DELAY INDEPENDENT STABILITY OF LINEAR SWITCHING SYSTEMS WITH TIME DELAY

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Abstract. We consider a switching system with time delay composed of a finite number of linear delay differential equations (DDEs). Each DDE consists of a sum of a linear ODE part and a linear DDE part. We study two particular cases: (a) all the ODE parts are stable and (b) all the ODE parts are unstable and determine conditions for delay independent stability. For case (a), we extend a standard result of linear DDEs via the multiple Lyapunov function and functional methods. For case (b) the standard DDE result is not directly applicable, however, we are able to obtain uniform asymptotic stability using the single Lyapunov function and functional methods.

Key words. switching systems, Lyapunov functional, delay differential equations

1. Introduction. A *switching system* is a type of *hybrid system* which is a combination of discrete and continuous dynamical systems. The discrete dynamical systems are usually called switching rules, logical decisions or finite automata. These systems arise as models for phenomena which cannot be described by exclusively continuous or exclusively discrete processes. Examples include the control of manufacturing systems [1, 2], communication networks, traffic control [3, 4, 5], chemical processing [6] and automotive engine control and air craft control [7].

Delay differential equations (DDEs) arise as models for systems where the rate of change of the state depends not only on the current state of the system but also on its state at some time(s) in the past (see e.g. [8, 9, 10, 11]). This is especially important for control systems where actuators, sensors and transmission lines may introduce time lags [12], and biological systems where there are large time delays associated with population growth of a species, transmission of disease [13], and learning process [14]. Since these biological systems can be modeled as switching systems [15, 16] and many of the applications of switching systems involve control, it is natural to consider the effect of the time delays in such systems, i.e., *switching systems with time delay.*

In some systems, the time delay is large or uncertain. For example, it may not be easy to measure the delay experimentally. Hence, establishing delay independent stability conditions for these systems would be beneficial. A standard result from the stability theory of DDEs states that a linear system will have a globally asymptotically stable equilibrium point at 0 if the system consists of a stable ODE part and a DDE part with sufficiently small norm. More precisely, we have the following:

Proposition 1.1. *[17] Let A, $B \in \mathcal{R}^n$ and consider the linear DDE*

$$\dot{x}(t) = Ax(t) + Bx(t-\tau), \tag{1.1}$$

where $\tau > 0$ is a time delay. If

(i) A is Hurwitz,

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(ii) $\|B\|$ is sufficiently small
then the trivial solution of (1.1) *is globally asymptotically stable for all* $\tau \geq 0$.
In this paper we will extend this idea to linear switching system with time delay. In particular, we consider the following linear switching system with delay

$$\dot{x}(t) = A_i x(t) + B_i x(t-\tau), \tag{1.2}$$

where $x \in \mathcal{R}^n$, A_i, $B_i \in \mathcal{R}^{n\times n}$, $\tau > 0$ a constant, and $i \in \mathcal{Q} = \{1, \cdots, N\}$.
We will examine two cases: (i) all A_i are stable and (ii) all A_i are unstable. For both cases we will establish delay independent stability conditions. For case (i) we use *multiple Lyapunov functional and function methods*. For case (ii), we use *single Lyapunov functional and function methods*.
The paper is organized as follows. In section 2 we provide background on Lyapunov stability for DDEs. Sections 3 and 4 contain our main results. In particular, in section 3 all ODE parts are stable and we establish stability conditions in terms of *multiple Lyapunov functionals and functions*. In section 4 we focus on the case when all ODE parts are unstable, and establish stability conditions in terms of a *single Lyapunov functional and function*. Finally, in Section 5 we provide an example, and in Section 6 we discuss our results and future work.

2. Background. Suppose that $f : \mathcal{R} \times \mathcal{C} \mapsto \mathcal{R}^n$ is continuous and consider the DDE given by

$$\dot{x}(t) = f(t, x_t), \tag{2.1}$$

where $x_t \in \mathcal{C} = C([-\tau, 0], \mathcal{R}^n)$ is defined by $x_t(\theta) = x(t+\theta)$, $\theta \in [-\tau, 0]$, $\tau > 0$, $\|x_t\|_\tau = \sup_{-\tau\leq\theta\leq 0} \|x_t(\theta)\|$ and $\|\cdot\|$ is Euclidean norm. Without loss of generality, we will restrict our stability investigation to the trivial solution of (2.1). Let $V : \mathcal{R} \times \mathcal{C} \mapsto \mathcal{R}^n$ be continuous and $x(t)$ be the solution of (2.1). Then, we define the upper right-hand derivative of $V(t, x_t)$ along the solution of (2.1) by

$$\dot{V}(t, x_t) = \lim \sup_{h\to 0^+} \frac{1}{h}[V(t+h, x_{t+h}(t) - V(t, x_t)]. \tag{2.2}$$

This leads to the following theorem.

Theorem 2.1. *[18, Section 5.3] Suppose that $f : \mathcal{R} \times \mathcal{C} \mapsto \mathcal{R}^n$ takes $\mathcal{R} \times$ (bounded sets of $\mathcal{C}$) into bounded sets of $\mathcal{R}^n$, and u, v, $w : \mathcal{R}^+ \mapsto \mathcal{R}^+$ are continuous nondecreasing functions, $u(s)$, $v(s)$, $w(s)$ are positive for $s > 0$, and $u(0) = v(0) = 0$. If there is continuous function $V : \mathcal{R} \times \mathcal{C} \mapsto \mathcal{R}$ such that*

$$u(\|x(t)\|) \leq V(t, x_t) \leq v(\|x_t\|_\tau)$$

$$\dot{V}(t, x_t) \leq -w(\|x(t)\|)$$

then the trivial solution $x = 0$ of (2.1) *is uniformly asymptotically stable.*
We will call a functional $V(t, x_t)$ satisfying the conditions of Theorem 2.1 a Lyapunov functional.
Let $V : \mathcal{R} \times \mathcal{R}^n \mapsto \mathcal{R}$ be a continuous function, and $x(t)$ be a solution of (2.1). Then the upper right-hand derivative of V along a solution of (2.1), $\dot{V}(t, x(t))$, is defined by

$$\dot{V}(t, x(t)) = \lim \sup_{h\to 0^+} \frac{1}{h}[V(t+h, x(t+h)) - V(t, x(t))].$$

This leads to the following theorem.

THEOREM 2.2. *[18, Section 5.4] Suppose that $f : \mathcal{R} \times \mathcal{C} \mapsto \mathcal{R}^n$ takes $\mathcal{R} \times$ (bounded sets of $\mathcal{C}$) into bounded sets of $\mathcal{R}^n$, and u, v, $w : \mathcal{R}^+ \mapsto \mathcal{R}^+$ are continuous nondecreasing functions, $u(s)$, $v(s)$, $w(s)$ are positive for $s > 0$, and $u(0) = v(0) = 0$. If there are a continuous function $V : \mathcal{R} \times \mathcal{R}^n \mapsto \mathcal{R}$ and a continuous and nondecreasing function $p(s) > s$ for $s > 0$ such that*

$$u(\|x(t)\|) \leq V(t, x_t) \leq v(\|x(t)\|)$$

and

$$\dot{V}(t, x(t)) \leq -w(\|x(t)\|) \tag{2.3}$$

if

$$V(t+\theta, x(t+\theta)) < p(V(t, x(t))), \tag{2.4}$$

for $\theta \in [-\tau, 0]$, then, the solution $x = 0$ of (2.1) *is uniformly asymptotically stable.* We will call a function $V(t, x(t))$ satisfying the conditions of Theorem 2.2 a Lyapunov function. Note that (2.4) is called a *Razumikhin Condition.* In the following sections we consider two cases: all A_i are stable and all A_i are unstable. By using the above two theorems we will provide sufficient conditions for *delay independent stability* of (1.2).

