\|^4)] + O(\|\mathbf{u}\|^4) \right\}.$$

Expanding each $\mathbf{F}_j$ in a Taylor series about $\mathbf{\Phi}(\theta)\mathbf{u}$ yields

$$\dot{\mathbf{u}} = B\mathbf{u} + \mathbf{\Psi}(0) \left[\mathbf{F}_2(\mathbf{\Phi}(\theta)\mathbf{u}) + D\mathbf{F}_2(\mathbf{\Phi}(\theta)\mathbf{u})\mathbf{h}_2(\theta, \mathbf{u}) + \mathbf{F}_3(\mathbf{\Phi}(\theta)\mathbf{u}) \right] + O(\|\mathbf{u}\|^4).$$

Thus we see that $\mathbf{h}_2$ is only needed to calculate the third order terms not the second order terms. A similar result holds for the higher order terms. Of particular note is the fact that if the lowest order terms we need in the centre manifold are the same as the lowest order terms in $\mathbf{F}$, then there is no need to calculate $\mathbf{h}$ at all! This is the case for a Hopf bifurcation when $\mathbf{F}_2 = 0$. Examples of this can be found in Wischert et al. (1994); Stépán and Haller (1995) and Landry et al. (2005). This is also the case when the normal form for a particular bifurcation is determined at second order, such as for the Bogdanov-Takens bifurcation (see, e.g., Campbell and Yuan (2008)) or a double Hopf bifurcation with 1:2 resonance (see, e.g., Campbell and LeBlanc (1998)).

Now let us return to solving (28). Substituting (29) and (30) into the first part of (28) and expanding the F_j about $\mathbf{\Phi}(\theta)\mathbf{u}$ yields

$$\frac{\partial \mathbf{h}_2}{\partial \theta} + O(\|\mathbf{u}\|^3) = \frac{\partial \mathbf{h}_2}{\partial \mathbf{u}}(\theta, \mathbf{u}) B\mathbf{u} + \mathbf{\Phi}(\theta)\mathbf{\Psi}(0)\mathbf{F}_2(\mathbf{\Phi}(\theta)\mathbf{u}) + O(\|\mathbf{u}\|^3). \tag{31}$$

Equating terms with like powers of $u_1, \ldots, u_m$ in this equation yields a system of ODEs for the $h_{jk}^i(\theta)$. The system is linear and is easily solved to find the general solutions for the $h_{jk}^i(\theta)$ in terms of arbitrary constants.

These arbitrary constants may be determined as follows. Substituting (29) and (30) into the second part of (28) and expanding the F_j about $\mathbf{\Phi}(\theta)\mathbf{u}$ yields

$$\frac{\partial \mathbf{h}_2}{\partial \mathbf{u}} \bigg|_{\theta=0} B\mathbf{u} + \mathbf{\Phi}(0)\mathbf{\Psi}(0)\mathbf{F}_2(\mathbf{\Phi}(\theta)\mathbf{u}) + O(\|\mathbf{u}\|^3)$$
$$= L(\mathbf{h}_2(\theta, \mathbf{u})) + \mathbf{F}_2(\mathbf{\Phi}(\theta)\mathbf{u}) + O(\|\mathbf{u}\|^3) \tag{32}$$

Equating terms with like powers of $u_1, \ldots, u_m$ in this equation yields a set of boundary conditions for the arbitrary constants.

Once one has determined $\mathbf{h}_2$ one may proceed to the next order of approximation and calculate $\mathbf{h}_3$. As discussed above, however, for most applications this is unnecessary.

3 Application

Now consider the model of Stone and Askari (2002):

$$\eta'' + \delta\eta' + \eta - \beta(1 - \mu(\eta - \eta(t-\tau)))(p_0 + p_1\eta' + p_2\eta'^2) = 0 \qquad (33)$$

which was developed to study the vibrations in drilling. This model is in dimensionless form, the model in physical variables can be found in Stone and Askari (2002) or Stone and Campbell (2004). The variable η corresponds to the amplitude of the vibrations and $'$ to derivative with respect to time. The parameter $1/\tau$ is proportional to the speed of rotation of the drill and β to the width of cut. Since these two parameters can be varied in practice, Stone and Campbell (2004) chose these as the bifurcation parameters. The other parameters, μ, p_0, p_1, p_2, can be related to other physical parameters (see Stone and Campbell (2004)).

Note that this equation has an equilibrium solution $\eta(t) = \beta p_0$. which corresponds to the steady cutting solution. The drilling process may exhibit *chatter* which is a self excited oscillation of the drill. The emergence of chatter in the physical system corresponds to a Hopf bifurcation in the model (33). In Stone and Campbell (2004) the criticality of this bifurcation was studied using the centre manifold construction described in section 1. We will reproduce the essence of the analysis here, including the relevant commands in Maple™11[3] used to perform the computations symbolically. Commands will be written in typewriter font and preceded by a >. Each command will be followed by the output produced when it is executed. If a command ends with a colon then no output is printed. More information on Maple can be found in the manual: Maple User Manual (2006).

