Fields Institute Communications
Volume **21**, 1999

Stability and bifurcation of a simple neural network with multiple time delays.

Sue Ann Campbell
Department of Applied Mathematics
University of Waterloo
Waterloo ON N2L 3G1
sacampbe@duchatelet.math.uwaterloo.ca

Abstract. A system of delay differential equations representing a simple model for a ring of neurons with time delayed connections between the neurons is studied. Conditions for the linear stability of fixed points of this system are represented in a parameter space consisting of the sum of the time delays between the elements and the product of the strengths of the connections between the elements. It is shown that both Hopf and steady state bifurcations may occur when a fixed point loses stability. Codimension two bifurcations are shown to exist and numerical simulations reveal the possibility of quasiperiodicity and multistability near such points.

1 Introduction

Hopfield [1984] considered a simplified neural network model in which each neuron is represented by a linear circuit consisting of a resistor and capacitor, and is connected to the other neurons via nonlinear sigmoidal activation functions. Assuming instantaneous updating of each neuron and communication between the neurons, Hopfield arrived at a system of first order ordinary differential equations. Not long afterward Marcus & Westervelt [1989] considered the effect of including discrete time delays in the connection terms to represent the propagation time between neurons and/or processing time at a given neuron. Due to the complexity of the analysis, Marcus & Westervelt [1989] and most subsequent work, for example Gopalsamy & Leung [1996], Ye, Michel & Wang [1994], Bélair, Campbell & van den Driessche [1996] (see also references therein), have focussed on the situation where all connection terms in the network have the same time delay. In the work which has been done on Hopfield neural networks with multiple time delays the analysis is usually simplified by either restricting the size of the network (e.g. Olien & Bélair [1997]), or considering networks with simple architectures (e.g. Baldi & Atiya

1991 *Mathematics Subject Classification.* Primary 92B20, 34K20; Secondary 34K15.

This work was supported by a grant from the Natural Sciences and Engineering Research Council of Canada.

[1994]). A notable exception is Ye, Michel & Wang [1995] who have considered the global stability of fixed points of an arbitrary sized network with different time delays in each connection term.

Here we are interested in studying not only the stability of fixed points of the network but also the bifurcation of new solutions when stability is lost. We thus consider a Hopfield network of arbitrary size with multiple time delays but with a simple architecture. Our network consists of a ring of neurons where the jth element receives two time delayed inputs: one from itself with delay τ_s, one from the previous element with delay τ_{j-1}. The architecture of this system is illustrated in fig. 1.

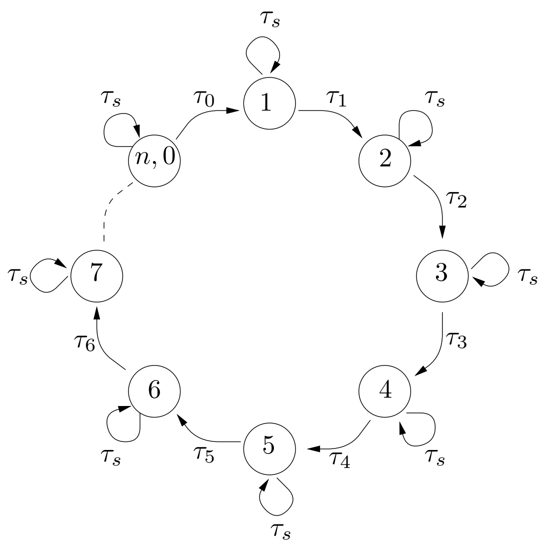

Figure 1 Architecture of the neural network

The equations for this system are

$$C_j \dot{u}_j(t) = -\frac{1}{R_j} u_j(t) + F_j(u_j(t-\tau_s)) + G_j(u_{j-1}(t-\tau_{j-1})),\ j = 1, \ldots, n \qquad (1.1)$$

where $C_j > 0,\ R_j > 0$ represent the capacitances and resistances of the individual neurons, and $F_j,\ G_j$ are nonlinear functions representing, respectively, the feedback from neuron j to itself, and the connection from neuron $j-1$ to neuron j; and the index 0 is taken equal to n. Normalizing this equation leads to

$$\dot{u}_j(t) = -d_j u_j(t) + f_j(u_j(t-\tau_s)) + g_j(u_{j-1}(t-\tau_{j-1})),\ j = 1, \ldots, n \qquad (1.2)$$

where $d_j > 0$.

The plan for the article is as follows. In section 2 we consider the linear stability analysis of eq. (1.2) and present some theorems about the region of stability of the fixed points as a function of the physical parameters in the model. In section 3 we discuss the codimension one and two bifurcations which can occur when stability is lost and illustrate these with numerical simulations of a particular system. In the final section we will discuss the implications of our results and put them in the context of the related work mentioned above.

2 Linear Stability Analysis

Fixed points of (1.2) are solutions $\mathbf{u}(t) = \mathbf{u}^*,\ \forall t$, where

$$\begin{aligned} \mathbf{u}^* &= [u_1^*, u_2^*, \ldots, u_n^*]^T \\ d_j u_j^* - f_j(u_j^*) &= g_j(u_{j-1}^*), \qquad j = 1, \ldots, n. \end{aligned} \tag{2.1}$$

The existence of such solutions depends, of course, on the particular functions f_j and g_j used in the model. Assuming that such a fixed point exists one can translate it to the origin via the transformation $\mathbf{u}(t) = \mathbf{u}^* + \mathbf{x}(t)$. If f_j and g_j are sufficiently smooth, one can expand these functions in Taylor series about $u(t) = u^*$. For $x(t)$ sufficiently small (solutions close to the fixed point) one can truncate this Taylor series to obtain the linearization of (1.2) about the fixed point:

$$\dot{x}_j(t) = -d_j x_j(t) + a_j x_j(t - \tau_s) + b_j x_{j-1}(t - \tau_{j-1}),\ j = 1, \ldots, n, \tag{2.2}$$

where $a_j = f_j'(u_j^*),\ b_j = g_j'(u_{j-1}^*)$. Physically, the a_j and b_j can be thought of as gains or strengths of the connections between neurons. To study the linearized stability of the fixed point $\mathbf{u}^*$ of (1.2), we consider solutions of (2.2) of the form $\mathbf{x}(t) = e^{\lambda t}\mathbf{c}$, where $\mathbf{c} = [c_1, c_2, \ldots, c_n]^T,\ c_j$ constants. Substituting this expression into (2.2) yields the matrix equation

$$\left[\lambda I + D - Ae^{-\lambda\tau_s} - B\right]\mathbf{c} = \mathbf{0}. \tag{2.3}$$

