DELAY INDEPENDENT STABILITY FOR ADDITIVE NEURAL NETWORKS

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Abstract. An additive neural network model consisting of n elements with arbitrary connections, nonlinear activation functions and multiple time delays is considered. Necessary and sufficient conditions for the delay independent local asymptotic stability of equilibrium solutions are determined. Slightly stronger conditions are shown to ensure the delay independent global stability of an equilibrium solution. These results and other delay independent stability results in the literature are compared.

1. Introduction. In 1978, Grossberg [21] considered the equation

$$\dot{u}_i = d_i(u_i)\left[-b_i(u_i) + \sum_{j=1}^{n} a_{ij} g_j(u_j)\right], \tag{1.1}$$

which, for appropriate choices of the functions $d_i,\ b_i,\ g_i$ encompasses a large variety of biological models, including several types of neural networks. In [21] and subsequent papers [9, 22] Grossberg et al. studied pattern formation and stability in this model. In 1984, Hopfield [26] considered a special case of model (1.1)

$$\dot{u}_i = -b_i u_i + \sum_{j=1}^{n} a_{ij} g_j(u_j), \tag{1.2}$$

as a continuous extension of a discrete, two state neural network model he had previously studied [25]. In equation (1.2), which we shall call the additive neural network model, u_i can be thought of as the mean soma potential of the ith neuron. Then each neuron is represented by a linear electrical circuit consisting of a resistor and capacitor, and is connected to the other neurons via nonlinear (usually sigmoidal) activation functions. The studies outlined above assume instantaneous updating of each neuron and instantaneous communication between the neurons, and thus the models are systems of ordinary differential equations.

Early work of Grossberg [16, 17, 18] considered the effect of including discrete time delays in neural network models to represent the signal propagation time between neurons. These models consist of equations for the neural activity, such as (1.2), with time delays in the connection terms, supplemented by equations modelling the variation of the synaptic weights, a_{ij}, due to learning. Although quite complicated systems of delay differential equations, results on the learning and pattern recognition properties of these networks were obtained. These are summarized and extended in [19, 20].

In 1989, Marcus and Westervelt [31] considered the simpler model of just (1.2) with time delays in the connection terms. The analysis of this model is still quite complicated, and thus [30, 31] and much of the initial work on these delayed, additive neural networks considered the situation where all the time delays in the network are the same. Examples include the work of Bélair et al. [4, 5, 6], Gopalsamy and Leung [15], Ye et al. [42] and Pakdaman et al. [33] (see also references therein).

Preliminary work on (1.2) with multiple time delays simplified the analysis by either restricting the size of the network (e.g. [32, 34, 37]), or considering networks with simple architectures (e.g. [2, 8, 35]). Recent work has focussed on the global stability of equilibrium points of an arbitrary sized network with different time delays in each connection term. These papers generally formulate conditions on the connection matrix under which there is global stability either independent of all the delays, as in Gopalsamy and He [13, 14], or under additional conditions on some delays, as in van den Driessche et al. [40, 41] and Ye et. al. [43]. (This last article considered the general model (1.1) with time delays in the connection terms). By contrast, the papers of Hofbauer and So [24] and of So, Tang and Zou [38] formulate conditions related to the linear asymptotic stability of additive neural networks with multiple time delays.

The purpose of this paper is to extend the results of [14] and [24] and to compare some of the delay independent and delay dependent criteria for stability found in the literature. We thus consider the nonlinear delay differential equation

$$\dot{u}_i(t) = -b_i u_i(t) + \sum_{j=1}^{n} a_{ij} g_j(u_j(t-\tau_{ij})) + J_i, \tag{1.3}$$

where the RC decay constants b_i are positive and the time delays τ_{ij} are nonnegative. We will allow the connection coefficients a_{ij} to be arbitrary real numbers, with positive (negative) numbers representing excitatory (inhibitory) connections. Similarly, the neuron inputs, J_i, can be arbitrary real numbers. We will assume that the nonlinear functions g_j satisfy

(H1) $g_j \in C^2(\mathbb{R}), \qquad g_j'(u) > 0, \qquad \sup_{u\in\mathbb{R}} g_j'(u) = g_j'(0) = 1,$

(H2) $g_j(0) = 0, \qquad \lim_{u\to\pm\infty} g_j(u) = \pm 1.$

The function $g(u) = \tanh(u)$, which is commonly used in the model (1.3), satisfies these conditions.

We will consider initial value problems consisting of (1.3) subject to the conditions

$$u_i(\theta) = \phi_i(\theta), \;\; h \le \theta \le 0, \tag{1.4}$$

where $h = \max_{i,j}\{\tau_{ij}\}$ and the ϕ_i are continuous. The local existence and uniqueness of solutions is then guaranteed (see e.g. the book of Hale and Lunel [23]).

The plan for the article is as follows. In section 2 we consider the linear stability analysis of equilibrium points of (1.3), giving necessary and sufficient conditions for them to be asymptotically stable for all $\tau_{ij} \ge 0$. In section 3 we show how related conditions give delay independent global stability of an equilibrium point. In section 4 we compare our results and those of [14, 38] and [40] and in section 5 we summarize our results and draw some conclusions.

2. Linear Stability Analysis. Equilibrium points of (1.3) are solutions $\mathbf{u}(t) = \mathbf{u}^*,\ t \ge -h$. We will study the existence of such equilibrium points in section 3. Assuming that such an equilibrium point exists one can translate it to the origin via the transformation $\mathbf{u}(t) = \mathbf{u}^* + \mathbf{x}(t)$, yielding the equations

$$\dot{x}_i = -b_i x_i(t) + \sum_{j=1}^{n} a_{ij}\left[g_j(x_j(t-\tau_{ij}) + u_j^*) - g_j(u_j^*)\right]. \tag{2.1}$$

Due to (H1) we can use the linearization

$$\dot{x}_i = -b_i x_i(t) + \sum_{j=1}^{n} c_{ij} x_j(t-\tau_{ij}), \tag{2.2}$$

where $c_{ij} = a_{ij} g_j'(u_j^*)$, to study the asymptotic stability of the trivial solution of (2.1) and hence of the equilibrium point $\mathbf{u}^*$ of (1.3). To do this we consider the roots of the characteristic equation,

$$\Delta(\lambda) \stackrel{def}{=} \det\begin{pmatrix} -b_1 + c_{11}e^{-\lambda\tau_{11}} - \lambda & c_{12}e^{-\lambda\tau_{12}} & \cdots & c_{1n}e^{-\lambda\tau_{1n}} \\ c_{21}e^{-\lambda\tau_{21}} & -b_2 + c_{22}e^{-\lambda\tau_{22}} - \lambda & \cdots & c_{2n}e^{-\lambda\tau_{2n}} \\ & \vdots & & \\ c_{n1}e^{-\lambda\tau_{n1}} & c_{n2}e^{-\lambda\tau_{n2}} & \cdots & -b_n + c_{nn}e^{-\lambda\tau_{nn}} - \lambda \end{pmatrix} = 0, \tag{2.3}$$

which correspond to nontrivial solutions of (2.2) of the form $\mathbf{x}(t) = e^{\lambda t}\mathbf{v}$.

Consider the matrices $B = diag(b_1, b_2, \ldots, b_n)$, $C = (c_{ij})$ and $|C| = (|c_{ij}|)$ associated with (2.2) and define the matrices $K = -B + C$ and $\hat{K} = -B + |C|$. In the following we will determine conditions on the matrices K and $\hat{K}$ which are necessary and sufficient for all the roots of (2.3) to have negative real parts, independent of the size of the delays τ_{ij}. It follows that these will be necessary and sufficient conditions for the trivial solution of eq. (2.2) to be asymptotically stable and sufficient conditions for the equilibrium point $\mathbf{u} = \mathbf{u}^*$ of nonlinear system (1.3) to be locally asymptotically stable (see the books of Kolmanovskii and Nosov [28], Stépán [39], or Hale and Lunel [23]). Our conditions are a slight generalization of those presented in [24], and our proofs similar to theirs. The main difference is that we allow delays in the self connection terms, i.e. the τ_{ii} are not necessarily all zero.