Shifting the equilibrium to the origin and rewriting the equation as a first order vector equation puts it in the form (4):

$$\mathbf{x}'(t) = A_0\mathbf{x}(t) + A_1\mathbf{x}(t-\tau) + \mathbf{f}(\mathbf{x}(t), \mathbf{x}(t-\tau)) \qquad (34)$$

where

$$\mathbf{x}(t) = \begin{bmatrix} x(t) \\ x'(t) \end{bmatrix} = \begin{bmatrix} \eta(t) - \beta\, p_0 \\ \eta'(t) \end{bmatrix} \qquad (35)$$

$$A_0 = \begin{bmatrix} 0 & 1 \\ -1 - \beta\, p_0\, \mu & \beta\, p_1 - \delta \end{bmatrix}, \ A_1 = \begin{bmatrix} 0 & 0 \\ \beta\, p_0\, \mu & 0 \end{bmatrix}, \qquad (36)$$

---

[3] The commands used are backward compatible to at least Maple™9.5.

and

$$\mathbf{f} = \begin{bmatrix} 0 \\ \beta \, p_2 x'(t)^2 - \beta \, \mu(x(t) - x(t-\tau))(p_1 x'(t) + p_2 x'(t)^2) \end{bmatrix}. \qquad (37)$$

Thus, in this example, the linear mapping of eq. (8) is given by

$$L(\phi(\theta)) = A_0 \phi(0) + A_1 \phi(-\tau), \qquad (38)$$

and $\mathbf{F} : \mathcal{C} \to \mathbb{R}^2$ is a nonlinear functional defined by

$$\mathbf{F}(\phi(\theta)) = \mathbf{f}(\phi(0), \phi(-\tau)). \qquad (39)$$

The characteristic matrix and equation for this example can be defined in Maple as follows.

```
> A0:=matrix(2,2,[[0,1],[-1-beta*p0*mu,beta*p1-delta]]);
```

$$A_0 := \begin{bmatrix} 0 & 1 \\ -1 - \beta \, p0 \, \mu & \beta \, p1 - \delta \end{bmatrix}$$

```
> A1:=matrix(2,2,[[0,0],[beta*p0*mu,0]]);
```

$$A_1 := \begin{bmatrix} 0 & 0 \\ \beta \, p0 \, \mu & 0 \end{bmatrix}$$

```
> ident:=evalm(array(1..2,1..2,identity));
```

$$ident := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

```
> Delta:=evalm(lambda*ident-A0-exp(-lambda*tau)*A1);
```

$$\Delta := \begin{bmatrix} \lambda & -1 \\ 1 + \beta \, p0\mu - \beta \, p0\mu e^{-\lambda \tau} & \lambda - \beta \, p1 + \delta \end{bmatrix}$$

```
> char_eq:=collect(det(Delta),lambda);
```

$$char\_eq := \lambda^2 + (-\beta \, p1 + \delta)\lambda + 1 + \beta \, p0 \, \mu - e^{(-\lambda \tau)} \beta \, p0 \, \mu$$

The work of Stone and Campbell (2004) described curves, in the τ, β parameter space, along which the equilibrium solution of eq. (33) loses stability. At each point on these curves, the characteristic equation (15) has a pair of pure imaginary roots and the rest of the roots have negative real parts. Equations describing where the characteristic equation has pure imaginary roots can be easily found in Maple:

```
> eq_im:=evalc(subs(lambda=I*omega,char_eq)):
  eq_Re:=coeff(eq_im,I,0);
  eq_Im:=coeff(eq_im,I,1);
```

$$eq\_Re := -\omega^2 + 1 + \beta\,p0\,\mu - \cos(\omega\,\tau)\,\beta\,p0\,\mu$$
$$eq\_Im := -\omega\,\beta\,p1 + \omega\,\delta + \sin(\omega\,\tau)\,\beta\,p0\,\mu$$

Stone and Campbell (2004) solved these equations to find expressions for τ and β in terms of ω and the other parameters. For fixed values of the other parameters, these expressions determined curves in the τ, β parameter space which are parametrized by ω.

It is straightforward to check that the DDE (33) satisfies the conditions for a Hopf bifurcation to occur as one passes through a point on these curves (see (Hale and Verduyn Lunel, 1993, pp. 331-333) or (Gilsinn, 2002, Section 2) for a statement of the Hopf bifurcation Theorem for FDE's). To determine the criticality of this Hopf bifurcation, we compute the centre manifold of the equilibrium point at the Hopf bifurcation, following the steps outlined in section 2.

To begin, we calculate a basis for the "centre eigenspace", N. To do this we need to find the eigenfunctions corresponding to the eigenvalues $\pm i\omega$. It Maple this may be done as follows. First solve $\Delta(i\omega)\mathbf{v} = 0$ where $\Delta(i\omega)$ is the characteristic matrix with $\lambda = i\omega$.

```
> v:=matrix([[v1],[v2]]);
  Dv:=subs(lambda=I*omega,evalm(multiply(Delta,v)));
  v2res:=v2=solve(Dv[1,1], v2);
```

$$v := \begin{bmatrix} v1 \\ v2 \end{bmatrix}$$

$$Dv := \begin{bmatrix} I\,\omega\,v1 - v2 \\ \left(1 + \beta\,p0\,\mu - e^{-I\,\omega\,\tau}\beta\,p0\,\mu\right) v1 + (I\,\omega - \beta\,p1 + \delta)\,v2 \end{bmatrix}$$
$$v2res := v2 = I\,\omega\,v1$$