Here I is the $n \times n$ identity matrix, $D = diag(d_1, d_2, \ldots, d_n)$, $A = diag(a_1, a_2, \ldots, a_n)$ and $B = B_{jk}$ where $B_{jk} = b_j e^{-\lambda\tau_{j-1}}$, if $k = j-1$, $B_{jk} = 0$ otherwise. For nontrivial solutions of (2.3) we require that the determinant of the coefficient matrix be zero,

$$det\left[\lambda I + D - Ae^{-\lambda\tau_s} - B\right] = 0, \tag{2.4}$$

which leads to the characteristic equation associated with the delay differential equation (2.2):

$$\Delta(\lambda) = \prod_{j=1}^{n}(\lambda + d_j - a_j e^{-\lambda\tau_s}) - \prod_{j=1}^{n}(b_j e^{-\lambda\tau_{j-1}}) = 0. \tag{2.5}$$

We note that the latter product in this equation may be expanded giving a simplified equation

$$\prod_{j=1}^{n}(\lambda + d_j - a_j e^{-\lambda\tau_s}) = \beta e^{-\lambda\tau} \tag{2.6}$$

where

$$\beta \stackrel{def}{=} \prod_{j=1}^{n} b_j \quad \text{and} \quad \tau \stackrel{def}{=} \sum_{j=1}^{n} \tau_j. \tag{2.7}$$

Thus the connections between the neurons act (so far as the linear stability is concerned) as a *single* feedback loop with gain β and delay τ. We shall see below that these are convenient and natural parameters to use in our stability and bifurcation analysis.

It is well known (see e.g. Kolmanovskii & Nosov [1986] or Stépán [1989]) that the trivial solution of eq. (2.2) will be asymptotically stable if all the roots of the characteristic equation (2.6) have negative real parts. This further implies (Hale & Lunel [1993]) that the fixed point $\mathbf{u} = \mathbf{u}^*$ of nonlinear system (1.2) is locally stable. For brevity, we will say that the network modelled by (2.2) (by (1.2)) is *stable* (*locally stable*) in this case.

Another standard result (Kolmanovskii & Nosov [1986]) tells us that the trivial solution of (2.2) can only lose stability as parameters are varied by having a root of the characteristic equation pass through the imaginary axis. Thus changes of stability will occur at points in parameter space where (2.6) has roots with zero real parts. Consider first the zero roots, $\lambda = 0$. These will occur where

$$\beta = \prod_{j=1}^{n} (d_j - a_j) \stackrel{def}{=} \alpha. \tag{2.8}$$

Locating the pure imaginary roots is slightly more complicated. Substituting $\lambda = i\omega$ into (2.6) and separating real and imaginary parts yields

$$\begin{array}{rcl} F_R & = & \beta \cos \omega\tau \\ F_I & = & -\beta \sin \omega\tau \end{array} \tag{2.9}$$

where F_R and F_I are, respectively, the real and imaginary parts of $\prod_{j=1}^{n}(i\omega + d_j - a_j e^{-i\omega\tau_s})$. These are defined by the following

$$\begin{aligned} F_R \stackrel{def}{=} & \prod_{j=1}^{n}(d_j - a_j \cos\omega\tau_s) \left[1 + \sum_{l=1}^{k} (-1)^l \left\{ \sum_{l1=1}^{n} \sum_{l2=1}^{n} \cdots \sum_{l2k=1}^{n} \right.\right. \\ & \left.\left. \frac{\omega + a_{l1}\sin\omega\tau_s}{d_{l1} - a_{l1}\cos\omega\tau_s} \, \frac{\omega + a_{l2}\sin\omega\tau_s}{d_{l2} - a_{l2}\cos\omega\tau_s} \cdots \frac{\omega + a_{l2k}\sin\omega\tau_s}{d_{l2k} - a_{l2k}\cos\omega\tau_s} \right\} \right], \end{aligned} \tag{2.10}$$

$$l2 \neq l1; \quad l3 \neq l1, l2; \quad \cdots \quad l2k \neq l1, l2, \ldots, l(2k-1).$$

$$\begin{aligned} F_I \stackrel{def}{=} & \prod_{j=1}^{n}(d_j - a_j \cos\omega\tau_s) \sum_{l=1}^{m} (-1)^{l-1} \left\{ \sum_{l1=1}^{n} \sum_{l2=1}^{n} \cdots \sum_{l(2m-1)=1}^{n} \right. \\ & \left. \frac{\omega + a_{l1}\sin\omega\tau_s}{d_{l1} - a_{l1}\cos\omega\tau_s} \, \frac{\omega + a_{l2}\sin\omega\tau_s}{d_{l2} - a_{l2}\cos\omega\tau_s} \cdots \frac{\omega + a_{l(2m-1)}\sin\omega\tau_s}{d_{l(2m-1)} - a_{l(2m-1)}\cos\omega\tau_s} \right\}, \\ & l2 \neq l1; \quad l3 \neq l1, l2; \quad \cdots \quad l(2m-1) \neq l1, l2, \ldots, l(2m-2). \end{aligned} \tag{2.11}$$

Here $k = \frac{n}{2} = m$ if n is even and $k = \frac{n-1}{2}, m = \frac{n+1}{2}$ if n is odd.

Equation (2.9) may be easily solved for τ and β to yield.

$$\beta = \beta^+ \stackrel{def}{=} \sqrt{F_R^2 + F_I^2} \tag{2.12}$$

$$\tau = \tau^+ \stackrel{def}{=} \begin{cases} \dfrac{1}{\omega}\left[\text{Arctan}\left(\dfrac{-F_I}{F_R} \right) + 2l\pi \right], & F_R > 0 \\ \dfrac{1}{\omega}\left[\text{Arctan}\left(\dfrac{-F_I}{F_R} \right) + (2l+1)\pi \right], & F_R < 0 \end{cases} \tag{2.13}$$

and

$$\beta = \beta^- \stackrel{def}{=} -\sqrt{F_R^2 + F_I^2} \tag{2.14}$$

$$\tau = \tau^- \stackrel{def}{=} \begin{cases} \dfrac{1}{\omega}\left[\text{Arctan}\left(\dfrac{F_I}{-F_R} \right) + 2l\pi \right], & F_R < 0 \\ \dfrac{1}{\omega}\left[\text{Arctan}\left(\dfrac{F_I}{-F_R} \right) + (2l+1)\pi \right], & F_R > 0. \end{cases} \tag{2.15}$$

Here $l = 0, \ldots, 1$ and Arctan denotes the inverse tangent function which has range $(-\frac{\pi}{2}, \frac{\pi}{2})$. For fixed a_j, d_j and τ_s eqs. (2.12)–(2.13) and (2.14)–(2.15) describe curves which lie in the right and left half of the β, τ plane, respectively, and which are parameterized by ω. If n is large, F_R and F_I may be quite complicated functions of ω and the physical parameters a_j, d_j, τ_s. Nevertheless, we can still make some assertions about the parameter values for which the trivial solution of (2.2) is asymptotically stable. We begin with some preliminary results about limiting cases.