3. All A_i are Stable. In this case, the stability of each mode will be guaranteed for all $\tau \geq 0$ if the norm of B_i in (1.2) is sufficiently small. However, switching among stable modes does not always imply stability of the switching system, as shown in [19]. Hence, we will establish delay independent stability conditions consisting of two parts: (a) delay independent conditions on the B_i which guarantee the existence of Lyapunov functionals or functions giving the stability of each subsystem; and (b) conditions which guarantee stability of the switching system given the stability of the subsystems.

3.1. Multiple Lyapunov Functional Method. Recall the switching system (1.2)

$$\dot{x}(t) = A_i x(t) + B_i x(t-\tau). \tag{3.1}$$

Consider the following conditions.

H_1. There exist continuous and increasing functions $\alpha(\cdot)$ and $\beta(\cdot)$ such that $\alpha(0) = \beta(0) = 0$ and

$$\alpha(\|x(t)\|) \leq V_i(x_t) \leq \beta(\|x_t\|_\tau).$$

H_2. For each i, there is a continuous and increasing function $\psi_i(\cdot)$ such that $\psi_i(s) > s$ for $s > 0$, $\psi_i(0) = 0$ and

$$\dot{V}_i(x_t) \leq -\psi_i(\|x(t)\|).$$

H_3. There is $\mu > 1$ such that

$$V_i(x_t) \leq \mu V_j(x_t) \quad \text{for all } x_t \in C([-\tau, 0], \mathcal{R}^n) \text{ and for } i \neq j.$$

H_4. For any pair of consecutive switching times $\{t_p, t_q\}$ of the i^{th} mode such that $t_p < t_q$ and the i^{th} mode is active at t_p and t_q, respectively, there is a constant $0 < \xi_i < 1$ such that

$$V_i(x_{t_q}) - V_i(x_{t_p}) \leq -\xi_i V_i(x_{t_p}).$$

THEOREM 3.1. *If* (3.1) *has functionals* V_i, $i = 1, 2, \cdots, N$, *satisfying* H_1–H_4 *then it is uniformly asymptotically stable for any switching rule.*

Proof. The proof can be found in Appendix A. □

Since A_i in (3.1) is stable for each i, we can construct a Lyapunov functional for each mode of the form

$$V_i(x_t) = x^T(t)P_i x(t) + \omega_i \int_{t-\tau}^{t} \|x(s)\|^2 ds, \tag{3.2}$$

where $\omega_i > 0$, and P_i is a symmetric positive definite matrix satisfying $A_i^T P_i + P_i A_i = -Q_i$ for some symmetric positive definite matrix Q_i.

PROPOSITION 3.2. *There exist continuous and increasing functions* $\alpha(\cdot)$, *and* $\beta(\cdot)$ *such that* $\alpha(0) = \beta(0) = 0$ *and*

$$\alpha(\|x(t)\|) \leq V_i(x_t) \leq \beta(\|x_t\|_\tau). \tag{3.3}$$

Proof. From (3.2) we can easily show that (3.5) is satisfied with $\alpha(\|x(t)\|) = \min_i \lambda_{min}(P_i)\|x(t)\|^2$ and $\beta(\|x_t\|_\tau) = \max_i(\lambda_{max}(P_i) + \omega_i\tau)\|x_t\|_\tau^2$. □

For stability we need the following propositions.

PROPOSITION 3.3. *Suppose that the* i^{th} *mode is active on* $[t_k, t_{k+1})$, *for some* $i \in \{1, 2, \cdots, N\}$. *Let* P_i *and* Q_i *be defined above. If*

$$\|P_i B_i\| < \frac{\lambda_{min}(Q_i)}{2}, \tag{3.4}$$

then there exist $\omega_i > 0$ *and a continuous, increasing function,* $\psi_i : \mathcal{R}^+ \mapsto \mathcal{R}^+$, *satisfying* $\psi(0) = 0$ *and* $\psi(s) > 0$ *for* $s > 0$ *such that*

$$\dot{V}_i(x_t) \leq -\psi_i(\|x(t)\|), \quad \textit{for } t \in [t_k, t_{k+1})$$

where V_i *is given by* (3.2).

Proof. Since $\|P_i B_i\| < \frac{1}{2}\lambda_{min}(Q_i)$, $\exists\ \omega_i > 0$ such that $\|P_i B_i\| \leq \omega_i < \frac{\lambda_{min}(Q_i)}{2}$. Let $V_i(x_t)$ be given by (3.2) with this ω_i.

It follows that

$$\begin{aligned}
\dot{V}_i(x_t) &= \dot{x}^T(t)P_i x(t) + x^T(t)P_i\dot{x}(t) + \omega_i(\|x(t)\|^2 - \|x(t-\tau)\|^2) \\
&= (A_i x(t) + B_i x(t-\tau))^T P_i x(t) + x^T(t)P_i(A_i x(t) + B_i x(t-\tau)) \\
&\quad +\omega_i(\|x(t)\|^2 - \|x(t-\tau)\|^2) \\
&= x^T(t)(A_i^T P_i + P_i A_i)x(t) + 2x^T(t)P_i B_i x(t-\tau) \\
&\quad +\omega_i(\|x(t)\|^2 - \|x(t-\tau)\|^2).
\end{aligned}$$

Using the definition of Q_i and ω_i we have

$$\begin{aligned} \dot{V}_i(x_t) &\leq -\lambda_{min}(Q_i)\|x(t)\|^2 + \|P_iB_i\|(\|x(t)\|^2 + \|x(t-\tau)\|^2) \\ &\quad +\omega_i(\|x(t)\|^2 - \|x(t-\tau)\|^2) \\ &\leq -\lambda_{min}(Q_i)\|x(t)\|^2 + \omega_i(\|x(t)\|^2 + \|x(t-\tau)\|^2) \\ &\quad +\omega_i(\|x(t)\|^2 - \|x(t-\tau)\|^2) \\ &= -(\lambda_{min}(Q_i) - 2\omega_i)\|x(t)\|^2. \end{aligned}$$

The result follows with $\psi_i(\|x(t)\|) = (\lambda_{min}(Q_i) - 2\omega_i)\|x(t)\|^2$. □

PROPOSITION 3.4. *There is $\mu > 1$ such that*

$$V_i(x_t) \leq \mu V_j(x_t) \text{ for all } x_t \in C([-\tau, 0], \mathcal{R}^n) \text{ and for } i \neq j. \tag{3.5}$$

Proof. From (3.2) we have

$$\begin{aligned} V_i(x_t) &= x^T(t)P_ix(t) + \omega_i \int_{t-\tau}^{t} \|x(s)\|^2 ds \\ &\leq \lambda_{max}(P_i)x^T(t)x(t) + \omega_i \int_{t-\tau}^{t} \|x(s)\|^2 ds \\ &= \tfrac{\lambda_{max}(P_i)}{\lambda_{min}(P_j)}\lambda_{min}(P_j)x^T(t)x(t) + \tfrac{\omega_i}{\omega_j}\omega_j \int_{t-\tau}^{t} \|x(s)\|^2 ds \\ &\leq \tfrac{\lambda_{max}(P_i)}{\lambda_{min}(P_j)}x^T(t)P_jx(t) + \tfrac{\omega_i}{\omega_j}\omega_j \int_{t-\tau}^{t} \|x(s)\|^2 ds \\ &\leq \mu(x^T(t)P_jx(t) + \omega_j \int_{t-\tau}^{t} \|x(s)\|^2 ds) \\ &= \mu V_j(x_t), \end{aligned}$$

where $\mu = \max\{\sup_{i,j} \frac{\lambda_{max}(P_i)}{\lambda_{min}(P_j)}, \sup_{i,j} \frac{\omega_i}{\omega_j}\}$. □

From Proposition 3.2 and Proposition 3.4 it is clear that the Lyapunov functionals defined by (3.2) satisfy H_1 and H_3, respectively. From Proposition 3.3, H_2 will be satisfied if (3.4) is satisfied. H_4 may be satisfied by an appropriate choice of switching rule, see [20] and example 1 in Section 5.