LEMMA 2.1. *$\lambda = 0$ is a solution of equation (2.3) if and only if* $\det K = 0$.

Proof. The proof follows immediately from the fact that $\Delta(0) = \det K$. □

Before proceeding further, we state some standard results from Matrix Theory.

DEFINITION 2.2. *Let A be an $n \times n$ matrix whose off-diagonal elements are all nonpositive. We shall call A an* **M-matrix** *if all its principal minors are nonnegative.*

There are many equivalent properties which can be used to define an M-matrix (see Fiedler [11, Theorem 5.3]), we have chosen the one most useful in our analysis. Note that our definition allows an M-matrix to be singular. In some texts [11] such matrices are referred to as class K_0 matrices. In [24] such matrices are called *weakly diagonally dominant*.

LEMMA 2.3. *If $-\hat{K}$ is an M-matrix and* $\det K \neq 0$ *then the all the roots of (2.3) have negative real parts for all $\tau_{ij} \geq 0,\ 1 \leq i, j \leq n$.*

Proof. First consider $-\hat{K}$ irreducible. Then by [11, Theorem 5.9], there exist $\gamma_i > 0,\ i = 1, \ldots, n$ such that

$$\hat{k}_{ii}\gamma_i + \sum_{j \neq i} \hat{k}_{ij}\gamma_j \leq 0, \qquad \text{for } i = 1, \ldots, n.$$

That is

$$(-b_i + |c_{ii}|)\gamma_i + \sum_{j \neq i} |c_{ij}|\gamma_j \leq 0, \qquad \text{for } i = 1, \ldots, n. \tag{2.4}$$

Let λ be a root of (2.3). Then λ is an eigenvalue of the matrix $D = (d_{ij})$, where $d_{ii} = -b_i + c_{ii}e^{-\lambda\tau_{ii}}$ and $d_{ij} = c_{ij}e^{-\lambda\tau_{ij}}$. Applying Geršgorin's theorem (cf. Lancaster and Tismenetsky [29, p. 371]) to the matrix $\hat{D} = (\gamma_i^{-1} d_{ij}\gamma_j)$ we know that each eigenvalue, $\hat{\lambda}$, of $\hat{D}$ satisfies

$$|\hat{\lambda} - d_{ii}| \leq \sum_{j \neq i} \gamma_i^{-1}|d_{ij}|\gamma_j,$$

for at least one $i \in \{1, 2, \ldots, n\}$. Now D is similar to $\hat{D}$ so they have the same eigenvalues. Thus for each eigenvalue λ of D there is an i such that

$$\mathrm{Re}(\lambda) \leq \mathrm{Re}(d_{ii}) + \sum_{j \neq i} \gamma_i^{-1}|d_{ij}|\gamma_j.$$

Now suppose that $\mathrm{Re}(\lambda) \geq 0$. Then $\mathrm{Re}(d_{ii}) \leq -b_i + |c_{ii}|$, $|d_{ij}| \leq |c_{ij}|$ and

$$\mathrm{Re}(\lambda) \leq -b_i + |c_{ii}| + \sum_{j \neq i} \gamma_i^{-1}|c_{ij}|\gamma_j \tag{2.5}$$

Hence

$$\begin{aligned}\gamma_i \mathrm{Re}(\lambda) &\le (-b_i + |c_{ii}|)\gamma_i + \sum_{j\neq i} |c_{ij}|\gamma_j \\ &\le 0,\end{aligned}$$

which implies $\mathrm{Re}(\lambda) \le 0$. Now equality occurs in (2.5) only when λ is real and $\det K \neq 0$ precludes $\lambda = 0$, hence we have a contradiction. We conclude that $\mathrm{Re}(\lambda) < 0$ for all eigenvalues λ of D.

Now consider $-\hat{K}$ reducible. Then there exists a permutation matrix P such that

$$P(-\hat{K})P^T = \begin{bmatrix} \tilde{K}_{11} & 0 & 0 & \cdots & 0 \\ \tilde{K}_{21} & \tilde{K}_{22} & 0 & \cdots & 0 \\ & & \vdots & & \\ \tilde{K}_{n1} & \tilde{K}_{n2} & \tilde{K}_{n3} & \cdots & \tilde{K}_{nn} \end{bmatrix},$$

where each $\tilde{K}_{ii}$ is square and either irreducible or a 1×1 null matrix (cf. Berman and Plemmons [7, p. 39]). Since the permutations won't change the principle minors, each $\tilde{K}_{ii}$ will be an M-matrix.

Let λ be a root of (2.3) and define D as above. Then PDP^T will also be lower block diagonal, with blocks $\tilde{D}_{ii}$ corresponding to the $\tilde{K}_{ii}$. Thus λ is root of

$$\begin{aligned}\Delta(\lambda) &= \det(D - \lambda I) \\ &= \pm \det(PDP^T - \lambda I) \\ &= \pm \prod_i \det(\tilde{D}_{ii} - \lambda I).\end{aligned}$$

The rest of the proof follows by applying the argument for the irreducible case to each block, $\tilde{K}_{ii}$.
$\square$

LEMMA 2.4. *If* $\det(-\hat{K}) < 0$, *then there exists delays* $\tau_{ij} \ge 0$ *such that (2.3) has a root* λ *with* $Re(\lambda) > 0$.

Proof. Consider the function

$$F_\epsilon(z) = \det \begin{pmatrix} -b_1 + c_{11}e^{-z\eta_{11}} - \epsilon z & c_{12}e^{-z\eta_{12}} & \cdots & c_{1n}e^{-z\eta_{1n}} \\ c_{21}e^{-z\eta_{21}} & -b_2 + c_{22}e^{-z\eta_{22}} - \epsilon z & \cdots & c_{2n}e^{-z\eta_{2n}} \\ \vdots & \vdots & & \vdots \\ c_{n1}e^{-z\eta_{n1}} & c_{n2}e^{-z\eta_{n2}} & \cdots & -b_n + c_{nn}e^{-z\eta_{nn}} - \epsilon z \end{pmatrix}$$

where

$$\eta_{ij} = \begin{cases} \frac{1}{2}, & \text{if } c_{ij} < 0 \\ 1, & \text{if } c_{ij} \ge 0. \end{cases}$$

For $z = x + 2\pi i$, where x is real, $F_0(z)$ becomes

$$D(x) = \det \begin{pmatrix} -b_1 + |c_{11}|e^{-x\eta_{11}} & |c_{12}|e^{-x\eta_{12}} & \cdots & |c_{1n}|e^{-x\eta_{1n}} \\ |c_{21}|e^{-x\eta_{21}} & -b_2 + |c_{22}|e^{-x\eta_{22}} & \cdots & |c_{2n}|e^{-x\eta_{2n}} \\ \vdots & \vdots & & \vdots \\ |c_{n1}|e^{-x\eta_{n1}} & |c_{n2}|e^{-x\eta_{n2}} & \cdots & -b_n + |c_{nn}|e^{-x\eta_{nn}} \end{pmatrix}.$$

By assumption, $(-1)^n D(0) = \det(-\hat{K}) < 0$. Further $(-1)^n \lim_{x\to\infty} D(x) = b_1 b_2 \cdots b_n > 0$. Hence, by the Intermediate Value Theorem, there exists $\hat{x} > 0$ such that $D(\hat{x}) = 0$ and $\hat{z} = \hat{x} + 2\pi i$ is a zero of

F_0. It follows from Rouché's Theorem (see Ahlfors [1, p. 153]) that the analytic function F_ϵ has a zero $\hat{z}(\epsilon)$ near $\hat{z}$ for small $\epsilon > 0$. Clearly, $\lambda = \hat{z}(\epsilon)\epsilon$ and $\tau_{ij} = \dfrac{\eta_{ij}}{\epsilon}$ satisfy (2.3), with $\text{Re}(\lambda) > 0$. ◻

LEMMA 2.5. *If* $\det K \neq 0$ *and* $-\hat{K}$ *is not an M-matrix, then there exists delays* $\tau_{ij} \geq 0$ *such that (2.3) has a root* λ *with* $Re(\lambda) > 0$.