Lemma 2.1 (Single Element) *Suppose there is just one element in the network. If*

$$-d_1 < a_1 < d_1 \quad \textit{and} \quad \tau_s \geq 0$$
$$\textit{or}$$
$$a_1 < -d_1 \quad \textit{and} \quad \tau_s < \left\{ \frac{1}{\sqrt{a_1^2 - d_1^2}} \left[\text{Arccos}\left(\frac{d_1}{a_1}\right)\right]\right\},$$

then the network is stable.

Proof If $n = 1$ then the characteristic equation reduces to

$$\lambda + d_1 - a_1 e^{-\lambda\tau_s} = 0. \tag{2.16}$$

This equation has been studied by many authors, most recently in the books by Kolmanovskii & Nosov [1986] and Stépán [1989]. For clarity we repeat the results here. When $a_1 = 0$ the equation has one negative real root $\lambda = -d_1$ and thus the trivial solution is stable. The trivial solution can only lose stability for $a_1 \neq 0$ by having a root which passes through the imaginary axis. Equation (2.16) has a zero root when $a_1 = d_1$ and pure imaginary roots, $\pm i\omega = \pm i\sqrt{a_1^2 - d_1^2}$, when

$$\begin{aligned} \tau_s &= \tau_s^+ \stackrel{def}{=} \frac{1}{\sqrt{a_1^2 - d_1^2}} \left[(2l+1)\pi - \text{Arccos}\left(\frac{d_1}{a_1}\right)\right], \qquad 0 < d_1 < a_1 \\ &\text{or} \\ \tau_s &= \tau_s^- \stackrel{def}{=} \frac{1}{\sqrt{a_1^2 - d_1^2}} \left[2l\pi + \text{Arccos}\left(\frac{d_1}{a_1}\right)\right], \qquad a_1 < -d_1 < 0. \end{aligned}$$

Here $l = 0, 1, 2, \ldots$ and Arccos denotes the inverse cosine function which has range $[0, \pi]$. For fixed d_1 the curves τ_s^+ are monotone decreasing in a_1 with $\lim_{a_1 \to d_1^+} \tau_s^+ = \infty$ and the τ_s^- are monotone increasing in a_1 with $\lim_{a_1 \to d_1^-} \tau_s^- = \infty$. Thus the trivial solution will lose stability for a_1 positive when $a_1 = d_1$ and for a_1 negative when a_1 crosses the τ_s^- curve closest to $a_1 = -d_1$. The result follows. $\square$

Corollary 2.2 (Chain of Neurons) *Suppose the network is a chain, i.e. at least one of the $b_j = 0$. If for each j*

$$-d_j < a_j < d_j \quad \textit{and} \quad \tau_s \geq 0$$
$$\textit{or}$$
$$a_j < -d_j \quad \textit{and} \quad \tau_s < \left\{ \frac{1}{\sqrt{a_j^2 - d_j^2}} \left[\text{Arccos}\left(\frac{d_j}{a_j}\right)\right]\right\},$$

then the network is stable.

Proof If one of the $b_j = 0$ then $\beta = 0$ and the characteristic equation (2.6) for the network becomes

$$\prod_{j=1}^{n} (\lambda + d_j - a_j e^{-\lambda\tau_s}) = 0.$$

All the roots of this equation will have negative real parts if and only if, for each j, all the roots of

$$\lambda + d_j - a_j e^{-\lambda\tau_s} = 0$$

have negative real parts. This is exactly the equation studied in Theorem 2.1 with $j = 1$. The result follows. □

Thus a chain of neurons with unidirectional connections will be stable if each element in the chain is stable when it is isolated. Note that under the conditions of this theorem, α as defined by (2.8) is always positive.

Lemma 2.3 *For fixed a_j, d_j and τ_s, if $\beta^{\pm}$ as defined by (2.12) and (2.14) are monotone in ω then there are no intersection points of the curves defined by (2.12)–(2.13) or of the curves defined by (2.14)–(2.15). If β^{+} is increasing in ω then there are no intersection points of the line (2.8) with the curves defined by (2.12)–(2.13).*

Proof Consider first intersections of the curves defined by (2.12)–(2.13) or between the curves defined by (2.14)–(2.15). Since these curves are defined parametrically in terms of ω intersection points of two different curves will occur when the *same* values of β and τ occur for two *different* values of ω. If $\beta^{\pm}$ is monotone in ω then this is impossible.

Now consider the curves defined by (2.12)–(2.13). Consideration of (2.10)–(2.11) and (2.12) shows that $\lim_{\omega\to 0^+} \beta^+(\omega) = |\alpha|_+$, thus if β^+ is increasing with ω then there can be no intersection points of the line (2.8) with these curves. □

Theorem 2.4 *Let a_j, d_j, τ_s, be fixed and let the conditions of Corollary 2.2 be satisfied. If β^+ is monotone increasing with ω and β^- is monotone decreasing with ω then the stability region of the network modelled by (2.2) is given by the following:*

$$-\alpha \le \beta < \alpha \quad \textit{and} \quad 0 < \tau,$$

$$\beta < -\alpha \quad \textit{and} \quad 0 < \tau < \begin{cases} \dfrac{1}{\omega}\left[\operatorname{Arctan}\left(\dfrac{F_I}{-F_R}\right)\right] & \textit{if } \; F_R < 0 \\[2ex] \dfrac{1}{\omega}\left[\operatorname{Arctan}\left(\dfrac{F_I}{-F_R}\right) + \pi\right] & \textit{if } \; F_R > 0 \end{cases},$$

where $\alpha > 0$ is defined as in (2.8).