Then define the complex eigenfunction and take the real and imaginary parts.

```
> yy:=map(evalc,subs(v2res,v1=1,evalm(exp(I*omega*theta)*v))):
  Phi:=array(1..2,1..2,[[coeff(yy[1,1],I,0),coeff(yy[1,1],I,1)],
    [coeff(yy[2,1],I,0),coeff(yy[2,1],I,1)]]);
```

$$\mathbf{\Phi} := \begin{bmatrix} \cos(\omega\theta) & \sin(\omega\theta) \\ -\sin(\omega\theta)\,\omega & \cos(\omega\theta)\,\omega \end{bmatrix}$$

Similarly we define $\mathbf{u}$ and $\mathbf{\Phi}\,\mathbf{u}$.

```
  u:=matrix([[u1],[u2]]);
```

$$u := \begin{bmatrix} u1 \\ u2 \end{bmatrix}$$

```
Phiu:=multiply(Phi,u);
```

$$Phiu := \begin{bmatrix} \cos(\omega\,\theta)\,u1 + \sin(\omega\,\theta)\,u2 \\ -\sin(\omega\,\theta)\,\omega\,u1 + \cos(\omega\,\theta)\,\omega\,u2 \end{bmatrix}$$

Next define the matrix B.

```
> B:=matrix([[0,omega],[-omega,0]]);
```

$$B := \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix}.$$

We need to define the basis, $\boldsymbol{\Psi}(\xi)$, $\xi \in [0,\tau]$, for the "centre eigenspace" of the transpose system. First we calculate a general basis $\boldsymbol{\Psi}_g$ in the same way as we set up the basis $\boldsymbol{\Phi}$.

```
> w:=array(1..2):
  wD:=subs(lambda=I*omega,multiply(w,Delta)):
  w1res:=w[1]=solve(wD[2],w[1]):
  yy:=map(evalc,subs(w1res,w[2]=1,lambda=I*omega,
      evalm(w*exp(-lambda*xi)))):
  Psi_g:=array(1..2,1..2,[[coeff(yy[1],I,0),coeff(yy[2],I,0)],
          [coeff(yy[1],I,1),coeff(yy[2],I,1)]]);
```

$$\Psi_g := \begin{bmatrix} \cos(\omega\,\xi)(-\beta p1 + \delta) + \sin(\omega\,\xi)\omega & \cos(\omega\,\xi) \\ -\sin(\omega\,\xi)(-\beta p1 + \delta) + \cos(\omega\,\xi)\omega & -\sin(\omega\,\xi) \end{bmatrix}$$

We now wish to find a basis $\boldsymbol{\Psi}$ such that $\langle \boldsymbol{\Psi}, \boldsymbol{\Phi} \rangle = \mathbf{I}$. The elements of $\boldsymbol{\Psi}$ will be linear combinations of those of $\boldsymbol{\Psi}_g$, i.e., $\boldsymbol{\Psi} = K\boldsymbol{\Psi}_g$ where K is a 2×2 matrix of constants. Thus we have

$$\mathbf{I} = \langle \boldsymbol{\Psi}, \boldsymbol{\Phi} \rangle$$
$$= \langle K\boldsymbol{\Psi}_g, \boldsymbol{\Phi} \rangle$$
$$= K \langle \boldsymbol{\Psi}_g, \boldsymbol{\Phi} \rangle$$

Which implies that $K = \langle \boldsymbol{\Psi}_g, \boldsymbol{\Phi} \rangle^{-1}$.

For this example the bilinear form (19) becomes

$$\langle \psi, \phi \rangle = \sum_{j=1}^{2} \psi_j(0)\phi_j(0) + \int_{-\tau}^{0} \psi(\sigma + \tau)A_1\phi(\sigma)\,d\sigma$$

$$= \psi(0) \cdot \phi(0) + \beta\,\mu\,p_0 \int_{-\tau}^{0} \psi_2(\sigma + \tau)\phi_1(\sigma)\,d\sigma$$

We define this bilinear form as a procedure as follows:

```
> bilinear_form:=proc(rowv,colv)
  local pstemp;
  pstemp:=subs(xi=0,theta=0,innerprod(rowv,colv))
  +int(subs(xi=sigma+tau,theta=sigma,
  innerprod(rowv,A1,colv)),sigma=-tau .. 0);
  RETURN(pstemp)
  end:
```

Note that the command `innerprod` calculates the dot product when given two vectors and the vector-matrix-vector product when given two vectors and a matrix.

Next we apply the bilinear form to each row of $\mathbf{\Psi_g}$ and each column of $\mathbf{\Phi}$ and store the result in the matrix `produit`. We then invert `produit` and multiply the result by $\mathbf{\Psi_g}$. [4]

```
> rowvec:=array(1..2): colvec:=array(1..2): produit:=matrix(2,2):
> for I1 from 1 to 2 do
    for I2 from 1 to 2 do
     rowvec:=row(Psi_g,I1);
     colvec:=col(Phi,I2);
     produit[I1,I2]:=eval(bilinear_form(rowvec,colvec));
    od;
  od;
> K:=inverse(produit):
> PPsi:=map(simplify,multiply(K,Psi_g)):
```