Proof. Since $-\hat{K}$ is not an M-matrix, it follows that some principle minor of order m of $-\hat{K}$ is negative. For notational convenience, we will assume that $\det(-\hat{K}_m) < 0$ where $\hat{K}_m$ is the $m \times m$ submatrix obtained by keeping the first m rows and columns of $\hat{K}$; the argument is the same for any principle minor. Applying Lemma 2.4 to $\hat{K}_m$ shows that the characteristic equation corresponding to this submatrix, i.e.

$$\Delta_m(\lambda) \stackrel{def}{=} \det \begin{pmatrix} -b_1 + c_{11}e^{-\lambda\tau_{11}} - \lambda & c_{12}e^{-\lambda\tau_{12}} & \cdots & c_{1m}e^{-\lambda\tau_{1m}} \\ c_{21}e^{-\lambda\tau_{21}} & -b_2 + c_{22}e^{-\lambda\tau_{22}} - \lambda & \cdots & c_{2m}e^{-\lambda\tau_{2m}} \\ & \vdots & & \\ c_{m1}e^{-\lambda\tau_{m1}} & c_{m2}e^{-\lambda\tau_{m2}} & \cdots & -b_m + c_{mm}e^{-\lambda\tau_{mm}} - \lambda \end{pmatrix} = 0,$$

has a root $\hat{\lambda}$ with $\text{Re}(\hat{\lambda}) > 0$.

Applying Laplace's Theorem [29, p. 37], $\Delta(\lambda)$ can be written

$$\Delta(\lambda) = \Delta_m(\lambda)\left[P(\lambda, e^{-\lambda\tau_{ii}}) + Q(\lambda, e^{-\lambda\tau_{ij}})\right] + R(\lambda, e^{-\lambda\tau_{ij}}),$$

where P, Q, R are quasipolynomials with

$$P(\lambda) = \prod_{i=m}^{n} (-b_i + c_{ii}e^{-\lambda\tau_{ii}} - \lambda)$$

and $|Q|$ and $|R|$ can be made arbitrarily small, for $\lambda > 0$, by taking $\tau_{ij},\ (m+1) \leq i, j \leq n$ large enough. Thus by Rouché's Theorem (cf. [10, p. 247]) (2.3) also has a root with positive real part. ◻

THEOREM 2.6. *The trivial solution of equation (2.2) is asymptotically stable for all delays* $\tau_{ij} \geq 0$ *if and only if* $-\hat{K}$ *is an M-matrix and* $\det K \neq 0$.

Proof. The trivial solution of (2.2) will be asymptotically stable if and only if all roots λ of (2.3) have negative real parts. Thus necessity follows from Lemmas 2.1 and 2.5 and sufficiency follows from Lemma 2.3. ◻

If $a_{ij} > 0$ for all i, j then the conditions of Theorem 2.6 reduce to:

$-K = -\hat{K}$ is a nonsingular M-matrix. (See Definition 3.1 below.)

COROLLARY 2.7. *Let* $\det K \neq 0$. *Then the equilibrium solution* $\mathbf{u}^*$ *of (1.3) is locally asymptotically stable for all delays* $\tau_{ij} \geq 0$ *if and only if* $-\hat{K}$ *is an M-matrix.*

Proof. The sufficiency follows directly from Theorem 2.6 and standard results on nonlinear delay differential equations [23]. The necessity follows from Lemma 2.5. ◻

The result for (1.3) is slightly weaker than that for (2.2) as the situation when $\det K = 0$ will depend on the exact nonlinearities in the model.

3. Global Stability. In this section we will show, in a manner similar to that in [14], that conditions slightly stronger than those of Theorem 2.6 imply global stability of an equilibrium point of the nonlinear equation (1.3). To begin, we introduce the following definition.

DEFINITION 3.1. *Let* A *be an* $n \times n$ *matrix whose off-diagonal elements are all nonpositive. We shall call* A *a* **nonsingular M-matrix** *if all its principal minors are positive.*

Consider the matrices $B = diag(b_1, b_2, \ldots, b_n)$ and $A = (a_{ij})$ associated with (1.3). Define the matrices $\mathcal{K}$ and $\hat{\mathcal{K}}$ via $\mathcal{K} = -B + A$ and $\mathcal{K} = -B + |A|$, where $|A|$ is the matrix with elements $|a_{ij}|$.

PROPOSITION 3.2. *Assume (H1) and (H2) hold. If* $-\hat{\mathcal{K}}$ *is a nonsingular M-matrix then (1.3) has a unique equilibrium point.*

Proof. If $-\hat{\mathcal{K}}$ is a nonsingular M-matrix then so is its transpose. Thus by a well-known Theorem ([7, Theorem 62.3], ([11, Theorem 5.1]) there are constants $\gamma_j > 0,\ j = 1, \ldots, n$ such that

$$-b_j\gamma_j + \sum_{i=1}^{n} |a_{ij}|\gamma_i < 0,\ j = 1, \ldots, n. \tag{3.1}$$

Hence it follows that

$$\beta \stackrel{def}{=} \max_{1\le j\le n} \left(\frac{1}{\gamma_j b_j} \sum_{i=1}^{n} |a_{ij}|\gamma_i \right) < 1.$$

Now equation (1.3) will have an equilibrium point, $\mathbf{u}^* = (u_1^*, u_2^*, \ldots, u_n^*)^T$, if the u_i^* satisfy

$$b_i u_i = \sum_{j=1}^{n} a_{ij} g_j(u_j) + J_i,\ i = 1, \ldots, n. \tag{3.2}$$

Multiplying equation i by $\gamma_i > 0$ yields the equivalent equations

$$\gamma_i b_i u_i = \sum_{j=1}^{n} \gamma_i a_{ij} g_j(u_j) + \gamma_i J_i,\ i = 1, \ldots, n.$$

Define $v_i = \gamma_i b_i u_i$. Then the u_i satisfy (3.2) if and only if the v_i satisfy

$$\begin{aligned} v_i &= \sum_{j=1}^{n} \gamma_i a_{ij} g_j\left(\frac{v_j}{\gamma_j b_j}\right) + \gamma_i J_i, && (3.3)\\ &\stackrel{def}{=} G_i(v_1, v_2, \ldots, v_n). && (3.4) \end{aligned}$$

Thus we seek fixed points of the map $\mathbf{G} : \mathbb{R}^n \to \mathbb{R}^n$, defined by $\mathbf{G} = (G_1(\mathbf{v}), \ldots, G_n(\mathbf{v}))$, $\mathbf{v} = (v_1, \ldots, v_n)$. Clearly, from the conditions (H1) and (H2) on the g_i, the G_i satisfy

$$\zeta_i^- = \gamma_i(J_i - \sum_{j=1}^{n} |a_{ij}|) \le G_i(\mathbf{v}) \le \gamma_i(J_i + \sum_{j=1}^{n} |a_{ij}|) = \zeta_i^+.$$

Thus $\mathbf{G}$ maps the set

$$S = \{(v_1, \ldots, v_n) | \zeta_i^- \le v_i \le \zeta_i^+,\ i = 1, \ldots, n\}$$

into itself.