3.2. Multiple Lyapunov Function Method. Let $V_i : \mathcal{R}^n \mapsto \mathcal{R}^n$ be continuous, and $x(t)$ be a solution of the switching system (3.1) for $i = 1, \cdots, N$. Consider the following conditions for stability of the switching system (3.1).

$\tilde{H}_1$. There are continuous and increasing functions $\alpha(\cdot)$ and $\beta(\cdot)$ such that $\alpha(0) = \beta(0) = 0$ and

$$\alpha(\|x(t)\|) \leq V_i(x(t)) \leq \beta(\|x(t)\|).$$

$\tilde{H}_2$. For each i, there are continuous and increasing functions $\psi_i(\cdot)$ and $p_i(\cdot)$ such that $\psi_i(s) > s$ for $s > 0$, $\psi_i(0) = 0$, $p_i(s) > s$ for $s > 0$ and

$$\dot{V}_i(x(t)) \leq -\psi_i(\|x(t)\|)$$

for $V_i(x(t+\theta)) < p_i(V_i(x(t)))$, $\theta \in [-\tau, 0]$.

$\tilde{H}_3$. There is $\mu > 1$ such that

$$V_i(x) \leq \mu V_j(x) \text{ for all } x \text{ and for } i \neq j.$$

$\tilde{H}_4$. For any pair of consecutive switchings $\{t_p, t_q\}$ of the mode i such that $t_p < t_q$, there is a constant $0 < \xi_i < 1$ such that

$$\overline{V_i}(t_q) - \overline{V_i}(t_p) \leq -\xi_i \overline{V_i}(t_p),$$

where $\overline{V}_i$ is defined as

$$\overline{V}_i(t) = \sup_{-\tau \leq \theta \leq 0} V_i(x(t+\theta)), \tag{3.6}$$

THEOREM 3.5. *If* (3.1) *has Lyapunov functions satisfying* $\tilde{H}_1 - \tilde{H}_4$ *then it is uniformly asymptotically stable for any switching rule.*

Proof. The proof can be found in Appendix B. □

Since each A_i is stable, we can define a Lyapunov function by

$$V_i(x) = x^T P_i x \tag{3.7}$$

where P_i is a symmetric positive definite matrix and satisfies $A_i^T P_i + P_i A_i = -Q_i$ for a given symmetric positive definite matrix Q_i. Clearly, V_i satisfies $\tilde{H}_1$ since

$$\lambda_{min}(P_i)\|x\|^2 \leq V_i(x) \leq \lambda_{max}(P_i)\|x\|^2. \tag{3.8}$$

$\tilde{H}_3$ may also be obtained from this equation. For $\tilde{H}_2$, we consider the following.

PROPOSITION 3.6. *Assume that the* i^{th} *mode is active on* $[t_k, t_{k+1})$ *for some* $i \in \{1, \cdots, N\}$, *and let* V *be given by* (3.7). *If*

$$\|P_i B_i\| < \frac{\lambda_{min}(Q_i)}{1 + \frac{\lambda_{max}(P_i)}{\lambda_{min}(P_i)}} \tag{3.9}$$

then, there exist a $q_i > 1$ *and* $\psi_i : \mathcal{R}^+ \mapsto \mathcal{R}^+$ *with* $\psi_i(0) = 0$ *and* $\psi_i(s) > s$ *for* $s > 0$ *such that*

$$\dot{V}_i(x(t)) \leq -\psi_i(\|x(t)\|)$$

when

$$q_i V_i(x(t)) > V_i(x(t+\theta)), \ \textit{for } \theta \in [-\tau, 0], \ i = 1, \cdots, N. \tag{3.10}$$

Proof. Let V_i be given by (3.7) and $x(t)$ be a solution of (3.1) for $t \in [t_k, t_{k+1})$. Then,

$$\begin{aligned}
\dot{V}_i &= \dot{x}^T(t) P_i x(t) + x^T(t) P_i \dot{x}(t) \\
&= (A_i x(t) + B_i x(t-\tau))^T P_i x(t) + x^T(t) P_i (A_i x(t) + B_i x(t-\tau)) \\
&= x^T(t)(A_i^T P_i + P_i A_i) x(t) + 2 x^T(t) P_i B_i x(t-\tau).
\end{aligned}$$

By (3.9) there exists $q_i > 1$ such that

$$\|P_i B_i\| \leq \frac{\lambda_{min}(Q_i)}{1 + q_i \frac{\lambda_{max}(P_i)}{\lambda_{min}(P_i)}} < \frac{\lambda_{min}(Q_i)}{1 + \frac{\lambda_{max}(P_i)}{\lambda_{min}(P_i)}}.$$

Choose this q_i and note that (3.10) can be rewritten as

$$q_i \frac{\lambda_{max}(P_i)}{\lambda_{min}(P_i)} \|x(t)\|^2 > \|x(t+\theta)\|^2 \text{ for } \theta \in [-\tau, 0]. \tag{3.11}$$

By using $A_i^T P_i + P_i A_i = -Q_i$ and (3.11), we have

$$\begin{aligned} \dot{V}_i &\leq -\lambda_{min}(Q_i)\|x(t)\|^2 + \|P_i B_i\|(\|x(t)\|^2 + \|x(t-\tau)\|^2) \\ &< -\lambda_{min}(Q_i)\|x(t)\|^2 + \|P_i B_i\|(\|x(t)\|^2 + q_i \tfrac{\lambda_{max}(P_i)}{\lambda_{min}(P_i)}\|x(t)\|^2) \\ &= -(\lambda_{min}(Q_i) - \|P_i B_i\|(1 + q_i \tfrac{\lambda_{max}(P_i)}{\lambda_{min}(P_i)}))\|x(t)\|^2 \end{aligned}$$

Hence,

$$\dot{V}_i \leq -\psi_i(\|x(t)\|)$$

when (3.10) holds, where $\psi_i(\|x(t)\|) = (\lambda_{min}(Q_i) - \|P_i B_i\|(1 + q_i \frac{\lambda_{max}(P_i)}{\lambda_{min}(P_i)}))\|x(t)\|^2$.
□

It follows from Proposition 3.6 that the Lyapunov function $V_i(x) = x^T P_i x$ satisfies $\tilde{H}_2$ if (3.9) is satisfied. As for the multiple Lyapunov functional case, $\tilde{H}_4$ can be satisfied by designing a proper switching rule. See [20] and the discussion in Section 6.