Proof From Corollary 2.2 the trivial solution of (2.2) will be asymptotically stable for $\beta = 0$. For $\beta \neq 0$ it can lose stability only when one of the roots of the characteristic equation (2.6) passes through the imaginary axis. Recall that these roots lie along the line (2.8) and the curves defined by (2.12)–(2.13) and (2.14)–(2.15). From Lemma 2.3 there can be no intersection points of the curves defined by (2.12)–(2.13) and (2.14)–(2.15) or of the line (2.8) and the curves defined by (2.12)–(2.13). From (2.10)–(2.11), $\lim_{\omega\to 0^+} F_R = \alpha = \lim_{\omega\to 0^+} F_I$; thus $\lim_{\omega\to 0^+} \beta^{\pm}(\omega) = \pm\alpha^{\pm}$ and $\lim_{\omega\to 0^+} \tau^{\pm} = \infty$. It follows that the curves will be nested and the boundary of the stability region will be defined by the line $\beta = \alpha$ in the right half plane and the curve defined by (2.14)–(2.15) closest to $\beta = -\alpha$ in the left half plane. The result follows. □

In general, $\beta^{\pm}$ may be quite complicated functions of ω, however, we can establish the following theorem about their monotonicity.

Theorem 2.5 *For fixed a_j and d_j if*

$$\tau_s < \min_j \frac{1}{d_j}\left(\sqrt{1+\frac{d_j}{|a_j|}}-1\right),$$

then β^+ is monotone increasing in ω and β^- is monotone decreasing in ω.

Proof From (2.12) and (2.14) we have

$$\begin{aligned}
\beta^{\pm 2} &= F_R^2 + F_I^2 \\
&= \left[\prod_{j=1}^{n}(d_j - a_j\cos\omega\tau_s) + i(\omega + a_j\sin\omega\tau_s)\right]\left[\prod_{j=1}^{n}(d_j - a_j\cos\omega\tau_s) - i(\omega + a_j\sin\omega\tau_s)\right] \\
&= \prod_{j=1}^{n}\left[(d_j - a_j\cos\omega\tau_s)^2 + (\omega + a_j\sin\omega\tau_s)^2\right]
\end{aligned}$$

Taking the derivative with respect to ω yields

$$\begin{aligned}
\beta^{\pm}\frac{d\beta^{\pm}}{d\omega} &= \left[\sum_{j=1}^{n}(d_j - a_j\cos\omega\tau_s)(a_j\tau_s\sin\omega\tau_s) + (\omega + a_j\sin\omega\tau_s)(1 + a_j\tau_s\cos\omega\tau_s)\right] \\
&\quad \times\left[\prod_{k=1,k\neq j}^{n}(d_k - a_k\cos\omega\tau_s)^2 + (\omega + a_k\sin\omega\tau_s)^2\right] \\
&= \beta^2\sum_{j=1}^{n}\frac{h_j(\omega)}{(d_j - a_j\cos\omega\tau_s)^2 + (\omega + a_j\sin\omega\tau_s)^2}
\end{aligned}$$

where

$$h_j(\omega) = \omega + a_j d_j\tau_s\sin\omega\tau_s + a_j\omega_j\tau_s\cos\omega\tau_s + a_j\sin\omega\tau_s$$

Note from eqs. (2.12) and (2.14) that β^+ is strictly positive and β^- strictly negative. Thus if each of the $h_j(\omega)$ is of the same sign for all ω then $\beta^{\pm}$ will be monotone in ω. Now,

$$\begin{aligned}
h_j(\omega) &= \omega + a_j(d_j\tau_s + 1)\sin\omega\tau_s + a_j\omega_j\tau_s\cos\omega\tau_s \\
&\geq \omega - |a_j|(d_j\tau_s + 1)\omega\tau_s - |a_j|\omega\tau_s \\
&= \omega(1 - 2|a_j|\tau_s - |a_j|d_j\tau_s^2) \\
&> \omega\left[1 - 2|a_j|\frac{1}{d_j}\left(\sqrt{1+\frac{d_j}{|a_j|}}-1\right) - |a_j|\frac{1}{d_j}\left(2+\frac{d_j}{|a_j|}-2\sqrt{1+\frac{d_j}{|a_j|}}\right)\right] \\
&= 0
\end{aligned}$$

Since $h_j(\omega) > 0$ for each j, β^+ is monotone increasing in ω and β^- is monotone decreasing in ω. □

This result is illustrated in the following figures which display the region of stability of the trivial solution of (2.2) in the $\beta,\ \tau$ plane for various values of the parameters a_j, d_j and τ_s. Figs. 2–4 show the results for a two element ring ($n = 2$) with $d_j = 1,\ j = 1, 2$ and $a_1 = -0.1,\ a_2 = -0.2$ (Fig. 2), $a_1 = 0.1,\ a_2 = -0.2$ (Fig. 3), and $a_1 = 0.1,\ a_2 = 0.2$ (Fig. 4). The stability plots for two values of τ_s are shown, illustrating the difference when the $\beta^{\pm}$ are monotone in ω from when they are not. The critical value of τ_s predicted by Theorem 2.5 is $\tau_s \approx 1.45$ for all cases.

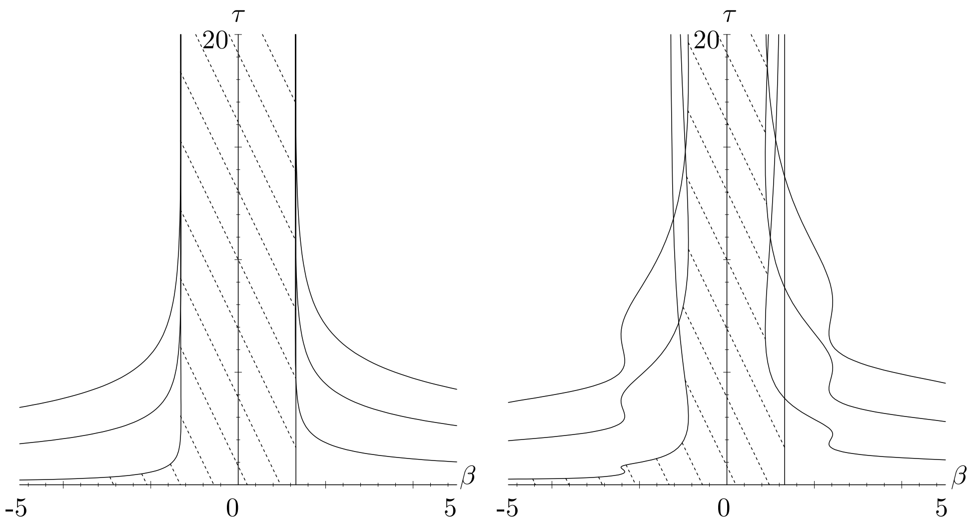

Figure 2 Region of stability in the $\beta,\ \tau$ plane for a two element network with negative self connections.(a) $\tau_s = 2$ (b) $\tau_s = 6$. Other parameter values are give in the text. Dashed lines indicate where the fixed point is stable.