Further

$$\begin{aligned} ||\mathbf{G}(\mathbf{v}) - \mathbf{G}(\mathbf{w})|| &= \sum_{i=1}^{n} |G_i(\mathbf{v}) - G_i(\mathbf{w})| \\ &\le \sum_{i=1}^{n}\sum_{j=1}^{n} \gamma_i |a_{ij}| \left| f_j\left(\frac{v_j}{\gamma_j b_j}\right) - f_j\left(\frac{w_j}{\gamma_j b_j}\right) \right| \\ &\le \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\gamma_i |a_{ij}|}{\gamma_j b_j} |v_j - w_j| \end{aligned}$$

where the latter follows from condition (H1) and the Mean Value Theorem. Rearranging the subscripts we see that

$$\begin{aligned}
||\mathbf{G}(\mathbf{v}) - \mathbf{G}(\mathbf{w})|| &\leq \sum_{j=1}^{n} \left(\frac{1}{\gamma_j b_j} \sum_{i=1}^{n} \gamma_i |a_{ij}| \right) |v_j - w_j| \\
&\leq \beta \sum_{j=1}^{n} |v_j - w_j| \\
&= \beta ||\mathbf{v} - \mathbf{w}||,
\end{aligned}$$

where $\beta < 1$. Thus $\mathbf{G}$ is a contraction on S and by the contraction mapping principle, $\mathbf{G}$ has a unique fixed point. It follows that (1.3) has a unique equilibrium point. ☐

THEOREM 3.3. *Assume (H1) and (H2) hold. If* $-\hat{\mathcal{K}}$ *is a nonsingular M-matrix then the equilibrium point of (1.3) is globally asymptotically stable.*

Proof. Let $\gamma_i > 0$ be as in the previous proof and define

$$\mu = \min_{1 \leq j \leq n} \left\{ b_j \gamma_j - \sum_{i=1}^{n} |a_{ij}| \gamma_i \right\}.$$

It follows from (3.1) that $\mu > 0$. Let $x_i(t) = u_i(t) - u_i^*$ and recall that the delay differential equation (2.1) governs the time evolution of $x_i(t)$. As mentioned in the introduction, we will assume that this equation is subject to continuous initial data of the form (1.4), in which case the local existence and uniqueness of solutions for $x_i(t)$ is guaranteed.

Consider a Lyapunov functional $V(t) = V(\mathbf{x})(t)$ defined by

$$V(\mathbf{x})(t) = \sum_{i=1}^{n} \gamma_i |x_i(t)| + \sum_{i=1}^{n} \gamma_i \sum_{j=1}^{n} |a_{ij}| \int_{t-\tau_{ij}}^{t} |x_j(s)|\, ds. \tag{3.5}$$

Then the upper righthand derivative of V along the solutions of (1.3) is given by

$$\begin{aligned}
D^+V(t) \quad &\leq \quad \sum_{i=1}^{n} \gamma_i \left(-b_i |x_i(t)| + \sum_{j=1}^{n} |a_{ij}| |g_j(x_j(t-\tau_{ij}) + u_j^*) - g_j(u_j^*)| \right) \\
&\quad\quad + \sum_{i=1}^{n} \gamma_i \sum_{j=1}^{n} |a_{ij}| \left(|x_j(t)| - |x_j(t-\tau_{ij})| \right) \\
&\leq \quad \sum_{i=1}^{n} (-\gamma_i b_i |x_i(t)|) + \sum_{i=1}^{n} \sum_{j=1}^{n} \gamma_i |a_{ij}| |x(t-\tau_{ij})| \\
&\quad\quad + \sum_{i=1}^{n} \sum_{j=1}^{n} \gamma_i |a_{ij}| |x_j(t)| - \sum_{i=1}^{n} \sum_{j=1}^{n} \gamma_i |a_{ij}| |x_j(t-\tau_{ij})| \\
&\leq \quad \sum_{j=1}^{n} \left(-\gamma_j b_j + \sum_{i=1}^{n} \gamma_i |a_{ij}| \right) |x_j(t)| \\
&\leq \quad -\mu \sum_{j=1}^{n} |x_j(t)|.
\end{aligned} \tag{3.6}$$

The boundedness of $\sum_{i=1}^{n} |x_i(t)|$ for all $t \geq 0$ follows from (3.5) and (3.6). Thus, the solutions of (2.1) exist for all $t \geq 0$. We have from (3.5) and (3.6) that

$$V(x)(t) + \mu \int_0^t \left(\sum_{i=1}^{n} |x_i(s)| \right) ds \leq V(x)(0).$$

It follows that

$$\sum_{i=1}^{n} |x_i(t)| \in \mathrm{L}_1(0,\infty). \tag{3.7}$$

The boundedness of $x_i(t)$ on $(0,\infty)$ implies that of the derivative of $x_i(t)$ on $(0,\infty)$; hence $x_i(t)$ is uniformly continuous on $(0,\infty)$. The uniform continuity of $\sum_{i=1}^{n} |x_i(t)|$ on $(0,\infty)$ together with (3.7) implies that

$$\sum_{i=1}^{n} |x_i(t)| \to 0 \qquad \text{as } t \to \infty. \tag{3.8}$$

This last assertion is a consequence of a lemma due to Barbălat [3]. (The statement and proof of this lemma can be found in the book of Gopalsamy [12, pp. 4–5]). □

It was pointed out in [41] that [14, Proposition 1] remains valid (with a slight modification of the Lyapunov function) under weaker assumptions on the nonlinearities: each of the g_j is globally Lipschitz and globally bounded. The same is true of Theorem 3.3.

4. Comparison of Results. Campbell [8] considered a delayed additive neural network which is a unidirectional ring, and is modelled by the following system:

$$\dot{u}_i(t) = -b_i u_i(t) + a_{ii} g_i(u_i(t-\tau_{ii})) + a_{i,i-1} g_{i-1}(u_{i-1}(t-\tau_{i-1})), \quad i = 1,\ldots,n. \tag{4.1}$$

Due to the structure of system, the characteristic equation is amenable to direct analysis and the exact region of linearized stability of the trivial solution of (4.1) can be described. In this section, we will use this model and its stability region as a benchmark for the criteria for local and global stability obtained in sections 2 and 3 and in the work of other authors. That is, we will compare the regions of stability predicted by each set of criteria when applied to this model.

4.1. Linear Stability. We begin our comparison of criteria for linearized stability by stating a slight generalization of a result of [8] concerning stability independent of all the delays.

THEOREM 4.1. *The trivial solution of (4.1) is linearly asymptotically stable for all* $\tau_{ij} \geq 0$ *if*

$$|a_{ii}| < b_i, \; i = 1,\ldots n \qquad \textit{and} \qquad \left|\prod_{i=1}^{n} a_{i,i-1}\right| < \prod_{i=1}^{n} (b_i - |a_{ii}|). \tag{4.2}$$

If $|a_{ii}| > b_i$ *for at least one* i*, then there no such region of delay independent stability.* The Theorem in [8] is stated for the case that the diagonal delays are all equal, i.e. $\tau_{ii} = \tau_s, \; i = 1,\ldots,n$. However, the proof there holds in the case that the τ_{ii} are not equal.

Consideration of the trivial solution of model (4.1) shows that for this model the matrix $\mathcal{K} = K$ and is given by

$$\begin{bmatrix} -b_1 + |a_{11}| & 0 & \cdots & 0 & |a_{1n}| \\ |a_{21}| & -b_2 + |a_{22}| & 0 & \cdots & 0 \\ & & \vdots & & \\ 0 & \cdots & 0 & |a_{n,n-1}| & -b_n + |a_{nn}| \end{bmatrix} \tag{4.3}$$

Applying Corollary 2.7 shows that the trivial solution of (4.1) will be linearly asymptotically stable for all $\tau_{ij} > 0$ if

$$\begin{gathered}
-b_i \le a_{ii} < b_i,\ i = 1, \dots n \\
\text{and} \\
\prod_{i=1}^{n}(b_i - a_{ii}) \le \prod_{i=1}^{n} a_{i,i-1} < \prod_{i=1}^{n}(b_i - a_{ii}), \qquad \text{if } a_{ii} > 0,\ i = 1, \dots, n, \\
\left|\prod_{i=1}^{n} a_{i,i-1}\right| \le \prod_{i=1}^{n}(b_i - |a_{ii}|), \qquad \text{otherwise.}
\end{gathered} \tag{4.4}$$

There will be no delay independent stability if $|a_{ii}| > b_i$ for at least one i or

$$\left|\prod_{i=1}^{n} a_{i,i-1}\right| > \prod_{i=1}^{n}(b_i - |a_{ii}|).$$

This gives a slight extension to Theorem 4.1.