In fact, all we need for subsequent calculations is $\mathbf{\Psi}(0)$. To keep the expressions from getting too large we will define an empty matrix $\Psi0$ to use as a place holder. We will store the actual values of $\mathbf{\Psi}(0)$ in the list `Psi0_vals`.

```
Psi0:=matrix(2,2);
Psi0_res:=map(simplify,map(eval,subs(xi=0,evalm(PPsi)))):
Psi0_vals:=[Psi0[1,1]=Psi0_res[1,1],Psi0[1,2]=Psi0_res[1,2],
            Psi0[2,1]=Psi0_res[2,1],Psi0[2,2]=Psi0_res[2,2]]:
```

$$\Psi0 = array(1..2, 1..2, [])$$

Now, to determine the criticality of the Hopf bifurcation, one need only find the terms up to and including those which are $O(\|\mathbf{u}(t)\|^3)$ in (27). Thus, as discussed in the previous section, we only need the quadratic terms in the series for $\mathbf{h}$. We thus define

```
h:=matrix([[h1_11(theta)*u1^2+h1_12(theta)*u1*u2
    +h1_22(theta)*u2^2],
  [h2_11(theta)*u1^2+h2_12(theta)*u1*u2+h2_22(theta)*u2^2]]);
```

---

[4] Note that `Psi` is a reserved word, so we use `PPsi` instead.

$$h := \begin{bmatrix} h1\_11(\theta)\,u1^2 + h1\_12(\theta)\,u1\,u2 + h1\_22(\theta)\,u2^2 \\ h2\_11(\theta)\,u1^2 + h2\_12(\theta)\,u1\,u2 + h2\_22(\theta)\,u2^2 \end{bmatrix} \qquad (40)$$

We define the linear and nonlinear parts of the DE as follows

```
> x:=matrix([[x1],[x2]]);
  xt:=matrix([[x1t],[x2t]]);
  lin:=evalm(multiply(A0,x)+multiply(A1,xt));
  f:= beta*p2*x2^2-beta*mu*p1*x1*x2+beta*mu*p1*x1t*x2
      -beta*mu*p2*x1*x2^2+beta*mu*p2*x1t*x2^2:
  nonlin:=matrix([[0],[f]]);
```

$$x := \begin{bmatrix} x1 \\ x2 \end{bmatrix}$$

$$xt := \begin{bmatrix} x1t \\ x2t \end{bmatrix}$$

$$lin := \begin{bmatrix} x2 \\ (-1 - \beta\,\mu\,p0)\,x1 + (\beta\,p1 - \delta)\,x2 + \beta\,\mu\,p0\,x1t \end{bmatrix}$$

$$nonlin := \begin{bmatrix} 0 \\ \beta\,p2\,x2^2 - \beta\mu\,p1\,x1\,x2 + \beta\,\mu\,p1\,x1t\,x2 - \beta\,\mu\,p2\,x1\,x2^2 + \beta\,\mu\,p2\,x1t\,x2^2 \end{bmatrix}$$

Then we define the expressions, in terms of the coordinates **u**, for points on the centre eigenspace, x_ce, on the centre manifold, x_cm, and on the nonlinear terms of the centre manifold, x_h.

```
> Phiu0:=map(eval,subs(theta=0,evalm(Phiu))):
  Phiut:=map(eval,subs(theta=-tau,evalm(Phiu))):
  x_ce:=[x1=Phiu0[1,1],x2=Phiu0[2,1],x1t=Phiut[1,1],
         x2t=Phiut[2,1]];
  h0:=map(eval,subs(theta=0,evalm(h))):
  ht:=map(eval,subs(theta=-tau,evalm(h))):
  x_cm:=[x1=Phiu0[1,1]+h0[1,1],x2=Phiu0[2,1]+h0[2,1],x1t=Phiut[1,1]+ht[1,1],
         x2t=Phiut[2,1]+ht[2,1]];
  x_h:=[x1=h0[1,1],x2=h0[2,1],x1t=ht[1,1],x2t=ht[2,1]]:
```

$$x\_ce := [x1 = u1,\ x2 = \omega\,u2,\ x1t = \cos(\omega\,\tau)\,u1 - \sin(\omega\,\tau)\,u2,$$
$$x2t = \sin(\omega\,\tau)\,\omega\,u1 + \cos(\omega\,\tau)\,\omega\,u2]$$

$$x\_cm := [x1 = u1 + h1\_11(0)\,u1^2 + h1\_12(0)\,u1\,u2 + h1\_22(0)\,u2^2,$$
$$x2 = \omega\,u2 + h2\_11(0)\,u1^2 + h2\_12(0)\,u1\,u2 + h2\_22(0)\,u2^2,$$
$$x1t = \cos(\omega\,\tau)\,u1 - \sin(\omega\,\tau)\,u2 + h1\_11(-\tau)\,u1^2$$
$$+ h1\_12(-\tau)\,u1\,u2 + h1\_22(-\tau)\,u2^2,$$
$$x2t = \sin(\omega\,\tau)\,\omega\,u1 + \cos(\omega\,\tau)\,\omega\,u2 + h2\_11(-\tau)\,u1^2$$
$$+ h2\_12(-\tau)\,u1\,u2 + h2\_22(-\tau)\,u2^2]$$