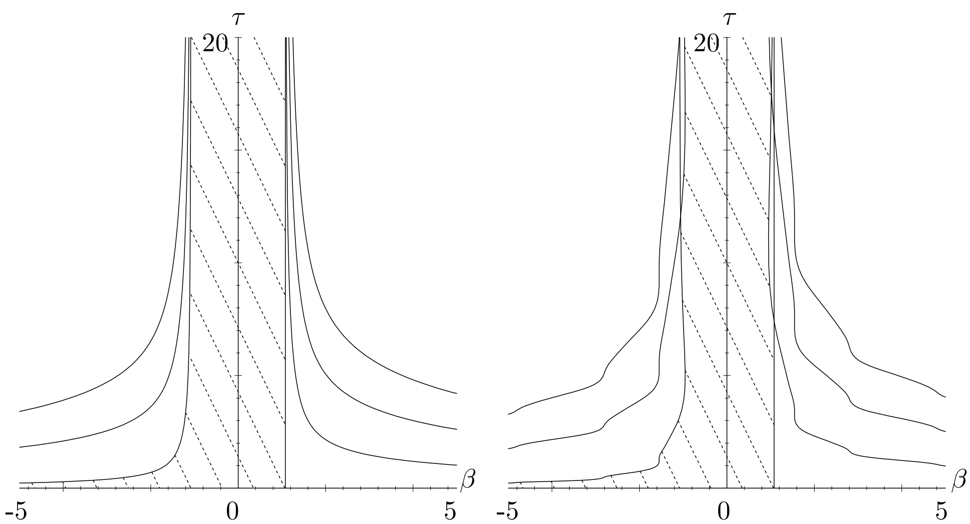

Figure 3 Region of stability in the $\beta,\ \tau$ plane for a two element network with one positive and one negative self connection.(a) $\tau_s = 5$ (b) $\tau_s = 9$. Dashed lines indicate where the fixed point is stable.

Figs. 5 and 6 show the results for a three element ring ($n = 3$) with $d_j = 1,\ j = 1, 2, 3$ and $a_1 = -0.1,\ a_2 = -0.2,\ a_3 = -0.3$ (Fig. 5) and $a_1 = 0.1,\ a_2 = 0.2,\ a_3 = 0.3$ (Fig. 6). The stability plots for two values of τ_s are shown, illustrating the difference when the $\beta^{\pm}$ are monotone in ω from when they are not. The critical value of τ_s predicted by Theorem 2.5 is $\tau_s \approx 1.08$ for both cases.

It is clear from Figs. 2(b)–6(b) that when $\beta^{\pm}$ are not monotone in ω the boundary of the stability region can be quite complicated as it is made up of pieces of the line (2.8) and arcs of the the curves defined by (2.12)–(2.13) and (2.14)–(2.15). It would be quite difficult to define this boundary analytically, however, we can state

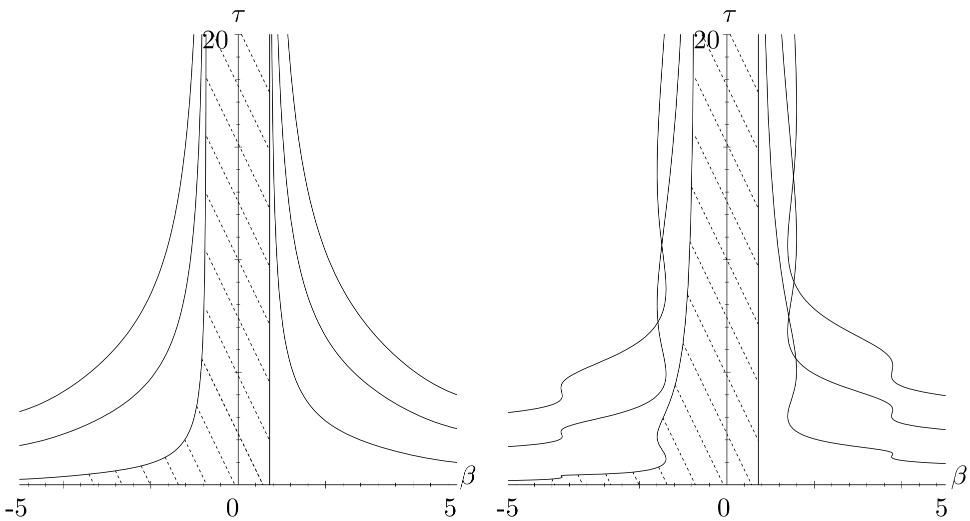

Figure 4 Region of stability in the β, τ plane for a two element network with positive self connections.(a) $\tau_s = 2$ (b) $\tau_s = 6$. Other parameter values are give in the text. Dashed lines indicate where the fixed point is stable.

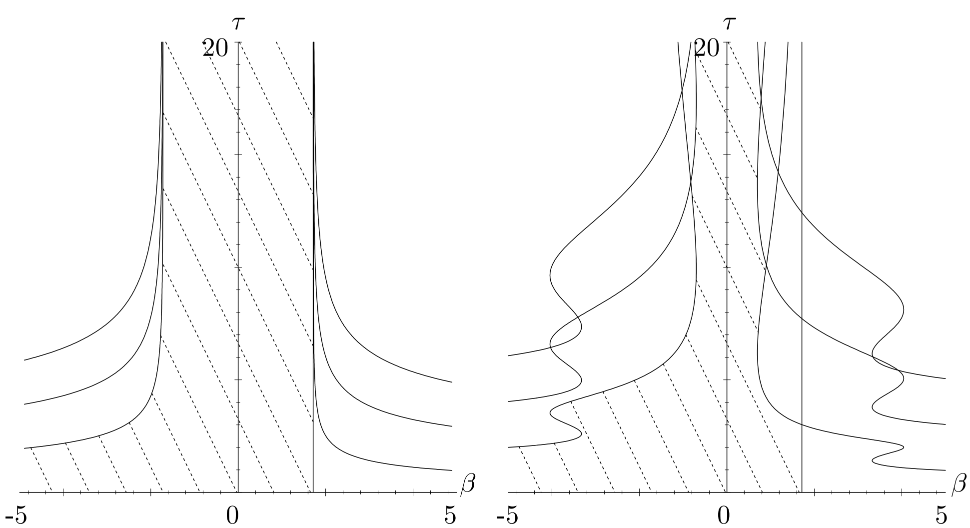

Figure 5 Region of stability in the β, τ plane for a three element network with negative self connections. (a) $\tau_s = 1.5$ (b) $\tau_s = 6$. Other parameter values are given in text. Dashed lines indicate where the fixed point is stable.

the following theorem about the region where the stability is independent of the delays τ_s and τ.