4. All A_i are Unstable. A well known result from the stability theory of linear DDEs is that, in a system of the form (1.2), if the ODE part is stable and the coefficient matrix of the DDE part is sufficiently small, then the delayed system will be stable regardless of the size of delay [17]. In this case A_i is unstable, so we may not be able to apply the theory directly. However, if we assume that there exists a Hurwitz convex combination of the A_i, viz.,

$$A = \sum_{i=1}^{N} \alpha_i A_i \quad \text{is Hurwitz}, \tag{4.1}$$

where $0 < \alpha_i < 1$, and $\sum_{i=1}^{N} \alpha_i = 1$, we will show that there is a way to achieve stability for the switching system (1.2) using this theory and the *single Lyapunov functional and function methods*.
Since $A = \sum_{i=1}^{N} \alpha_i A_i$ is Hurwitz, we know there exists a symmetric positive definite matrix P such that

$$A^T P + PA = -Q, \tag{4.2}$$

for a given symmetric positive definite matrix Q. Then, for all $x \neq 0$

$$\sum_{i=1}^{N} \alpha_i x^T (A_i^T P + PA_i) x = x^T (A^T P + PA) x = -x^T Q x.$$

Since $\alpha_i > 0$, $x^T(A_i^T P + PA_i)x < 0$ for at least one i. So, we construct Ω_i as follows

$$\Omega_i = \{x \in \mathcal{R}^n : x^T (A_i^T P + PA_i) x \leq -x^T Q x\}. \tag{4.3}$$

It can be shown that $\mathcal{R}^n = \cup_{i=1}^N \Omega_i$ [21]. Now we have N subregions for the given switching system. To prevent a sliding motion (a motion of a trajectory along a boundary between two switching regions) or chattering phenomenon (many fast switchings across a boundary) we would like to construct a set of overlapping regions, $\tilde{\Omega}_i$, such that each of Ω_i is contained in exactly one of the $\tilde{\Omega}_i$. We thus define

$$\tilde{\Omega}_i = \{x \in \mathcal{R}^n : x^T(A_i^T P + PA_i)x \leq -\frac{1}{\xi} x^T Q x\}, \tag{4.4}$$

for some $\xi > 1$, and note that $\tilde{\Omega}_i \subset \Omega_i$ for each $i \in \{1, \cdots, N\}$. Then, we can describe our switching rule.

Definition 4.1. *(Minimum rule) At each switching we decide the next mode by the minimum rule given by*

$$i(x) = \arg\min x^T(A_i^T P + PA_i)x.$$

To obtain the asymptotic stability of the given switching system we propose the switching rule $\mathbf{S}$ given by:

- S_0 Choose the initial mode, i_o, by the minimum rule applied to $x(t_o)$.
- S_1 Stay in the i^{th} mode as long as the state satisfies $x \in \tilde{\Omega}_i = \{x \in \mathcal{R}^n : x^T(A_i^T P + PA_i)x \leq -\frac{1}{\xi} x^T Q x\}$.
- S_2 If the state hits the boundary of $\tilde{\Omega}_i$, determine the j^{th} mode according to the *minimum rule* and switch to the j^{th} mode.

Theorem 4.2. *[22] The system* (1.2) *with $B = 0$ and the switching rule* $\mathbf{S}$ *is uniformly asymptotically stable.*

Our goal in the following is to show that this result may be extended to (1.2) with B sufficiently small, for any delay, $\tau > 0$. We will do this using both Lyapunov functionals and functions.

4.1. Single Lyapunov Functional Method. Let $V(x_t)$ be a Lyapunov functional given by

$$V(x_t) = x^T(t)Px(t) + \omega \int_{t-\tau}^{t} \|x(s)\|^2 ds \tag{4.5}$$

where P is as defined as above, and $\omega > 0$. Note that

$$\alpha(\|x(t)\|) \leq V(x_t) \leq \beta(\|x_t\|_\tau) \tag{4.6}$$

with $\alpha(\|x(t)\|) = \lambda_{min}(P)\|x(t)\|^2$ and $\beta(\|x_t\|_\tau) = (\lambda_{max}(P) + \omega\tau)\|x_t\|_\tau^2$.

Proposition 4.3. *Suppose that the i^{th} mode is active on $[t_k, t_{k+1})$, i.e., $x \in \tilde{\Omega}_i = \{x \in \mathcal{R}^n : x^T(A_i^T P + PA_i)x \leq -\frac{1}{\xi} x^T Q x\}$ for $t \in [t_k, t_{k+1})$, for some $i \in \{1, \cdots, N\}$. If*

$$\|PB_i\| < \frac{\lambda_{min}(Q)}{2\xi} \tag{4.7}$$

then there exists $\omega > 0$, and a continuous, increasing function, $\gamma : \mathcal{R}^+ \mapsto \mathcal{R}^+$, satisfying $\gamma(0) = 0$ and $\gamma(s) > 0$ for $s > 0$ such that

$$\dot{V}(x_t) \leq -\gamma(\|x(t)\|), \quad \textit{for } t \in [t_k, t_{k+1})$$

where V is given by (4.5).

Proof. Since $\|PB_i\| < \lambda_{min}(Q)/2\xi$ there exists $\omega > 0$ such that $\|PB_i\| \leq \omega < \lambda_{min}(Q)/2\xi$. Let V given by (4.5) with this ω. The rest of the proof is essentially same as that of Proposition 3.3, hence we omit it. Here, $\gamma(\|x(t)\|) = (\frac{\lambda_{min}(Q)}{\xi} - 2\omega)\|x(t)\|^2$. $\square$

THEOREM 4.4. *If $\|PB_i\| < \lambda_{min}(Q)/2\xi$ for all i, then the switching system* (1.2) *with the switching rule* **S** *is uniformly asymptotically stable.*

Proof. Let V be given by (4.5) where ω is such that $\|PB_i\| \leq \omega < \lambda_{min}(Q)/2\xi$ for all i. Suppose that the i_o^{th} mode is active on $[t_o, t_1)$. Then, $x \in \tilde{\Omega}_{i_o}$. Until the boundary of $\tilde{\Omega}_{i_o}$ is hit, $x^T(A_{i_o}^T P + PA_{i_o})x \leq -\frac{1}{\xi}x^T Qx$ and by S_1 and Proposition 4.3 $\dot{V}(x_t) < -\gamma(\|x(t)\|)$ for $t \in [t_o, t_1)$. Now, consider $[t_k, t_{k+1})$ for some $k > 0$. Then, the minimum rule determines the i_k^{th} mode at t_k such that $x(t) \in \tilde{\Omega}_{i_k}$ on $[t_k, t_{k+1})$. Then, by S_1 and Proposition 4.3 $\dot{V}(x_t) < -\gamma(\|x(t)\|)$ for $t \in [t_k, t_{k+1})$. Hence, it is true that $\dot{V}(x_t) < 0$ for all $t \geq t_o$. The rest of the proof is essentially the same as that of Theorem 2.1 in Sec 2. $\square$

4.2. Single Lyapunov Function Method. We consider a Lyapunov function given by

$$V(x) = x^T Px, \tag{4.8}$$

where P satisfies (4.2). Then, we have the following.

PROPOSITION 4.5. *Suppose that the i^{th} mode is active on $[t_k, t_{k+1})$ for some $i \in \{1, \cdots, N\}$ and let V be given by* (4.8). *If*

$$\|PB_i\| < \frac{\lambda_{min}(Q)}{\xi(\frac{\lambda_{max}(P)}{\lambda_{min}(P)} + 1)} \tag{4.9}$$

then there exist a constant $q > 1$ and a continuous and increasing function $\gamma : \mathcal{R}^+ \mapsto \mathcal{R}^+$, satisfying $\gamma(0) = 0$ and $\gamma(s) > s$ for $s > 0$ such that

$$\dot{V}(x(t)) \leq -\gamma(\|x(t)\|) \text{ for } t \in [t_k, t_{k+1})$$

whenever

$$qV(x(t)) > V(x(t+\theta)), \;\; \theta \in [-\tau, 0]. \tag{4.10}$$

Proof. The proof is essentially the same as that of Proposition 3.6, hence we omit it. Here, $\gamma(\|x(t)\|) = (\frac{\lambda_{min}(Q)}{\xi} - \|PB_i\|(q\frac{\lambda_{max}(P)}{\lambda_{min}(P)} + 1))\|x(t)\|^2$. $\square$

THEOREM 4.6. *If $\|PB_i\| < \frac{\lambda_{min}(Q)}{\xi(\frac{\lambda_{max}(P)}{\lambda_{min}(P)}+1)}$ for all i, then, the switching system* (1.2) *with the switching rule* **S** *is uniformly asymptotically stable.*