The following is a restatement and generalization of a result of [8] concerning stability independent of the *off-diagonal* delays.

Theorem 4.2. *The trivial solution of (4.1) is linearly asymptotically stable for all $\tau_{ij} > 0,\ i \ne j$ if*

$$a_{ii} < b_i,\ i = 1, \dots n; \qquad \tau_{ii} < \frac{1}{b_i}\left(\sqrt{1 + \frac{b_i}{|a_{ii}|}} - 1\right),\ i \text{ such that } a_{ii} < 0; \tag{4.5}$$

and

$$\left|\prod_{i=1}^{n} a_{i,i-1}\right| < \prod_{i=1}^{n}(b_i - a_{ii}). \tag{4.6}$$

Proof. Let $\beta \stackrel{def}{=} \prod_{i=1}^{n} a_{i,i-1}$ and $\alpha \stackrel{def}{=} \prod_{i=1}^{n}(b_i - a_{ii})$. Note that if $-b_i < a_{ii} < b_i$ or $a_{ii} < -b_i$ and $\tau_{ii} < \frac{1}{\sqrt{a_{ii}^2 - b_i^2}} \operatorname{Arccos}\left(\frac{b_i}{a_{ii}}\right) < \frac{1}{b_i}\left(\sqrt{1 + \frac{b_i}{|a_{ii}|}} - 1\right)$, then the trivial solution will be stable for $\beta^- < \beta < \min\{\beta^+,\ \alpha\}$, where $\beta = \beta^{\pm}$ correspond to curves along which the characteristic equation of (4.1) has a pair of pure imaginary roots.

It is shown in [8] that

$$\begin{aligned}
\beta^{\pm 2} \ &= \prod_{i=1}^{n}\left[(b_i - a_{ii}\cos\omega\tau_{ii})^2 + (\omega + a_{ii}\sin\omega\tau_{ii})^2\right] \\
&\stackrel{def}{=} \prod_{i=1}^{n} \beta_i^2(\omega)
\end{aligned}$$

Now if $a_{ii} > 0$ then $(b_i - a_{ii}\cos\omega\tau_{ii})^2 \ge (b_i - a_{ii})^2$ and $(\omega + a_{ii}\sin\omega\tau_{ii})^2 \ge 0$ thus $\beta_i^2(\omega) \ge \beta_i^2(0)$, with equality only when $\omega = 0$. Further, by [8, Theorem 2.5], if $\tau_{ii} < \frac{1}{b_i}\left(\sqrt{1 + \frac{b_i}{|a_{ii}|}} - 1\right)$ then $\beta_i'(\omega) \ge 0$ with equality at $\omega = 0$ implying $\beta_i^2(\omega) \ge \beta_i^2(0)$. Thus if this condition is satisfied for each i such that $a_{ii} < 0$ then we have $\beta^{\pm 2} \ge \beta^{\pm}(0) = \alpha^2$ and the result follows. ∎

Note that in the case $b_i = 0,\ i = 1, \dots, n$ conditions (4.5)–(4.6) become

$$a_{ii} < 0, \qquad -a_{ii}\tau_{ii} < \frac{1}{2}, \qquad \left|\prod_{i=1}^{n} a_{i,i-1}\right| < \prod_{i=1}^{n}(-a_{ii}), \qquad i = 1, \dots, n. \tag{4.7}$$

Recently, So, Tang and Zou [38] obtained the following result for general networks of the form (1.3), with $b_i = 0$.

THEOREM 4.3. *[38, Theorem 1.2] Let $\tilde{C}$ be the matrix with elements $\tilde{c}_{ii} = -c_{ii}$,*

$$\tilde{c}_{ij} = -\frac{1 - \frac{1}{9}c_{ii}\tau_{ii}(3 - 2c_{ii}\tau_{ii})}{1 + \frac{1}{9}c_{ii}\tau_{ii}(3 - 2c_{ii}\tau_{ii})}|c_{ij}|, \;\; i \neq j,$$

and assume that $-c_{ii}\tau_{ii} < \frac{3}{2}$, $i = 1, \ldots, n$. If $\tilde{C}$ is a non-singular M-matrix, then every solution $(x_1(t),\ x_2(t), \ldots,\ x_n(t))$ of (2.2) tends to 0 as $t \to \infty$.

In the case of the ring network (4.1) these conditions reduce to

$$a_{ii} < 0, \quad -a_{ii}\tau_{ii} < \frac{3}{2}, \qquad \left|\prod_{i=1}^{n} a_{i,i-1}\right| < \prod_{i=1}^{n}(-a_{ii})\frac{1 + \frac{1}{9}a_{ii}\tau_{ii}(3 - 2a_{ii}\tau_{ii})}{1 - \frac{1}{9}a_{ii}\tau_{ii}(3 - 2a_{ii}\tau_{ii})}, \tag{4.8}$$

$i = 1, \ldots, n$.

There is playoff between the conditions on the connection strengths, a_{ij} and those on the diagonal delays τ_{ii}, which can be seen in Figure 4.1. In this figure we portray the stability regions of the trivial solution of (4.1) with $b_i = 0$ in the β, τ plane ($\beta = \prod_{i=1}^{n} a_{i,i-1}$ and $\tau = \sum_{i=1}^{n} \tau_{i,i-1}$). The exact linear stability region lies between the solid curves. The lines with negative slope mark the region corresponding to (4.7) and lines with positive slope mark that corresponding to (4.8). It would appear that (4.7) gives the region with the weakest condition on the $a_{i,i-1}$. A simple analysis of the characteristic equation shows that the region with the weakest condition on the τ_{ii} is

$$a_{ii} < 0, \qquad \tau_{ii} < \frac{\pi}{2}, \qquad \left|\prod_{i=1}^{n} a_{i,i-1}\right| = 0.$$

Condition (4.8) gives an interesting compromise between the two.

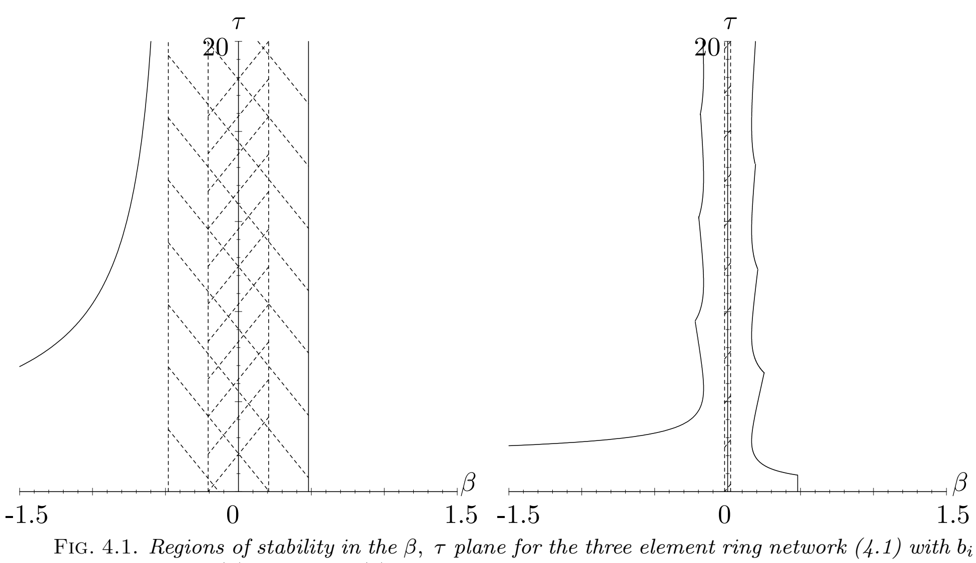

FIG. 4.1. *Regions of stability in the β, τ plane for the three element ring network (4.1) with $b_i = 0$, $a_{11} = -0.5$, $a_{22} = -0.8$, $a_{33} = -1.2$. (a) $\tau_{ii} = 0.4$ (b) $\tau_{ii} = 1.0$. The trivial solution is linearly asymptotically stable between the solid curves. Dashed lines with negative slope indicate the region of off-diagonal delay independent linear stability as predicted by Theorem 4.2 [8]. Dashed lines with positive slope indicate the region as predicted by Theorem 4.3 [38].*