Theorem 2.6 *Let a_j, d_j, be fixed. If $|a_j| < d_j$, $j = 1, \ldots n$ then the trivial solution of (2.2) is stable for $-\beta_{min} < \beta < \beta_{min}$ and all $\tau_s > 0, \tau > 0$, where $\beta_{min} \stackrel{def}{=} \sqrt{\prod_{j=1}^{n}(d_j - |a_j|)^2}$. If $|a_j| > d_j$ for at least one j then there is no such region of delay independent stability.*

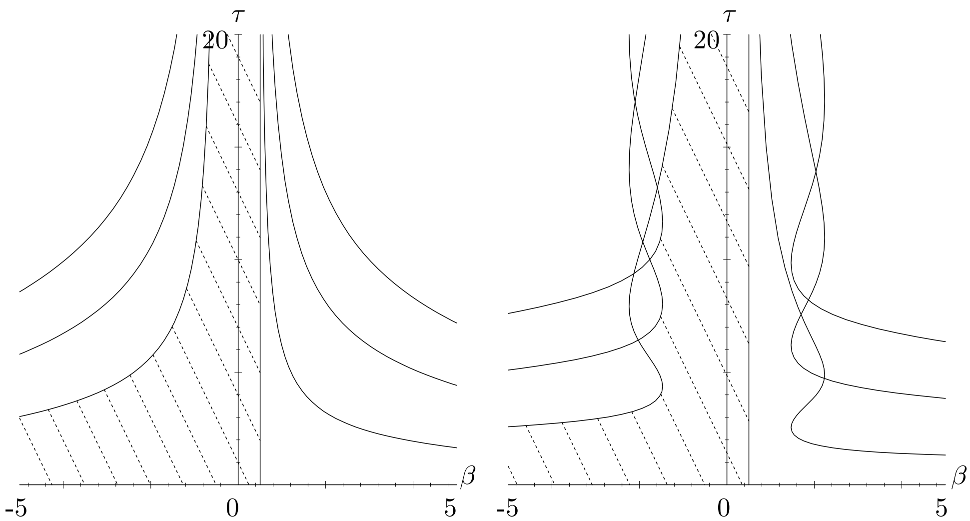

Figure 6 Region of stability in the β, τ plane for a three element network with positive self connections. (a) $\tau_s = 2$ (b) $\tau_s = 6$. Other parameter values are given in text. Dashed lines indicate where the fixed point is stable.

Proof Recall from Theorem 2.5 that

$$\beta^{\pm 2} = \prod_{j=1}^{n} \left[(d_j - a_j \cos \omega\tau_s)^2 + (\omega + a_j \sin \omega\tau_s)^2 \right].$$

Now if $|a_j| < d_j, \ j = 1, \ldots n$, then

$$(d_j - a_j \cos \omega\tau_s)^2 + (\omega + a_j \sin \omega\tau_s)^2 \geq (d_j - |a_j|)^2, \qquad j = 1, \ldots n,$$

and

$$\beta^{\pm 2} \geq \prod_{j=1}^{n} (d_j - |a_j|)^2.$$

Using a similar argument to that of Theorem 2.4 we see that the trivial solution of (2.2) is stable for $-\beta_{min} < \beta < \beta_{min}$ and all $\tau_s > 0, \tau > 0$.

However, if $|a_j| > d_j$ for some j, then $\beta^{\pm 2} = 0$ when $\omega = \sqrt{a_j^2 - d_j^2}$, and

$$\tau_s = \begin{cases} \dfrac{1}{\sqrt{a_j^2 - d_j^2}} \left[(2l+1)\pi - \mathrm{Arccos}\left(\dfrac{d_j}{a_j}\right) \right], & 0 < d_j < a_j \\ \dfrac{1}{\sqrt{a_j^2 - d_j^2}} \left[2l\pi + \mathrm{Arccos}\left(\dfrac{d_j}{a_j}\right) \right], & a_j < -d_j < 0 \end{cases},$$

for $l = 0, 1, 2, \ldots$. Hence there is no region for which the trivial solution is stable for all positive values of τ_s and τ. $\square$

Remark 2.7 Suppose that $|a_j| < d_j, \ j = 1, \ldots n$. Note that if $a_j > 0, \ j = 1, \ldots, n$ then $\beta_{min} = \alpha$ and this line will be the boundary of the region of stability for $\beta > 0$ for all values of τ_s. This situation is shown in Figs. 4(b) and 6(b). Alternatively, if at least one $a_j < 0$ then $d_j - |a_j| < d_j - a_j$ for this j and $\beta_{min} < \alpha$. In this case the boundary of the region of stability for $\beta > 0$ will be more complicated. This situation is shown in the other figures.

3 Bifurcations

One can verify that the roots of the characteristic equation as described above are simple and that they cross the imaginary axis with non zero speed. To do this we consider the following derivatives of the characteristic polynomial $\Delta(\lambda)$ as defined by (2.5):

$$\frac{d\Delta}{d\lambda} = \beta e^{-\lambda\tau}\left[\tau + \sum_{j=1}^{n}\frac{1+\tau_s a_j e^{-\lambda\tau_s}}{\lambda + d_j - a_j e^{-\lambda\tau_s}}\right] \qquad \text{and} \qquad \frac{d\Delta}{d\beta} = -e^{-\lambda\tau}, \tag{3.1}$$

where β and τ are as defined in (2.7). Consideration of these derivatives where the characteristic equation has a zero root, i.e. along the line $\beta = \alpha$ yields:

$$\left.\frac{d\Delta}{d\lambda}\right|_{\lambda=0} = \alpha\left[\tau + \sum_{j=1}^{n}\frac{1+\tau_s a_j}{d_j - a_j}\right] \tag{3.2}$$

and

$$\left.\frac{d\lambda}{d\beta}\right|_{\lambda=0} = -\left[\left.\frac{d\Delta}{d\lambda}\right|_{\lambda=0}\right]^{-1} \tag{3.3}$$