We can now define differential equations for the h^i_{jk}. First define the left hand side of (31).

```
> delhs:=map(diff,h,theta);
```

$$
delhs := \left[\begin{array}{c} \left(\dfrac{d}{d\theta}h1\_11(\theta)\right) u1^2 \;+\; \left(\dfrac{d}{d\theta}h1\_12(\theta)\right) u1\,u2 \;+\; \left(\dfrac{d}{d\theta}h1\_22(\theta)\right) u2^2 \\[2ex] \left(\dfrac{d}{d\theta}h2\_11(\theta)\right) u1^2 \;+\; \left(\dfrac{d}{d\theta}h2\_12(\theta)\right) u1\,u2 \;+\; \left(\dfrac{d}{d\theta}h2\_22(\theta)\right) u2^2 \end{array} \right]
$$

Now define the right hand side of (31).

```
> dhdu:=matrix([[diff(h[1,1],u1),diff(h[1,1],u2)],
                [diff(h[2,1],u1),diff(h[2,1],u2)]]);
```

$$
dhdu := \left[\begin{array}{cc} 2\,h1\_11(\theta)\,u1 \,+\, h1\_12(\theta)\,u2 & h1\_12(\theta)\,u1 \,+\, 2\,h1\_22(\theta)\,u2 \\ 2\,h2\_11(\theta)\,u1 \,+\, h2\_12(\theta)\,u2 & h2\_12(\theta)\,u1 \,+\, 2\,h2\_22(\theta)\,u2 \end{array} \right]
$$

```
> derhs:=map(collect,map(expand,evalm(multiply(dhdu,multiply(B,u))+
          multiply(Phi,multiply(Psi0,[0,subs(x_ce,f)])))),
          [u2,u2],distributed,factor):
```

The expression for `derhs` is quite long, so we don't display it. Now we put together the right hand side and left hand side. The coefficient of each distinct monomial, $u1^k u2^j$, $j+k=2$, determines one differential equation. We display two of them as examples.

```
> hdes:=delhs-derhs:
  de1:=coeff(coeff(hdes[1,1],u1^2),u2,0);
  de2:=coeff(coeff(hdes[1,1],u1),u2);
  de3:=coeff(coeff(hdes[1,1],u2^2),u1,0):
  de4:=coeff(coeff(hdes[2,1],u1^2),u2,0):
  de5:=coeff(coeff(hdes[2,1],u1),u2):
  de6:=coeff(coeff(hdes[2,1],u2^2),u1,0):
```

$$
de1 := \left(\frac{d}{d\theta}h1\_11(\theta)\right) + \omega\,h1\_12(\theta)
$$

$$
de2 := \left(\frac{d}{d\theta}h1\_12(\theta)\right) - 2\omega\,(h1\_11(\theta) - h1\_22(\theta)) + \omega\,\beta\,\mu\,p1(1-\cos(\omega\tau))
$$
$$
(\cos(\omega\theta)\,\mathbf{\Psi}_{1,2}(0) + \sin(\omega\theta)\,\mathbf{\Psi}_{2,2}(0))
$$

Now define the list of differential equations and functions to solve for.

```
> des:={de1,de2,de3,de4,de5,de6}:
  fns:={coeff(h[1,1],u1^2),coeff(coeff(h[1,1],u1),u2),
  coeff(h[1,1],u2^2), coeff(h[2,1],u1^2),
  coeff(coeff(h[2,1],u1),u2),coeff(h[2,1],u2^2)};
```

$$fns := \{h1\_11(\theta),\ h1\_12(\theta),\ h1\_22(\theta),\ h2\_11(\theta),\ h2\_12(\theta),\ h2\_22(\theta)\}$$

The DEs are linear and are easily solved to find the general solutions for the $h_{jk}^i(\theta)$ in terms of six arbitrary constants using the command `dsolve`. For convenience, we rename the arbitrary constants.

```
> temp:=dsolve(des,fns):
  changeC:=[_C1=C1,_C2=C2,_C3=C3,_C4=C4,_C5=C5,_C6=C6];
  hsoln:=simplify(expand(evalc(subs(changeC,value(temp))))):
```

The solutions are quite long, so we show only one example.

```
> collect(hsoln[6],[Psi0[1,1],Psi0[1,2],Psi0[2,1],Psi0[2,2],
  p1,p2],factor);
```

$$
\begin{aligned}
h1\_22(\theta) = {}& \left(-\frac{1}{3}\ \beta\,\mu\left(\cos(\omega\theta)+\sin(\omega\theta)\sin(\omega\tau)-\cos(\omega\theta)\cos(\omega\tau)\right)p1 \right. \\
&\left. +\frac{1}{3}\beta\,\sin(\omega\theta)\,\omega\,p2\right)\mathbf{\Psi}0_{1,2} + \left(\frac{1}{3}\ \beta\,\mu(\cos(\omega\theta)\sin(\omega\tau)\right. \\
&\left. -\sin(\omega\theta)+\sin(\omega\theta)\cos(\omega\tau))p1 -\frac{1}{3}\beta\,\cos(\omega\theta)\,\omega\,p2\right)\mathbf{\Psi}0_{2,2} \\
&-C6\,\cos(\omega\theta)^2 + C2 + \frac{1}{2}\,C6 + C5\,\sin(\omega\theta)\cos(\omega\theta)
\end{aligned}
$$