4.2. Global Stability. Now consider the result in section 3. When applied to the ring network (4.1), Theorem 3.3 predicts global stability exactly in the region given by (4.2). In [14], Gopalsamy and He obtained the following sufficient condition for global stability independent of the delays:

$$\max_{1\le i\le n}\left\{\frac{1}{b_i}\sum_{j=1}^{n}|a_{ji}|\right\}<1 \tag{4.9}$$

which for system (4.1) can be written as

$$|a_{i,i-1}|<b_i-|a_{ii}|,\ 1\le i\le n. \tag{4.10}$$

Clearly conditions (4.10) imply conditions (4.2).

Since (4.2) is close to being sharp, in the sense that the condition for delay independent local stability is almost the same, it seems likely that to get any weaker conditions on the a_{ij} which guarantee global stability, one will need to have conditions involving the delays. For example, recent work of van den Driessche, Wu and Zou [40, 41] has obtained conditions for global stability in the network (1.3) (in the case $b_i = 1$) which depend on the *diagonal* delays. This work uses the techniques of monotone dynamical systems theory to obtain the following result:

THEOREM 4.4. *[40, Theorem 2.1] Assume (H1) and (H2) hold, and $\tau_{ij}>0$. If either*

$$\max_{1\le i\le n}\left\{a_{ii}+\sum_{j\ne i}^{n}|a_{ij}|\right\}<1 \qquad or \qquad \max_{1\le i\le n}\left\{a_{ii}+\sum_{j\ne i}^{n}\frac{|a_{ij}|+|a_{ji}|}{2}\right\}<1,$$

then for every input $J=(J_1,\ldots,J_n)^T$, system (1.3) has a unique equilibrium, which is globally asymptotically stable provided that the diagonal delays τ_{ii} corresponding to the negative a_{ii} are sufficiently small such that $0\le\tau_{ii}\le\frac{1}{1-ea_{ii}}$.

If these conditions are applied to system (4.1), they become

$$|a_{i,i-1}|<1-a_{ii},\ 1\le i\le n. \tag{4.11}$$

and

$$0\le\tau_{ii}\le\frac{1}{1-ea_{ii}}, \tag{4.12}$$

for each i such that $a_{ii}<0$. Clearly condition (4.11) is weaker than either of the conditions (4.2), (4.10), except in the case $a_{ii}\ge 0,\ i=1,\ldots,n$ in which case (4.10) and (4.11) are equivalent. This is not surprising; van den Driessche et al. [40] have been able to weaken the condition on the a_{ii} by adding a condition on the τ_{ii}. Note further that both (4.11),(4.12) are stronger than the analogous ones obtained for off-diagonal delay independent local stability in [8], i.e (4.5), (4.6).

We illustrate these results in the β,τ plane in Figures 4.2 and 4.3 ($\beta=\prod_{i=1}^n a_{i,i-1}$ and $\tau=\sum_{i=1}^n\tau_{i,i-1}$). In this parameter space the result of [14] appears the same as the result of Theorem 3.3.

5. Conclusions. We have given necessary and sufficient conditions for delay independent, asymptotic stability of the trivial solution of the linear delay differential equation (2.2) and shown how they can be modified to give necessary and sufficient conditions for the local asymptotic stability of equilibrium points of the general additive neural network (1.3). We further showed that a slightly strengthening of these conditions yields sufficient conditions for the nonlinear system to possess a unique equilibrium solution, and for this solution to be global asymptotically stable independent of the delays. Our linear results are slightly stronger than those of [24] as we allow delays in the self feedback terms. They are complementary to those of [38] who impose weaker conditions on the connection matrix, but stronger

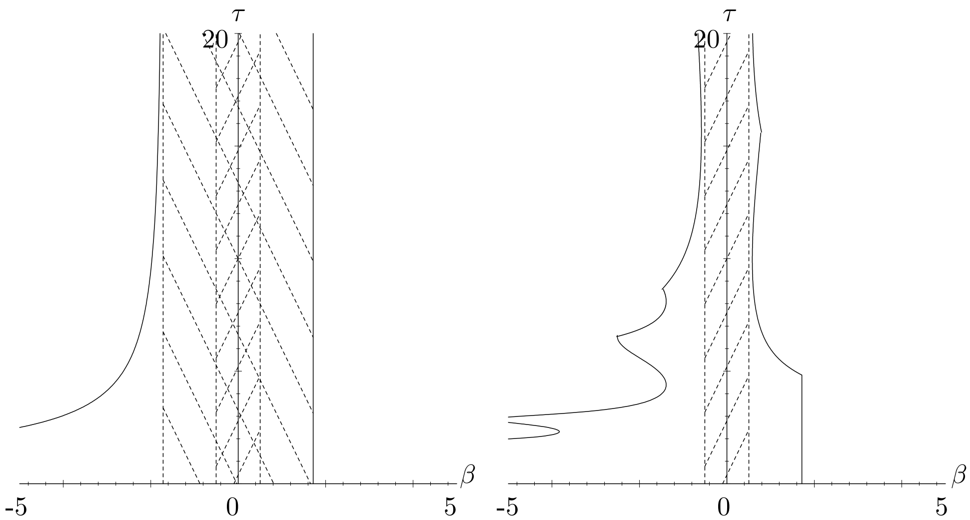

FIG. 4.2. *Region of stability in the β, τ plane for the three element ring network (4.1) with* $b_i = 1,\ a_{11} = -0.1,\ a_{22} = -0.2,\ a_{33} = -0.3$. *(a)* $\tau_{ii} = 0.5$ *(b)* $\tau_{ii} = 10$. *The trivial solution is linearly asymptotically stable between the solid curves. Dashed lines with negative slope indicate the region of global stability as predicted by Theorem 4.4 [40]. Dashed lines with positive slope indicate the region of global stability as predicted by Theorem 3.3.*

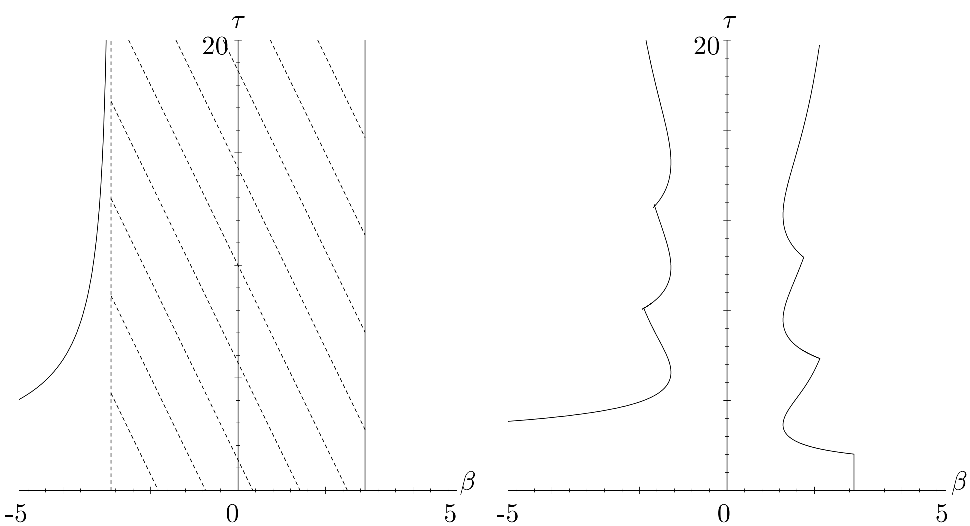

FIG. 4.3. *Region of stability in the β, τ plane for the three element ring network (4.1) with* $b_i = 1,\ a_{11} = -0.1,\ a_{22} = -0.2,\ a_{33} = -1.2$. *(a)* $\tau_{ii} = 0.2$ *(b)* $\tau_{ii} = 1.5$. *The trivial solution is linearly asymptotically stable between the solid curves. Dashed lines indicate the region of global stability as predicted by [40]*

conditions on the diagonal delays. Our global results are stronger to those of [14] and complementary to those of [40], as described above.