Thus for fixed a_j, b_j, τ_s the roots are simple and pass through zero with non zero speed everywhere on the line $\beta = \alpha$ except possibly for one isolated point given by

$$\tau = -\sum_{j=1}^{n}\frac{1+\tau_s a_j}{d_j - a_j}. \tag{3.4}$$

Consideration of the derivatives (3.1) when the characteristic equation has a pure imaginary root, i.e. along the curves defined by (2.12)–(2.13) and (2.14)–(2.15) yields:

$$\begin{aligned} Re\left[\left.\frac{d\lambda}{d\beta}\right|_{\lambda=i\omega}\right] \quad &= \quad -\frac{1}{\beta S^2}\Bigg[\tau \\ &+ \quad \sum_{j=1}^{n}\frac{d_j - \tau_s a_j^2 + (d_j\tau_s - 1)a_j\cos\omega\tau_s - \omega\tau_s a_j \sin\omega\tau_s}{K_j^2}\Bigg] \end{aligned} \tag{3.5}$$

where

$$\begin{aligned} S^2 \quad &= \quad \left[\sum_{j=1}^{n}\frac{d_j - \tau_s a_j^2 + (d_j\tau_s - 1)a_j\cos\omega\tau_s - \omega\tau_s a_j\sin\omega\tau_s}{K_j^2} + \tau\right]^2 \\ &+ \quad \left[\sum_{j=1}^{n}\frac{\omega + (d_j\tau_s + 1)a_j\sin\omega\tau_s + \omega\tau_s a_j\cos\omega\tau_s}{K_j^2}\right]^2 \\ K_j^2 \quad &= \quad (d_j - a_j\cos\omega\tau_s)^2 + (\omega + a_j\sin\omega\tau_s)^2 \end{aligned}$$

Thus the pure imaginary roots will pass through the imaginary axis will nonzero speed everywhere except at isolated points $(\beta^{\pm}(\omega_s), \tau^{\pm}(\omega_s))$ where ω_s is found by solving

$$\tau^{\pm}(\omega) = -\sum_{j=1}^{n}\frac{d_j - \tau_s a_j^2 + (d_j\tau_s - 1)a_j\cos\omega\tau_s - \omega\tau_s a_j\sin\omega\tau_s}{K_j^2},$$

$\beta^{\pm}$ are given by (2.12) and (2.14), $\tau^{\pm}$ are given by (2.13) and (2.15). From (3.1) if $Re\left[\left.\frac{d\lambda}{d\beta}\right|_{\lambda=i\omega}\right] \neq 0$ then $\left.\frac{d\Delta}{d\lambda}\right|_{\lambda=i\omega} \neq 0$ also, thus the pure imaginary roots will certainly be simple if they cross the imaginary axis with nonzero speed.

We conclude that if the nonlinearities of equation (1.2) satisfy appropriate non-degeneracy conditions, the curves (2.12)–(2.13) and(2.14)–(2.15) define Hopf bifurcation curves in the β, τ plane, and the vertical line $\beta = \alpha$ defines a steady state bifurcation. The exact nature of these bifurcations, i.e. whether the Hopf bifurcations are supercritical or subcritical, and what type of steady state bifurcation occurs, would depend on the nonlinearities f_j, g_j. As an example, consider the case when $n = 2$ and the nonlinearities are given by $f_j(u) = a_j \tanh(u)$, $g_j(u) = b_j \tanh(u)$. Then $\mathbf{u}^* = \mathbf{0}$ and it can be shown (Shayer [1998]) that the steady state bifurcation is a supercritical pitchfork. One way to determine the nature of the Hopf bifurcations is to analyze the center manifold for equations. In Campbell & Bélair [1995] we describe a Maple program which will carry out such an analysis. Applying our program to this example shows that for fixed d_j, a_j and τ_s the criticality of the Hopf bifurcation may vary with ω along the curves shown in Figs 2–Figs 6. More significantly, even if τ and β are fixed, corresponding to a particular point on one of these curves, the criticality may still change if b_j or τ_j change.

Codimension two bifurcation points occur when two of these different bifurcations happen simultaneously. Points of Hopf-Hopf interaction exist when the characteristic equation has two pairs of pure imaginary roots $\pm i\omega_1,\ \pm i\omega_2$. For the system we are considering these points can occur for any set of values of the parameters a_j, d_j, τ_s such that $\beta^{\pm}(\omega_1) = \beta^{\pm}(\omega_2)$ and $\tau_j^{\pm}(\omega_1) = \tau_k^{\pm}(\omega_1)$, for some $j, k \in \mathbb{Z}$. These points cannot, in general, be solved for in closed form. They can, however, be computed numerically and are easily seen in Figs. 2(b)–6(b), where they appear as intersection points of two of the Hopf bifurcation curves. Hopf-steady state interactions exist when the characteristic equations has both a zero root and a pure imaginary pair. For our system these points may be found by solving $\beta(\omega) = \alpha$ for ω and substituting in τ^+ as given by (2.13). Again these are readily visible in Figs. 2(b)–6(b) where they appear as intersection points of the Hopf bifurcation curves with the line $\beta = \alpha$. From Theorem 2.5 it is clear that neither type of codimension two point can occur in our system if

$$\tau_s < \min_j \frac{1}{d_j}\left(\sqrt{1 + \frac{d_j}{|a_j|}} - 1\right).$$

Codimension two points can be the source of more complicated dynamics such as multistability and quasiperiodicity. To illustrate what may occur in the system we are studying, we performed numerical simulations near one Hopf-Hopf and the Hopf-steady state interaction points which occur next to the stability region in Fig. 2(b). We chose our nonlinearities as in the example discussed above. Plots in the $u_1,\ u_2$ plane of two of these simulations are shown in Fig. 3. Fig. 3(a) shows the existence of a stable 2-torus near the Hopf-Hopf interaction at $\tau \approx 7$, $\beta \approx -1.2$. Fig. 3(b) shows the coexistence of a stable limit cycle with two stable fixed points near the Hopf-pitchfork interaction at $\tau \approx 4$, $\beta = 1.32$.