Recall that the values for $\mathbf{\Psi}0_{i,j}$ are stored in the list `Psi0_vals`. Later we will need the values of $h_{jk}^i(0)$ and $h_{jk}^i(-\tau)$, so we store them in the sets `hsoln0` and `hsolnt`.

```
> hsoln0:=simplify(eval(subs(theta=0,hsoln))):
  hsolnt:=simplify(eval(subs(theta=-tau,hsoln))):
```

We now set up the boundary conditions to solve for the arbitrary constants, $C1, C2, \ldots$. Note that the left hand side of (32) is just the right hand side of (31) with $\theta = 0$.

```
> bclhs:=map(eval,subs(theta=0,evalm(derhs))):
  bcrhs:=map(collect,evalm(subs(x_h,evalm(lin))+
  subs(x_ce,evalm(nonlin)))),[u1,u2]);
```

Now we put together the right hand side and left hand side. The coefficient of each distinct monomial, $u1^k u2^j$, $j + k = 2$, determines one boundary condition.

```
> consts:=[C1,C2,C3,C4,C5,C6];
  bceq:=subs(hsoln0,hsolnt,evalm(bclhs-bcrhs)):
  bc1:=collect(coeff(coeff(bceq[1,1],u1,2),u2,0),consts);
  bc2:=collect(coeff(coeff(bceq[1,1],u1,1),u2,1),consts):
```

```
bc3:=collect(coeff(coeff(bceq[1,1],u1,0),u2,2),consts):
bc4:=collect(coeff(coeff(bceq[2,1],u1,2),u2,0),consts);
bc5:=collect(coeff(coeff(bceq[2,1],u1,1),u2,1),consts):
bc6:=collect(coeff(coeff(bceq[2,1],u1,0),u2,2),consts):
```

Form the list of boundary conditions and solve using `solve`.

```
> bcs:={bc1,bc2,bc3,bc4,bc5,bc6}:
  consts:=convert(const,set);
  Csoln:=map(simplify,solve(bcs,consts)):
```

The solutions are quite long, so we show only one example.

```
> collect(Csoln[2],[Psi0[1,1],Psi0[1,2],Psi0[2,1],Psi0[2,2],
  p1,p2],factor);
```

$$
\begin{aligned}
C2 := &\left(-\frac{1}{2}\,\beta^2 \omega\,\mu\,\sin(\omega\,\tau)p1^2 + \left(\frac{1}{2}\,\beta^2\omega^2\,p2 + \frac{1}{2}\,\mu\,(\delta\,\omega\,\sin(\omega\,\tau) + \beta\,p0\,\mu \right.\right. \\
&\left.\left. -\beta\,p0\,\mu\,\cos(\omega\,\tau)^2\,)\,\beta \right) p1 - \frac{1}{2}\,\omega\,(\beta\,p0\,\mu\,\sin(\omega\,\tau) + \delta\,\omega)\,\beta\,p2 \right) \Psi0_{1,2} \\
&+ \left(\frac{1}{2}\,\mu\,\sin(\omega\,\tau)\,\left(-1 - \beta\,p0\,\mu + \beta\,p0\,\mu\,\cos(\omega\,\tau) + \omega^2 \right)\,\beta\,p1 \right. \\
&\left. -\frac{1}{2}\,\omega\,\left(-1 - \beta\,p0\,\mu + \beta\,p0\,\mu\,\cos(\omega\,\tau) + \omega^2 \right)\,\beta\,p2 \right) \Psi0_{2,2} \\
&-\frac{1}{2}\,\beta\,p1\,\mu\,\sin(\omega\,\tau)\,\omega + \frac{1}{2}\,\beta\,\omega^2\,p2
\end{aligned}
$$

The final step is to use the expressions for $\mathbf{\Psi}(0)$, $\mathbf{\Phi}$ and $\mathbf{h}$ to calculate the nonlinear terms of (27).

```
> fu:=collect(subs(x_cm,f),[u1,u2],distributed,factor);
  nonlinu:=matrix([[0],[fu]]):
  ODE_nonlin:=multiply(Psi0,nonlinu):
```

Note that we have used the fact that the first component of the nonlinearity in our example equation (34) is 0.

Recalling our expression for the matrix B, we can see that for our example, the general equation on the centre manifold (27) becomes (to $O(\|\mathbf{u}\|^3)$)

$$
\begin{aligned}
\dot{u}_1 &= \omega u_2 + f^1_{11}u_1^2 + f^1_{12}u_1 u_2 + f^1_{22}u_2^2 + f^1_{111}u_1^3 + f^1_{112}u_1^2 u_2 \\
&\quad + f^1_{122}u_1 u_2^2 + f^1_{222}u_2^3, \\
\dot{u}_2 &= -\omega u_1 + f^2_{11}u_1^2 + f^2_{12}u_1 u_2 + f^2_{22}u_2^2 + f^2_{111}u_1^3 + f^2_{112}u_1^2 u_2 \\
&\quad + f^2_{122}u_1 u_2^2 + f^2_{222}u_2^3.
\end{aligned}
\tag{41}
$$