The stability results derived and reviewed in this paper all contain the following:

(A1) conditions on some or all of the "diagonal parameters" b_i, a_{ii}, and τ_{ii} ;

(A2) a condition forcing the off-diagonal connection coefficients, a_{ij}, to be "small" in comparison with the diagonal parameters.

When these conditions are independent of τ_{ii}, as in Theorems 2.6, 3.3 and 4.1 and in [14], [24] one obtains what is usually called *delay independent stability*. When they depend on the τ_{ii}, as in Theorem 4.2 and in [8, 38, 40], one obtains what can be described as *off-diagonal delay independent stability.*

To see why such conditions might lead to stability, we return to the simple case of a group of unconnected elements, i.e. (1.3) with $a_{ij} = 0, i \neq j$. In this situation it is well known [28, 39] that equilibrium points $\mathbf{u}^*$ of the system will be locally asymptotically stable for

$$-b_i \leq a_{ii} < b_i,\ \tau_{ii} \geq 0 \tag{5.1}$$

and for

$$a_{ii} < -b_i,\ \tau_{ii} < \frac{1}{\sqrt{a_{ii}^2 - b_i^2}} \operatorname{Arccos} \frac{b_i}{a_{ii}}. \tag{5.2}$$

In Theorem 2.6, condition (A1) above is exactly (5.1); in Theorems 3.3,4.1 and in [14] it is slightly stronger. In Theorem 4.2 and in [8, 38, 40] condition (A2) is similar to (5.2), but stronger. Thus it seems that we can obtain (off-diagonal) delay independent local asymptotic stability of an equilibrium point of the network by choosing the diagonal parameters so that the (corresponding) equilibrium point is locally asymptotically stable when the neurons are unconnected, and choosing the off-diagonal connection weights to be "small" or "weak". (Similar ideas are discussed for neural networks without time delays in the work of Hoppensteadt and Izhikevich [27].) Global stability results can be obtained by adding some monotonicity conditions, e.g. (H1) and (H2), on the activation functions.

We note that necessary and sufficient conditions have only been found for delay independent local stability in (1.3). The off-diagonal delay independent case is more delicate as there is a compromise between the conditions imposed on the diagonal parameters and those on the off-diagonal parameters. As might be expected, one can find weaker conditions on one set of parameters by imposing stronger conditions on the other set. For Lotka-Volterra systems it has been shown [24, 36] that delay independent conditions such as those discussed in this paper are actually necessary and sufficient for delay independent *global* stability. To our knowledge, no such results have yet been obtained for the model (1.3).

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REFERENCES

[1] L.V. Ahlfors, *Complex analysis: an introduction to the theory of analytic functions of one complex variable (3rd ed.)*, McGraw-Hill, New York, 1979.

[2] P. Baldi and A.F. Atiya, *How delays affect neural dynamics and learning*, IEEE Transactions on Neural Networks, 5(1994), pp. 612–621.

[3] I. Barbălat, *Systèmes d'équations différentielles d'oscillations non linéaires*, Rev. Math. Pures Appl., 4(1959), pp. 267–270.

[4] J. Bélair, *Stability in a model of a delayed neural network*, Journal of Dynamics and Differential Equations, 5(1993), pp. 607–623.

[5] J. Bélair, S. A. Campbell, and P. van den Driessche, *Frustration, stability and delay-induced oscillations in a neural network model*, SIAM Journal on Applied Mathematics, 56(1996), pp. 245–255.

[6] J. Bélair and S. Dufour, *Stability in a three-dimensional system of delay-differential equations*, Canadian Applied Mathematics Quarterly, 4(1996), pp. 135–156.

[7] A. Berman and R. Plemmons, *Nonnegative matrices in the mathematical sciences*, SIAM, Philadelphia, 1994.

[8] S. A. Campbell, *Stability and bifurcation of a simple neural network with multiple time delays*, in Differential Equations with Application to Biology, *Fields Institute Communications* 21, S. Ruan, G. S. K. Wolkowicz, and J. Wu, eds., pp. 65–79, AMS, 1999.

[9] M. Cohen and S. Grossberg, *Absolute stability of global pattern formation and parallel memory storage by competitive neural networks.*, IEEE Trans. Systems Man Cybernet., 13(1983), pp. 815–826.

[10] J. Dieudonné, *Foundations of Modern Analysis*, Academic Press, New York, 1969.

[11] M. Fiedler, *Special matrices and their applications in numerical mathematics*, Martinus Nijhoff Publ. (Kluwer), Dordrecht, 1986.

[12] K. Gopalsamy, *Stability and Oscillations in Delay Differential Equations of Population Dynamics*, Kluwer, Dordrecht, 1992.

[13] K. Gopalsamy and X-Z He, *Delay independent stability in bidirectional associative memory networks*, IEEE Transactions on Neural Networks, 5(1994), p. 998.

[14] K. Gopalsamy and X-Z He, *Stability in asymmetric Hopfield nets with transmission delays*, PhysicaD, 76(1994), pp. 344–358.

[15] K. Gopalsamy and I. Leung, *Delay induced periodicity in a neural netlet of excitation and inhibition*, PhysicaD, 89(1996), pp. 395–426.

[16] S. Grossberg, *Nonlinear difference-differential equations in prediction and learning theory*, Proc. Nat. Acad. Sci. U.S.A., 58(1967), pp. 1329–1334.

[17] S. Grossberg, *Global ratio limit theorems for some nonlinear functional-differential equations. I, II*, Bull. Amer. Math. Soc., 74(1968), pp. 95–99; 100–105.

[18] S. Grossberg, *Some nonlinear networks capable of learning a spatial pattern of arbitrary complexity*, Proc. Nat. Acad. Sci. U.S.A., 59(1968), pp. 368–372.

[19] S. Grossberg, *Pattern learning by functional-differential neural networks with arbitrary path weights*, in Delay and functional differential equations and their applications (Proc. Conf., Park City, Utah, 1972), pp. 121–160, Academic Press, New York, 1972.

[20] S. Grossberg, *Classical and instrumental learning by neural networks*, Prog. Theor. Biol., 3(1974), pp. 51–141.

[21] S. Grossberg, *Competition, decision, and consensus*, J. Math. Anal. Appl., 66(1978), pp. 470–493.

[22] S. Grossberg, *Biological competition: Decision rules, pattern formation, and oscillations*, Proc. Nat. Acad. Sci. U.S.A., 77(1980), pp. 2338–2342.

[23] J.K. Hale and S.M. Verduyn Lunel, *Introduction to Functional Differential Equations*, Springer Verlag, New York, 1993.

[24] J. Hofbauer and J. So, *Diagonal dominance and harmless off-diagonal delays*, Proceedings of the American Mathematical Society, 128(2000), pp. 2675–2682.