Proof. Let V be given by (4.8). Suppose at $t = t_o$, an initial time, the i_o^{th} mode is active on $[t_o, t_1)$, i.e. $x \in \tilde{\Omega}_{i_o}$. Since (4.9) is satisfied for all i, there exists $q > 1$ satisfying (4.10) for all i. Let q take this value. Then $\dot{V} < 0$ for $qV(x(t)) \geq V(x(t+\theta))$, $\theta \in [-\tau, 0]$ by the Proposition 4.5 and S_1. Now, consider any $[t_k, t_{k+1})$ for $k > 0$, and assume that the j^{th} mode is active on $[t_k, t_{k+1})$ according to the minimum rule. Then, $x \in \Omega_j$, and by Proposition 4.5 and S_1, $\dot{V} < 0$ for $qV(x(t)) \geq V(x(t+\theta))$,

$\theta \in [-\tau, 0]$. When the boundary of $\tilde{\Omega}_j$ is reached, the minimum rule is applied and we switch to the i^{th} mode. Then,

$$x^T(A_i^T P + PA_i)x \le -x^T Qx \le -\frac{1}{\xi} x^T Qx \text{ on } [t_{k+1}, t_{k+2}),$$

and

$$\dot{V} < 0 \quad \text{for } qV(x(t)) \ge V(x(t+\theta)),\, \theta \in [-\tau, 0], \text{ for } x \in \tilde{\Omega}_i$$

by the Proposition 4.5. Hence, after each switching $\dot{V} < 0$ whenever $qV(x(t)) \ge V(x(t+\theta))$ for $\theta \in [-\tau, 0]$, for all $t \in [t_k, t_{k+1})$, $k = 0, 1, \cdots$.
The rest of the proof is similar to that for Theorem 2.2 in Sec. 2. □

Corollary 4.7. *If*

$$\|PB_i\| < \max\{\frac{\lambda_{min}(Q)}{2\xi}, \frac{\lambda_{min}(Q)}{\xi(\frac{\lambda_{max}(P)}{\lambda_{min}(P)} + 1)}\} \textit{ for all } i,$$

then the system (1.2) *is uniformly asymptotically stable.*

5. Examples.

5.1. Example 1: Multiple Lyapunov functional method.

Consider the switching system given by

$$\dot{x} = A_i x(t) + B_i x(t-\tau), \; i = 1, 2 \tag{5.1}$$

where A_i is Hurwitz, in particular,

$$\begin{aligned} (A_1, B_1) &= \left(\begin{bmatrix} -2 & 2 \\ -20 & -2 \end{bmatrix}, \begin{bmatrix} 0.0655 & 0.2292 \\ -0.4123 & 1.0307 \end{bmatrix} \right), \\ (A_2, B_2) &= \left(\begin{bmatrix} -2 & 10 \\ -4 & -2 \end{bmatrix}, \begin{bmatrix} -0.375 & 0.9375 \\ 0.1186 & 0.0594 \end{bmatrix} \right). \end{aligned} \tag{5.2}$$

Each of A_1 and A_2 has a pair of complex eigenvalues with negative real part, and each B_i has small norm. Since A_i is Hurwitz, there exists symmetric positive definite P_i such that $A_i^T P_i + P_i A_i = -Q_i = -\frac{1}{2} I$, $i = 1, 2$, for some symmetric positive definite Q_i. In particular,

$$(P_1, P_2) = \left(\begin{bmatrix} 7/11 & -9/176 \\ -9/176 & 13/176 \end{bmatrix}, \begin{bmatrix} 1/11 & 3/176 \\ 3/176 & 37/176 \end{bmatrix} \right).$$

Now $\|P_1 B_1\| = 0.1558 < \lambda_{min}(Q_1)/2 = 1/4$, and $\|P_2 B_2\| = 0.1183 < \lambda_{min}(Q_2)/2 = 1/4$, so the condition in Proposition 3.3 is satisfied. Here, for the matrix norm, $\|\cdot\|$, we use the maximum row sum. With these P_i we define Lyapunov functionals by

$$V_i(x_t) = x^T(t) P_i x(t) + \omega_i \int_{t-\tau}^{t} \|x(s)\|^2 ds \tag{5.3}$$

where $\omega_i > 0$ such that $\max_i \|P_i B_i\| \le \omega_i < \lambda_{min}(Q_i)/2$. We can choose any $\omega \in [0.1558, 0.25)$ and let $\omega_1 = \omega_2 = \omega$. Hence, (5.3) is written as

$$V_i(x_t) = x^T(t) P_i x(t) + \omega \int_{t-\tau}^{t} \|x(s)\|^2 ds \tag{5.4}$$

for $i = 1, 2$. Thus, from Propositions 3.2–3.4, V_i in (5.4) satisfies the conditions H_1–H_3. In order to satisfy H_4, we propose the following switching rule **S1** [20]:

S1: Choose the mode 1 if $V_1 \le V_2$ or the mode 2 if $V_2 \le V_1$.

Let Ω_1 and Ω_2 be sets defined by

$$\Omega_1 = \{x \in \mathcal{R}^2 | V_1 \le V_2\}, \quad \Omega_2 = \{x \in \mathcal{R}^2 | V_2 \le V_1\}. \tag{5.5}$$

Since in each V_i only the term $x^T P_i x$ is different, the above Ω_1 and Ω_2 can be written as

$$\begin{aligned} \Omega_1 &= \{x \in \mathcal{R}^2 | x^T (P_1 - P_2) x \le 0\}, \\ \Omega_2 &= \{x \in \mathcal{R}^2 | x^T (P_2 - P_1) x \le 0\}. \end{aligned} \tag{5.6}$$

Figure 5.1 shows Ω_1 and Ω_2. Figure 5.2 shows a numerical simulation of the system

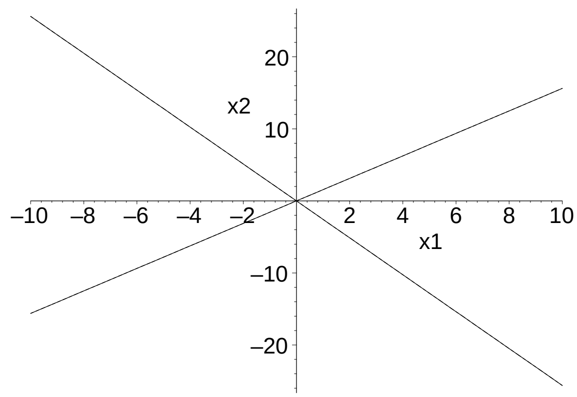

FIG. 5.1. *Switching rule: choose mode 1 in the regions surrounding the x_2 axis, choose mode 2 in the other two regions.*

for small delay, illustrating that stability is achieved. In Figure 5.3 we show numerical simulations for larger delays. These confirm that the given switching system (5.1), with the switching rule **S1** and with B_i of sufficiently small norm, is uniformly asymptotically stable regardless of the size of the time delay.