The f^i_{jk} and f^i_{jkl} are functions of the parameters $\beta, \tau, \delta, \theta, p_0, p_1, p_2$, the Hopf frequency ω, and the centre manifold coefficients $h^i_{jk}(0)$ and $h^i_{jk}(-\tau)$. As should be expected, eq. (41) is an ODE at a Hopf bifurcation. The criticality

of this bifurcation (and hence of the Hopf bifurcation in the original system of DDE's) may be determined by applying standard approaches. For example, one can show that the criticality of the Hopf bifurcation of (41) is determined by the sign of the quantity (Guckenheimer and Holmes, 1983, p. 152)

$$a = \tfrac{1}{8}\left(3f_{111}^1 + f_{122}^1 + f_{112}^2 + 3f_{222}^2\right)$$
$$-\tfrac{1}{8\omega}\left(f_{12}^1(f_{11}^1 + f_{22}^1) - f_{12}^2(f_{11}^2 + f_{22}^2) - 2f_{11}^1 f_{11}^2 + 2f_{22}^1 f_{22}^2\right). \tag{42}$$

To evaluate this expression, we first find the coefficients of the quadratic terms.

```
> quad:=array(1..2,1..3):
> quad[1,1]:=coeff(coeff(ODE_nonlin[1,1],u1,2),u2,0);
> quad[1,2]:=coeff(coeff(ODE_nonlin[1,1],u1,1),u2,1);
> quad[1,3]:=coeff(coeff(ODE_nonlin[1,1],u1,0),u2,2);
> quad[2,1]:=coeff(coeff(ODE_nonlin[2,1],u1,2),u2,0);
> quad[2,2]:=coeff(coeff(ODE_nonlin[2,1],u1,1),u2,1);
> quad[2,3]:=coeff(coeff(ODE_nonlin[2,1],u1,0),u2,2);
```

$$quad_{1,1} := 0$$
$$quad_{1,2} := \Psi 0_{1,2}\,\omega\,\beta\,\mu\,p1(\cos(\omega\tau) - 1)$$
$$quad_{1,3} := \Psi 0_{1,2}\,\beta\,\omega\,(\omega\,p2 - \mu\,p1\,\sin(\omega\tau))$$
$$quad_{2,1} := 0$$
$$quad_{1,2} := \Psi 0_{2,2}\,\omega\,\beta\,\mu\,p1(\cos(\omega\tau) - 1)$$
$$quad_{1,3} := \Psi 0_{2,2}\,\beta\,\omega\,(\omega\,p2 - \mu\,p1\,\sin(\omega\tau))$$

The necessary cubic coefficients are found in a similar way.

```
>cub:=array(1..2,1..4):
>cub[1,1]:=coeff(coeff(ODE_nonlin[1,1],u1,3),u2,0);
>cub[1,3]:=coeff(coeff(ODE_nonlin[1,1],u1,1),u2,2):
>cub[2,2]:=coeff(coeff(ODE_nonlin[2,1],u1,2),u2,1):
>cub[2,4]:=coeff(coeff(ODE_nonlin[2,1],u1,0),u2,3);
```

$$cub_{1,1} := \Psi 0_{1,2}\,\beta\,\mu\,p1\,h2\_11(0)(\cos(\omega\tau) - 1)$$
$$cub_{1,3} := \Psi 0_{1,2}\,\beta\,\left(\mu\,p2\,\cos(\omega\,\tau)\omega^2 - \mu\,p2\,\omega^2 + p1\,\mu\,h1\_12(-\tau)\,\omega - p1\,\mu\,h2\_22(0)\right.$$
$$+2\,p2\,h2\_12(0)\,\omega - p1\,\mu\,h1\_12(0)\,\omega - p1\,\mu\,\sin(\omega\,\tau)\,h2\_12(0)$$
$$\left.+p1\,\mu\,\cos(\omega\,\tau)\,h2\_22(0)\right)$$
$$cub_{2,2} := -\Psi 0_{2,2}\,\beta\,\left(p1\,\mu\,h2\_12(0) - p1\,\mu\,h1\_11(-\tau)\,\omega + p1\,\mu\,\sin(\omega\,\tau)\,h2\_11(0)\right.$$
$$\left.-p1\,\mu\,\cos(\omega\,\tau)\,h2\_12(0) + p1\,\mu\,h1\_11(0)\,\omega - 2\,p2\,h2\_11(0)\,\omega\right)$$
$$cub_{2,4} := -\Psi 0_{2,2}\,\beta\,\left(\mu\,p2\,\sin(\omega\,\tau)\,\omega^2 + p1\,\mu\,\sin(\omega\,\tau)\,h2\_22(0) + p1\mu\,h1\_22(0)\,\omega\right.$$
$$\left.-p1\,\mu\,h1\_22(-\tau)\,\omega - 2\,p2\,h2\_22(0)\,\omega\right)$$

Note that only the cubic terms depend on the h_{jk}^i, as expected. The quantity a is evaluated using the formula of eq. (42)

```
a:=collect(simplify(1/8*(3*cub[1,1]+cub[1,3]+cub[2,2]
    +3*cub[2,4])-1/(8*omega)*(quad[1,2]*(quad[1,1]
    +quad[1,3])-quad[2,2]*(quad[2,1]+quad[2,3])
    -2*quad[1,1]*quad[2,1]+2*quad[1,3]*quad[2,3]))),
    [Psi0[1,2],Psi0[2,2]],distributed,factor);
```

$$a := \frac{1}{64}\,\beta^2\omega\,p1\,\mu\,(\cos{(\omega\,\tau)} - 1)\,(-\omega\,p2 + \mu\,p1\,\sin{(\omega\,\tau)})\,\Psi0_{1,2}^2 + \frac{1}{32}\,\beta^2\omega$$