[25] J. J. Hopfield, *Neural networks and physical systems with emergent collective computational abilities*, Proc. Nat. Acad. Sci. U.S.A., 79(1982), pp. 2554–2558.

[26] J.J. Hopfield, *Neurons with graded response have collective computational properties like those of two-state neurons*, Proc. Nat. Acad. Sci. U.S.A., 81(1984), pp. 3088–3092.

[27] F.C. Hoppensteadt and E.M. Izhikevich, *Weakly connected neural networks*, Springer-Verlag, New York, 1997.

[28] V.B. Kolmanovskii and V.R. Nosov, *Stability of functional differential equations*, in Mathematics in Science and Engineering 180, Academic Press, 1986.

[29] P. Lancaster and M. Tismenetsky, *The Theory of Matrices (2nd edition)*, Academic Press, New York, 1985.

[30] C.M. Marcus, F.R. Waugh, and R.M. Westervelt, *Nonlinear dynamics and stability of analog neural networks*, Physica D, 51(1991), pp. 234–247.

[31] C.M. Marcus and R.M. Westervelt, *Stability of analog neural networks with delay*, Physical Review A, 39(1989), pp. 347–359.

[32] L. Olien and J. Bélair, *Bifurcations, stability, and monotonicity properties of a delayed neural network model*, Physica D, 102(1997), pp. 349–363.

[33] K. Pakdaman, C. Grotta-Ragazzo, and C. P. Malta, *Transient regime duration in continuous-time neural networks with delay*, Physical Review E, 58(1998), pp. 3623–3627.

[34] K. Pakdaman, C. Grotta-Ragazzo, C. P. Malta, O. Arino, and J.-F. Vibert, *Effect of delay on the boundary of the basin of attraction in a system of two neurons*, Neural Networks, 11(1998), pp. 509–519.

[35] K. Pakdaman, C. P. Malta, C. Grotta-Ragazzo, O. Arino, and J.-F. Vibert, *Transient oscillations in continuous-time excitatory ring neural networks with delay*, Physical Review E, 55(1997), pp. 3234–3248.

[36] Yasuhisa Saito, Tadayuki Hara, and Wanbiao Ma, *Necessary and sufficient conditions for permanence and global stability of a Lotka-Volterra system with two delays*, J. Math. Anal. Appl., 236(1999), pp. 534–556.

[37] L.P. Shayer and S.A. Campbell, *Stability, bifurcation and multistability in a system of two coupled neurons with multiple time delays*, SIAM Journal on Applied Mathematics, 61(2000), pp. 673–700.

[38] Joseph So, Xianhua Tang, and Xingfu Zou, *Stability in Linear Delay System without Instantaneous Negative Feedback*, Preprint, 2001.

[39] G. Stépán, *Retarded Dynamical Systems*, Vol. 210 of *Pitman Research Notes in Mathematics*, Longman Group, Essex, 1989.

[40] P. van den Driessche, J. Wu, and X. Zou, *Stabilization role of inhibitory self-connections in a delayed neural network*, Physica D, 150(2001), pp. 84–90.

[41] P. van den Driessche and X. Zou, *Global attractivity in delayed Hopfield neural network models*, SIAM Journal on Applied Mathematics, 58(1998), pp. 1878–1890.

[42] H. Ye, A. N. Michel, and K. Wang, *Global stability and local stability of Hopfield neural networks with delays*, Physical Review E, 50(1994), pp. 4206–4213.

[43] H. Ye, A. N. Michel, and K. Wang, *Qualitative analysis of Cohen-Grossberg neural networks with multiple delays*, Physical Review E, 51(1995), pp. 2611–2618.

7. Appendix. Lemma 7.1. *Let $-\hat{K}$ be an M-matrix. If $|c_{ii}| = b_i$ for some i, then for each $j \neq i$ either $c_{ij} = 0$ or $c_{ji} = 0$. If $c_{ii} = b_i$, then* $\det K = 0$.

Proof. Since $-\hat{K}$ is an M-matrix then all its 2×2 principle minors are nonnegative. In particular, for each $j \neq i$

$$(b_i - |c_{ii}|)(b_j - |c_{jj}|) - |c_{ij}||c_{ji}| \geq 0.$$

Then $|c_{ii}| = b_i$ implies

$$-|c_{ij}||c_{ji}| \geq 0,$$

i.e. for each $j \neq i$ either $c_{ij} = 0$ or $c_{ji} = 0$. If $c_{ii} = b_i$ then for a suitable permutation matrix P, PKP^T has a row of zeros, which implies $\det K = 0$. ▯

Theorem 7.2. *If $|c_{kk}| > b_k$ for some $k \in \{1, 2, \ldots, n\}$, then there exist delays $\tau_{ij} > 0$ such that the characteristic equation (2.3) has a root with positive real part.*

Proof. Note that the characteristic equation (2.3) can be written

$$\Delta(\lambda) = P(\lambda) + Q(\lambda, e^{-\lambda\tau_{ij}}) = 0$$

where

$$P(\lambda) = \prod_{i=1}^{n}(-b_i + c_{ii}e^{-\lambda\tau_{ii}} - \lambda) \stackrel{def}{=} \prod_{i=1}^{n} p_i(\lambda) = 0.$$

Note that as $\tau_{ij},\ i \neq j$ approach ∞, $Q(\lambda)$ approaches 0.

Fix the τ_{ii} and the c_{ij} and suppose $c_{kk} > b_k > 0$, for some k. Then we have $p_k(0) = -b_k + c_{kk} > 0$. Further, for λ real, we have $\lim_{\lambda \to \infty} p_k(\lambda) = -\infty$. Since $p_k(\lambda)$ is a continuous function, we may apply the Intermediate Value Theorem to conclude that $p_k(\lambda)$ and hence $P(\lambda)$ has a positive real root.

Now fix the c_{ij} with $c_{kk} < -b_k < 0$ and the τ_{ii} with

$$\tau_{kk} > \left\{ \frac{1}{\sqrt{c_{kk}^2 - b_k^2}} \left[\operatorname{Arccos}\left(\frac{b_k}{c_{kk}}\right)\right]\right\}.$$

Then it is well known [28, 39] that $p_k(\lambda)$ (and hence $P(\lambda)$ has at least one pair of complex conjugate roots with positive real parts.

Now applying Rouché's Theorem, we conclude that for τ_{ij} sufficiently large the characteristic equation (2.3) also has a root with positive real part. ▯

Theorem 7.3. *If $|c_{ij}|,\ i \neq j$ are sufficiently small and $|c_{kk}|$ sufficiently large for some $k \in \{1, 2, \ldots, n\}$, then the characteristic equation (2.3) has a root with positive real part, for any set of delays $\tau_{ij} > 0$.*

Proof. Fix $\tau_{ij},\ i \neq j$ at some arbitrary values. Setting $c_{ij} = 0$ for $i \neq j$ in (2.3) yields

$$P(\lambda) = \prod_{i=1}^{n}(-b_i + c_{ii}e^{-\lambda\tau_{ii}} - \lambda) \stackrel{def}{=} \prod_{i=1}^{n} p_i(\lambda) = 0.$$

This is just the product of the characteristic equations for the n decoupled neurons.

Suppose that $c_{kk} > b_k > 0$ for some k. In exactly the same manner as in the proof of Theorem7.2 one may show that $P(\lambda)$ has a positive real, root.

Now suppose for some k $c_{kk} < -\sqrt{b_k^2 - \omega^2}$, where ω is the first solution of the equation $\tan\omega\tau_{kk} = b_k/\omega$. Then it is well known (see e.g. [28, 39]) that $p_k(\lambda)$ (and hence $P(\lambda)$) has at least one pair of complex conjugate roots with positive real parts.

Now by Rouché's Theorem if the $c_{ij},\ i \neq j$ vary slightly from zero in each case the characteristic equation will still have a root with positive real part. ▯