5.2. Example 2. Single Lyapunov functional Method. Consider the linear switching system (5.1) with

$$\begin{aligned} (A_1, B_1) &= \left(\begin{bmatrix} -3 & -5 \\ 3 & 4 \end{bmatrix}, \begin{bmatrix} 0.0582 & 0.286 \\ -0.0303 & 0.23 \end{bmatrix} \right) \\ (A_2, B_2) &= \left(\begin{bmatrix} 2 & 5 \\ -3 & -10 \end{bmatrix}, \begin{bmatrix} 0.1163 & 0.349 \\ -0.0227 & 0.4766 \end{bmatrix} \right) \end{aligned} . \tag{5.7}$$

It is easy to check that each A_i is unstable. In particular, A_1 has a pair of complex conjugate eigenvalues with positive real part and A_2 has one positive and one negative

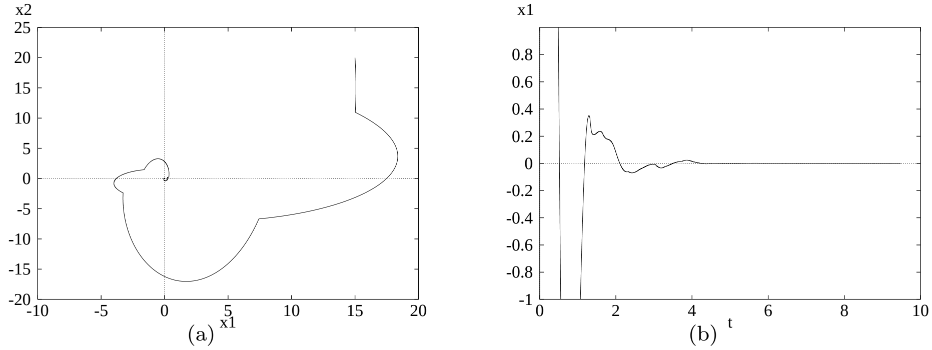

FIG. 5.2. *Numerical simulation of* (5.1) *with* A_i*,* B_i *given by* (5.2)*,* $\tau = 1$ *and the switching rule* **S1**. *The initial condition is* $(x(t), y(t))^T = (15, 20)$ *for* $-\tau \le t \le 0$*. (a)* x_2 *vs.* x_1*. (b)* x_1 *vs. time.*

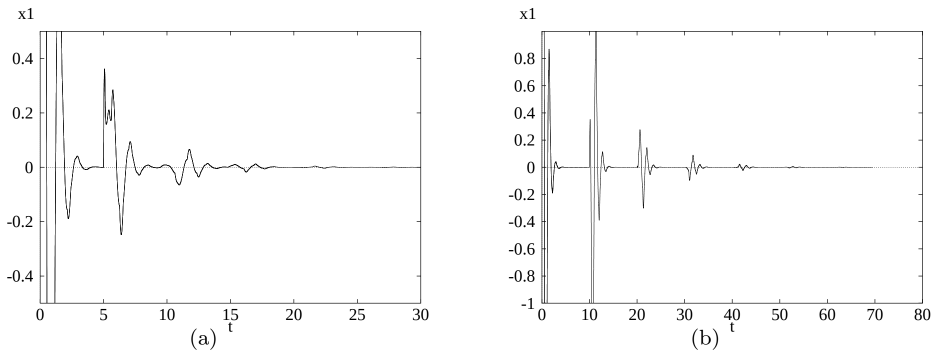

FIG. 5.3. *Numerical simulations of* (5.1) *with* A_i*,* B_i *given by* (5.2)*,* $\tau = 5$ *and the switching rule* **S1**. *The initial condition is* $(x(t), y(t))^T = (15, 20)$ *for* $-\tau \le t \le 0$*. (a)* $\tau = 5$*. (b)* $\tau = 10$*.*

eigenvalue. There is Hurwitz convex combination A given by

$$A = \frac{3}{5}A_1 + \frac{2}{5}A_2 = \begin{bmatrix} -1 & -1 \\ 3/5 & -8/5 \end{bmatrix}. \tag{5.8}$$

Hence, for $Q = \frac{1}{3}I$ there exists a symmetric positive definite matrix P such that $A^T P + PA = -Q$, viz.

$$P = \begin{bmatrix} 64/429 & -25/858 \\ -25/858 & 35/286 \end{bmatrix}. \tag{5.9}$$

Following the procedure described in section IV, we set up $\tilde{\Omega}_i$ with $\xi = 1.5$ and $Q = \frac{1}{3}I$ as

$$\tilde{\Omega}_i = \{x \in \mathcal{R}^2 | x^T(A_i^T P + PA_i)x \le -\frac{1}{\xi}x^T Q x\}.$$

This gives

$$\tilde{\Omega}_1 = \{x \in \mathcal{R}^2 | (545/429)x_2^2 - (350/429)x_1x_2$$
$$-(153/143)x_1^2 \leq -\tfrac{1}{4.5}(x_1^2 + x_2^2)\},$$
$$\tilde{\Omega}_2 = \{x \in \mathcal{R}^2 | -(1175/429)x_2^2 + (175/143)x_1x_2$$
$$+(331/429)x_1^2 \leq -\tfrac{1}{4.5}(x_1^2 + x_2^2)\}.$$

The switching rule is illustrated in Figure 5.4. Now $\|PB_1\| = 0.0455 < \lambda_{min}(Q)/2\xi =$

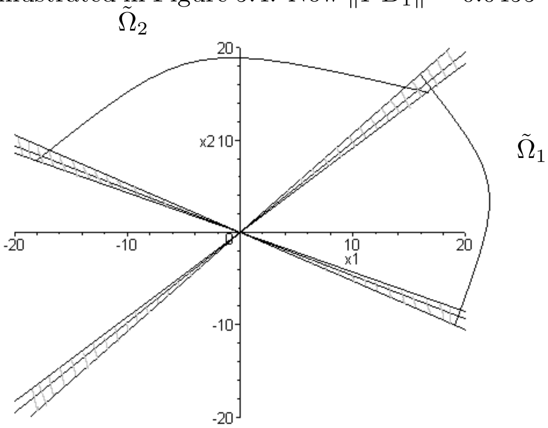

FIG. 5.4. *Switching rule: choose the mode 1 in $\tilde{\Omega}_1$ and the mode 2 in $\tilde{\Omega}_2$. The shaded regions are overlapping regions.*

$1/9$ and $\|PB_2\| = 0.0562 < \lambda_{min}(Q)/2\xi = 1/9$. Hence, with this P we can define a Lyapunov functional as in Proposition 4.3 with $\omega \in [\max_i \|PB_i\|, \lambda_{min}(Q)/2\xi) = [0.0562, 1/9)$. Then, according to Theorem 4.4, system (5.1) with the switching rule **S** is uniformly asymptotically stable for any $\tau > 0$. This is confirmed by numerical simulations, examples of which are shown in Figure 5.5 and Figure 5.6.

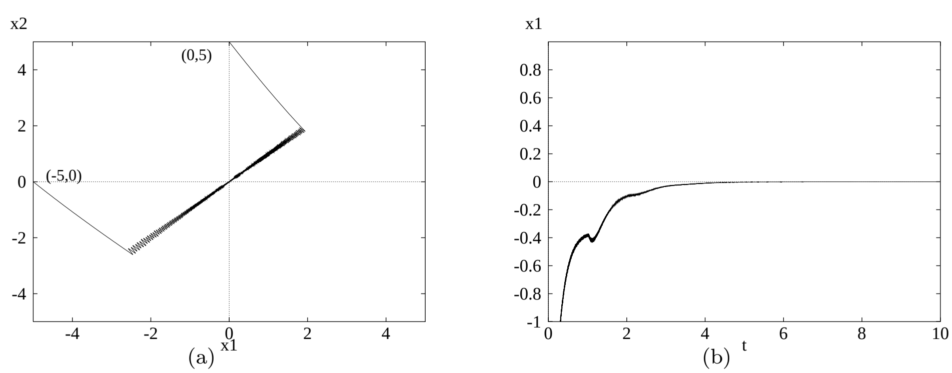

FIG. 5.5. *Numerical simulation of* (5.1) *with A_i, B_i given by* (5.7)*, $\tau = 3$, and the switching rule* **S**. *Two initial conditions are shown: $(x_1(t), x_2(t))^T = (x_{10}, x_{20})$, $-\tau \leq t \leq 0$ with $(x_{10}, x_{20}) = (-5, 0)$, and $(0, 5)$. (a) x_1 vs. x_2 (b) x_1 vs. time.*