$$\left(-\omega^2p2^2 - \mu^2p1^2 + 2\,\omega\,p2\,\mu\,p1\,\sin{(\omega\,\tau)} + \mu^2p1^2\,(\cos{(\omega\,\tau)})^2\right)\Psi0_{1,2}\Psi0_{2,2}$$

$$+\frac{1}{8}\,\beta\,\Big(\ p1\,\mu\,\cos{(\omega\,\tau)}\,h2\_22(0) - 3\,\mu\,p1\,h2\_11(0)$$

$$+3\,\mu\,p1\,h2\_11(0)\,\cos{(\omega\,\tau)} + 2\,p2\,h2\_12(0)\omega - p1\,\mu\,h2\_22(0)$$

$$+p1\,\mu\,h1\_12(-\tau)\omega - p1\,\mu\,\sin{(\omega\,\tau)}\,h2\_12(0) - \mu\,p2\,\omega^2 - p1\,\mu\,h1\_12(0)\omega$$

$$+\mu\,p2\,\cos{(\omega\,\tau)}\,\omega^2\ \Big)\,\Psi0_{1,2} - \frac{1}{64}\,\beta^2\omega\,p1\,\mu\,(\cos{(\omega\,\tau)} - 1)\,\Big(-\omega\,p2$$

$$+\mu\,p1\,\sin{(\omega\,\tau)}\ \Big)\Psi0_{2,2}^2 - \frac{1}{8}\,\beta\,\Big(-2\,p2\,h2\_11(0)\omega + p1\,\mu\,h2\_12(0)$$

$$-6\,p2\,h2\_22(0)\omega + p1\,\mu\,h1\_11(0)\omega + p1\,\mu\,\sin{(\omega\,\tau)}\,h2\_11(0)$$

$$-p1\,\mu\,\cos{(\omega\,\tau)}\,h2\_12(0) - p1\,\mu\,h1\_11(-\tau)\omega - 3\,p1\,\mu\,h1\_22(-\tau)\omega$$

$$+3\,\mu\,p2\,\sin{(\omega\,\tau)}\,\omega^2 + 3\,p1\,\mu\,h1\_22(0)\omega + 3\,p1\,\mu\,\sin{(\omega\,\tau)}\,h2\_22(0)\ \Big)\,\Psi0_{2,2}$$

To get the final expression for a we need to substitute in the actual values for $\Psi0$, $h^i_{jk}(0)$ and $h^i_{jk}(\tau)$. The expression is very large, so we do not print it out.

```
afinal:=subs(Psi0_vals,simplify(subs(Csoln,
        simplify(subs(hsoln0,hsolnt,a)))):
```

4 Discussion

In this chapter, we have shown how the symbolic algebra package Maple™can be used to calculate the centre manifold for a delay differential equation at a Hopf bifurcation. The commands involved are fairly simple, and thus it should be fairly easily to adapt them to other computer algebra systems.

The emphasis of this chapter was on a system at a Hopf bifurcation. The implementation for other bifurcations is similar, and just requires the modifying the following parts:

1. The calculation of the basis functions for the centre eigenspace for the original and transpose systems ($\mathbf{\Phi}$, $\mathbf{\Psi}$).
2. The calculation of the quantity that determines the criticality. This depends on the normal form for the bifurcation involved.

Also, as discussed in section 2, for some systems and some bifurcations, it may not be necessary to compute the nonlinear terms of the centre manifold.

This chapter has focused on systems *at* a bifurcation. This means that our predictions of the stability of the bifurcating limit cycle will only be valid in some neighbourhood of the bifurcation point. To get predictions which valid in a larger region, one can use the approach of *parameter dependent* centre manifolds. For a general outline of this approach and some specific examples for ordinary differential equations see (Guckenheimer and Holmes, 1983, p. 134) or (Wiggins, 1990, p. 198). For the application of the approach to DDEs with a Hopf bifurcation see Faria and Magalhães (1995b) or Qesmi et al. (2006b). For the application of this approach to DDEs with a Bogdanov-Takens singularity see Faria and Magalhães (1995a) or Qesmi et al. (2007). For the application of this approach to delay differential equations with a Fold-Hopf singularity see Qesmi et al. (2006a). Note that the papers of Qesmi et al. have some discussion of the implementation of their algorithms in Maple.

Finally, we would note that there are other approaches for studying the dynamics of a delay differential equation near a nonhyperbolic equilibrium point. Perturbation techniques (multiple scales, Poincaré-Lindstedt, averaging) have been used to study Hopf bifurcation by Campbell et al. (2006); Chow and Mallet-Paret (1977); Das and Chatterjee (2002); Gopalsamy and Leung (1996); Nayfeh (2008); Rand and Verdugo (2007); Stech (1979) and Wirkus and Rand (2004). Often the computations for such methods are as extensive than for the centre manifold, however the mathematical theory is more approachable. The Liapunov-Schmidt reduction has also been implemented for delay differential equations, both numerically by Stech (1985a,b,c); Aboud et al. (1988) and using symbolic algebra by Franke and Stech (1991). However, this method only determines existence of the bifurcating solutions. Some other method must be used to determined stability. Stech (1985b,c) discuss such a method.

5 Acknowledgements

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