4 Conclusions

We have studied a system of delay differential equations representing a simple model for a ring of neurons with time delayed connections between each neuron

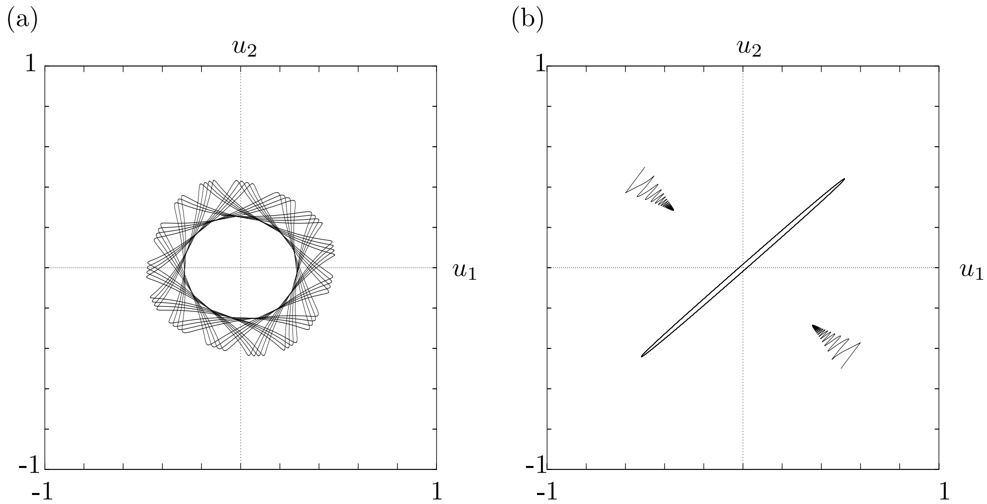

Figure 7 Numerical simulations of a two element network given by (1.2) with $d_j = 1$, $\tau_s = 6$, $f_1(x) = -0.1\tanh(x)$,$f_2(x) = -0.2\tanh(x)$ and $g_j(x) = b_j\tanh(x)$. Other parameter values are (a) $b_1 = 1.2$, $b_2 = -1$, (giving $\beta = -1.2$) and $\tau = 6$ (b) $b_1 = -1.4$, $b_2 = -1$ (giving $\beta = 1.4$) and $\tau = 4$.

and its immediate predecessor in the ring and time delayed feedback from each neuron to itself. We showed how conditions for the linear stability of fixed points of this system may be easily represented in a parameter space consisting of the sum of the time delays between the elements and the product of the strengths of the connections between the elements. A connection was made between the stability of the fixed point in the ring and the stability of fixed points in the individual neurons which comprise it. It was shown that both Hopf and steady state bifurcations may take place when a fixed point loses stability. Conditions under which interactions between these bifurcations may take place were given and numerical simulations revealed the possibility of quasiperiodicity and multistability near such points.

Our work is complementary to that of Ye, Michel & Wang [1995] who analyzed the global stability of a general network of n neurons with different time delays in each connection. One should expect their regions of global stability to lie within our region of local stability. This has been checked (and found to be true) by Shayer [1998] in the case $n = 2$ with nonlinearity $f_j(u) = a_j\tanh(u)$, $g_j(u) = b_j\tanh(u)$. Olien & Bélair [1997] have made a detailed study of the stability and bifurcations of a network consisting of two neurons with two time delays. However, as they choose to identify a particular time delay with a given neuron (representing the processing time at that neuron) their work is not directly comparable to ours, even in the case $n = 2$. Baldi & Atiya [1994] have considered a ring of n neurons with n time delays. The main difference between their work and our is that we include delayed self connection terms (represented by $f_j(u_j(t - \tau_s))$ in (1.2)). As far as the linear stability analysis is concerned, their system effectively has one time delay: τ, the product of the delays in the connections between neurons, whereas ours effectively has two: τ and τ_s. Our analysis can be applied to their system by taking $a_j = 0$ for all j. Consideration of Theorems 2.4 and 2.5 shows that the stability diagrams in this case will always be qualitatively like Fig. 2(a). That is, the stability region will be bounded on one side by the steady state bifurcation curve and on the other by one curve of Hopf bifurcation. As the codimension two points cannot exist one would not expect more complex dynamics than periodic solutions and fixed points. This is consistent with the results given by Baldi & Atiya [1994].

While ring networks of are of limited biological relevance, they may be regarded as building blocks for networks with more realistic connection topologies. One approach which seems promising is to decompose larger networks into sets of connected rings and use studies such as ours to gain insight into the dynamics the networks. For example, many of our results have shown how stability of the ring depends on the stability of the individual neurons. It is possible this may be extended to show how the stability of a network depends on the stability of the individual rings. See Baldi & Atiya [1994] for further discussion and example applications of similar ideas.

Acknowlegements

This work has benefitted from the support of the Natural Sciences and Engineering Research Council of Canada and from useful discussions with L. Shayer and J. Wainwright. The numerical simulations were performed with the package XPP written by Bard Ermentrout.

References

Baldi, P. and Atiya, A. [1994] *How delays affect neural dynamics and learning*, IEEE Trans. Neural Networks, **5**(4), 612–621.

Bélair, J., Campbell, S., and van den Driessche, P. [1996] *Frustration, stability and delay-induced oscillations in a neural network model*, SIAM J. Appl. Math., **56**(1), 245–255.

Campbell, S. and Bélair, J. [1995] *Analytical and symbolically-assisted investigation of Hopf bifurcations in delay-differential equations*, Can. Appl. Math. Quart., **3**(2), 137–154.

Gopalsamy, K. and Leung, I. [1996] *Delay induced periodicity in a neural netlet of excitation and inhibition*, PhysicaD, **89**, 395–426.

Hale, J. and Lunel, S. V. [1993] *Introduction to Functional Differential Equations*. Springer Verlag, New York.

Hopfield, J. [1984] *Neurons with graded response have collective computational properties like those of two-state neurons*, Proc. Nat. Acad. Sci. USA, **81**, 3088–3092.

Kolmanovskii, V. and Nosov, V. [1986] *Stability of Functional Differential Equations*, Academic Press, London.

Marcus, C. and Westervelt, R. [1989] *Stability of analog neural networks with delay*, Phys. Rev. A, **39**(1), 347–359.

Olien, L. and Bélair, J. [1997] *Bifurcations, stability, and monotonicity properties of a delayed neural network model*, Physica D, **102**(3-4), 349–363.

Shayer, L.P. [1998] *Analysis of a system of two coupled neurons with two time delays*, Master's thesis, University of Waterloo, (in preparation).

Stépán, G. [1989] *Retarded Dynamical Systems*, volume 210 of *Pitman Research Notes in Mathematics*. Longman Group, Essex.

Ye, H., Michel, A. N., and Wang, K. [1994] *Global stability and local stability of Hopfield neural networks with delays*, Phys. Rev. E, **50**(5), 4206–4213.

Ye, H., Michel, A. N., and Wang, K. [1995] *Qualitative analysis of Cohen-Grossberg neural networks with multiple delays*, Phys. Rev. E, **51**(3), 2611–2618.