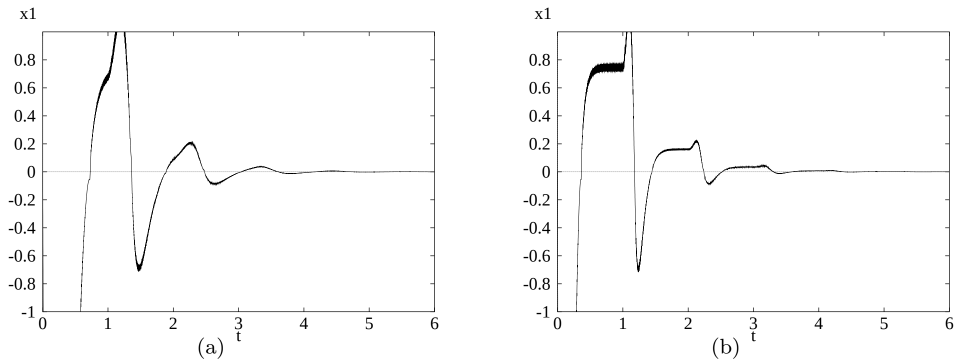

FIG. 5.6. *Numerical simulations of* (5.1) *with* A_i, B_i *given by* (5.7), *and the initial condition* $(x_1(t), x_2(t))^T = (0, 5)$, $-\tau \leq t \leq 0$. *(a)* $\tau = 5$. *(b)* $\tau = 10$.

6. Discussion. In this paper we achieved *delay independent stability* of a linear switching system with time delay. In particular we considered two cases of (1.2): (a) All A_i are stable and (b) all A_i are unstable. The main achievements of our work are as follows.

(i) We extended the standard theory of DDEs stated in Section 1 to the case (a) via the multiple Lyapunov function and functional methods and established delay independent stability conditions.

(ii) We established delay independent stability conditions even when A_i is unstable in each subsystem (case (b)) via the single Lyapunov function and functional methods.

We note that there is one case we have not discussed: when (1.2) has both stable and unstable A_i. Stability in this case can be achieved via the *average dwell time* approach [20].

For the single Lyapunov method, the delay independent stability condition is based on the assumption that there exists a linear convex combination of the unstable A_i from (1.2) and focuses how to obtain $\dot{V} < 0$ (whether V is Lyapunov functional or function) all $t \geq t_o$. This is accomplished via the switching rule **S**. On the other hand, for the multiple Lyapunov method each subsystem is stable due to the A_i being stable and the B_i having sufficiently small norm. However, this does not guarantee the stability of (1.2) and hence we need another condition: $H4$ or $\tilde{H}4$. These are called *peak decreasing conditions*. Thus, the stability of (1.2) can be obtained by constructing Lyapunov functionals or functions satisfying a *peak decreasing condition* or designing a switching rule in order that Lyapunov functionals or functions satisfy a *peak decreasing condition*.

Uniform asymptotic stability of (1.2) is obtained for any $\tau > 0$ by using either method. However, we observe some transient behavior when there is a large time delay in the switching system. This is why Figure 5.3(a), 5.3(b) and 5.6(a), 5.6(b) show irregular oscillating behaviour before converging to zero. In simulations for a given time delay, the larger the norm of B_i, the more irregular the transient behaviour observed. Hence, choosing B_i with a smaller norm relative to the conditions in Section 3 and Section 4 will give better performance.

There are three issues for implementing switching rule **S1** in example 1 in Sec. 5

(1) In example 1 the switching rule **S1** is stated as:

$$\textbf{S1}\text{: Choose} \begin{cases} \text{mode 1 if } V_1 \leq V_2, \text{ or} \\ \\ \text{mode 2 if } V_2 \leq V_1. \end{cases}$$

Then, we construct Ω_i in (5.5). If the ω_i in the Lyapunov functional (5.3) are the same, then Ω_i in (5.5) can be reduced to that in (5.6). In order to have the same ω_i for all i we can choose either Q_i the same for all i as in example 1 showed, or choose B_i such that

$$\|P_i B_i\| < \min_i \frac{\lambda_{min}(Q_i)}{2}$$

Then, choose $\omega = \omega_i$ for all i.

(2) Once ω_i is same for all i, then Ω_i in (5.6) does not depend on the time delay. Thus, the partition by Ω_i is the same as ODE switching systems. This is much easier to implement.

(3) For the multiple Lyapunov function method **S1** can be modified as

$$\textbf{S2}\text{: Choose} \begin{cases} \text{mode 1 if } \overline{V}_1 \leq \overline{V}_2, \text{ or} \\ \\ \text{mode 2 if } \overline{V}_2 \leq \overline{V}_1. \end{cases}$$

In future work we plan to investigate the relation between the size of delay and the transient behaviour in the switching system. Other possible areas for future research include the implementation of the multiple Lyapunov function method since the rule described in (3) above may not be as simple to implement as the rule for the multiple Lyapunov functional method.

Acknowledgment. The numerical simulations were done with **XPPAUT** [23] which is a software package that performs numerical integration of delay differential equations.

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Appendix A. Proof of Theorem 3.1.

Let $N = 2$ for simplicity. Then, let $V_1(x_t)$ and $V_2(x_t)$ be the Lyapunov functionals for the switching system satisfying conditions H_1–H_4 as stated in subsection 3.1. From H_1 for any $\rho > 0$ there exists $\delta > 0$ such that

$$\beta(\delta) = \frac{\alpha(\rho)}{\mu}$$

by the continuity of α and β. Now, assume that the mode 1 is active on $[t_o, t_1)$ with $\|x(t_o + \theta)\| \leq \delta$ where t_o is an initial time and $\theta \in [-\tau, 0]$. Since $\dot{V}_1 < 0$ on $[t_o, t_1)$,

$$V_1(x_t) < V_1(x_{t_o}) \leq \beta(\delta) = \frac{\alpha(\rho)}{\mu} \quad \text{for } t \in [t_o, t_1).$$

Then, the mode 2 is active on $[t_1, t_2)$. Since $V_2(x_{t_1}) \leq \mu V_1(x_{t_1})$,

$$V_2(x_{t_1}) \leq \mu V_1(x_{t_1}) < \mu \frac{\alpha(\rho)}{\mu} = \alpha(\rho)$$

Since $\dot{V}_2 < 0$ on $[t_1, t_2)$,

$$V_2(x_t) < V_2(x_{t_1}) < \alpha(\rho) \quad \text{for } t \in [t_1, t_2).$$

By H_4 at each switching moment $V_1(x_{t_{2k+2}}) < V_1(x_{t_o}) < \alpha(\rho)$ and $V_2(x_{t_{2k+3}}) < V_2(x_{t_1}) < \alpha(\rho)$, $k = 0, 1, \cdots$. Then, by H_2

$$\begin{cases} V_1(x_t) < V_1(x_{t_{2k+2}}) < V_1(x_{t_o}) & \text{for } t \in [t_{2k+2}, t_{2k+3}) \\ V_2(x_t) < V_2(x_{t_{2k+3}}) < V_2(x_{t_1}) & \text{for } t \in [t_{2k+3}, t_{2k+4}), \end{cases}$$

for $k = 0, 1, \cdots$. This implies that

$$V_i(x_t) \leq \alpha(\rho) \quad \text{for } t \geq t_o,\ i = 1, 2.$$

Now, want to show that for any $\eta \in (0, \rho)$ there exists $T \geq 0$ such that

$$V_i(x_t) < \alpha(\eta) \quad \text{for all } t \geq t_o + T.$$

Note that by H_4 we have

$$\begin{aligned} V_1(x_{t_{2k}}) &\leq (1 - \xi_1)^k V_1(x_{t_o}) \\ &< (1 - \xi_1)^k \alpha(\rho) < \alpha(\eta) \text{ for } t_{2k} \geq t_{2M}, \end{aligned} \tag{A.1}$$

and

$$\begin{aligned} V_2(x_{t_{2k+1}}) &\leq (1 - \xi_2)^k V_2(x_{t_1}) \\ &< (1 - \xi_2)^k \alpha(\rho) < \alpha(\eta) \text{ for } t_{2k+1} \geq t_{2M} \end{aligned} \tag{A.2}$$

since $0 < 1 - \xi_i < 1$ for $i = 1, 2$, where

$$M > \max_i \frac{\ln(\alpha(\eta)) - \ln(\alpha(\rho))}{\ln(1 - \xi_i)}.$$

Moreover, by H_2, i.e. $\dot{V}_i < 0$, we have

$$V_i(x_t) < V_i(x_{t_k}) \quad \text{for } t \in [t_k, t_{k+1}). \tag{A.3}$$

Therefore, let $T \geq t_{2M} - t_o$ then we have

$$V_i(x_t) < \alpha(\eta) \quad \text{for } t \geq t_o + T.$$

This completes the proof.

Appendix B. Proof of Theorem 3.5.

To prove Theorem 3.5 we need the following lemma.

LEMMA B.1. *[18, Section 5.4] Let V be a Lyapunov function satisfying the conditions $\tilde{H}_1$–$\tilde{H}_4$ as stated in subsection 3.2. For any $\omega > 0$ choose $\gamma(\omega) > 0$ such that $\beta(\gamma) \leq \alpha(\omega)$. Suppose that $\|x_{t^*}\|_\tau \leq \gamma$ for some $t^* \geq 0$. Then,*

$$V(x(t)) \leq \alpha(\omega) \quad \textit{for all } t \geq t^*.$$

Proof. Recall that

$$\overline{V}(t) = \sup_{-\tau \leq \theta \leq 0} V(x(t + \theta)) \quad \text{for } t \geq t^*. \tag{B.1}$$

Then, there is a $\theta_o \in [-\tau, 0]$ such that $\overline{V}(t) = V(x(t+\theta_o))$ and either $\theta_o = 0$, $\theta_o < 0$, or $\theta_o = -\tau$. In fact, if $\theta_o < \theta \leq 0$ then $V(x(t+\theta)) \leq V(x(t+\theta_o))$.
If $\theta_o < 0$ then for $h > 0$ small enough

$$\overline{V}(t+h) = \overline{V}(t) \Rightarrow \dot{\overline{V}}(t) = 0$$

If $\theta_o = 0$ then, we use the Razumikhin condition and hence $\dot{V}(x(t)) < 0$ and this implies that $\dot{\overline{V}}(t) = 0$. If $\theta_o = -\tau$, then for small enough h

$$\overline{V}(t+h) < \overline{V}(t) \Rightarrow \dot{\overline{V}}(t) < 0$$

Hence, $\dot{\overline{V}} \leq 0$ for all $t \geq t^*$. Thus, we have

$$V(x(t)) \leq \overline{V}(t) \leq \overline{V}(t^*) \leq \beta(\gamma) \leq \alpha(\omega) \quad \text{for } t \geq t^*.$$

since $\|x_{t^*}\|_\tau \leq \beta(\gamma)$. ◻

Proof of Theorem 3.5. For simplicity we will consider $N = 2$. Recall that

$$\overline{V}_i(t) = \sup_{-\tau \leq \theta \leq 0} V_i(x(t+\theta)).$$

For any $\epsilon > 0$ there is $\rho > 0$ such that

$$\beta(\rho) \leq \frac{\alpha(\epsilon)}{\mu}$$

and for such ρ there is $\delta > 0$ such that

$$\beta(\delta) = \frac{\alpha(\rho)}{\mu},$$

by continuity of α and β. Let the mode 1 be on at t_o, with $\|x_{t_o}\|_\tau \leq \delta$. Then,

$$\overline{V}_1(t_o) \leq \beta(\delta).$$

By Lemma B.1 we have $\dot{\overline{V}}_1 \leq 0$ and hence

$$V_1(x(t)) \leq \overline{V}_1(t) \leq \overline{V}_1(t_o) \leq \frac{\alpha(\rho)}{\mu} \quad \text{for } t \in [t_o, t_1).$$

Note that $\overline{V}_1(t_1) \leq \frac{\alpha(\rho)}{\mu}$. In fact, by $\tilde{H}_4$,

$$\begin{aligned} \overline{V}_1(t_{2k}) &< (1-\xi_1)^k \overline{V}_1(t_o) \\ &\leq (1-\xi_1)^k \tfrac{\alpha(\rho)}{\mu} \quad k = 1, 2, \cdots \end{aligned}$$

Then, by Lemma B.1 and the fact that $0 < (1-\xi_1)^k < 1$

$$V_1(x(t)) \leq \frac{\alpha(\rho)}{\mu} \quad \text{for } t \in [t_{2k}, t_{2k+1}),\ k = 0, 1, 2, \cdots$$

At $t = t_1$ the mode 2 is on and hence, by $\tilde{H}_3$

$$\overline{V}_2(t_1) \le \mu \overline{V}_1(t_1) \le \alpha(\rho) \le \beta(\rho) (\le \frac{\alpha(\epsilon)}{\mu}).$$

Then, by Lemma B.1

$$V_2(x(t)) \le \frac{\alpha(\epsilon)}{\mu} \quad \text{for } t \in [t_1, t_2).$$

In addition by $\tilde{H}_4$

$$\begin{aligned} \overline{V}_2(t_{2k+1}) &< (1-\xi_2)^k \overline{V}_2(t_1) \\ &\le (1-\xi_2)^k \tfrac{\alpha(\epsilon)}{\mu} \quad \text{for } k = 1, 2, \cdots . \end{aligned}$$

By Lemma B.1 and the fact that $0 < 1 - \xi_2 < 1$

$$V_2(x(t)) \le \frac{\alpha(\epsilon)}{\mu} \quad \text{for } t \in [t_{2k+1}, t_{2k+2}),\ k = 0, 1, 2, \cdots .$$

Thus,

$$V_i(x(t)) \le \frac{\alpha(\epsilon)}{\mu} \quad \text{for all } t \ge t_o.$$

This implies that

$$\|x(t)\| < \epsilon \quad \text{for } t \ge t_o$$

if $\|x_{t_o}\|_\tau \le \delta$.
Now, we need to show that for any $\eta \in (0, \rho)$ there is $T \ge 0$ such that

$$V_i(x(t)) \le \frac{\alpha(\eta)}{\mu^2} \quad \text{for } t \ge t_o + T.$$

Now, from $\tilde{H}_4$ and $0 < 1 - \xi_i < 1$, $i = 1, 2$, we have

$$\begin{aligned} \overline{V}_1(t_{2k}) &\le (1-\xi_1)^k \overline{V}_1(t_o) \\ &\le (1-\xi_1)^k \alpha(\epsilon) \le \tfrac{\alpha(\eta)}{\mu^2} \quad \text{for } t_{2k} \ge t_{2M}, \end{aligned} \tag{B.2}$$

and

$$\begin{aligned} \overline{V}_2(t_{2k+1}) &\le (1-\xi_2)^k \overline{V}_2(t_1) \\ &\le (1-\xi_2)^k \alpha(\epsilon) \le \tfrac{\alpha(\eta)}{\mu^2} \quad \text{for } t_{2k+1} \ge t_{2M}, \end{aligned} \tag{B.3}$$

where

$$M > \max_i \frac{\ln \alpha(\eta) - \ln \alpha(\epsilon) \mu^2}{\ln(1 - \xi_i)}.$$

By Lemma B.1, $\overline{V}_i \le 0$, thus

$$V_i(x(t)) \le \overline{V}_i(t) \le \overline{V}_i(t_k) \quad \text{for } t \in [t_k, t_{k+1}).$$

Set $T \ge t_{2M} - t_o$. Then,

$$V_i(x(t)) \le \frac{\alpha(\eta)}{\mu^2} \quad \text{for } t \ge t_o + T.$$

This completes the proof.