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FRUSTRATION, STABILITY, AND DELAY-INDUCED OSCILLATIONS IN A NEURAL NETWORK MODEL*

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Abstract. The effect of time delays on the linear stability of equilibria in an artificial neural network of Hopfield type is analyzed. The possibility of delay-induced oscillations occurring is characterized in terms of properties of the (not necessarily symmetric) connection matrix of the network. Such oscillations are possible exactly when the network is frustrated, equivalently when the signed digraph of the matrix does not require the Perron property. Nonlinear analysis (centre manifold computation) of a three-unit frustrated network is presented, giving the nature of the bifurcations taking place. A supercritical Hopf bifurcation is shown to occur, and a codimension-two bifurcation is unfolded.

Key words. neural networks, frustration, Perron property, stability, Hopf bifurcation

AMS subject classifications. 92B20, 34K15, 15A48, 05C20

1. Introduction.

In 1984, Hopfield [11] introduced a continuous version of a circuit equation for a network of n saturating voltage amplifiers (neurons). The system of ordinary differential equations describes the time evolution of the voltage on the input of each neuron. Updating and propagation are assumed to occur instantaneously. A few years later in a series of papers, the processing time in each neuron was incorporated into Hopfield's (by then well-known) equations. The resulting delay-differential equation system has been the starting point of several recent investigations (see, for example, [1], [2], [4], [8], [16]–[18], [20]).

In § 2, we give the circuit equations as in [18] together with a careful analysis of their normalisation. This leads to a matrix equation governing linear stability of the equilibria. Under certain assumptions, this uncouples to a set of n scalar equations. If an equilibrium solution of a delay-differential system is linearly stable when the delay is zero but there exists a value of the delay for which this solution becomes linearly unstable, then *delay-induced instability* occurs. We review the linearized stability analysis [2], [18] and study delay-induced instability of the uncoupled system expressing results succinctly in terms of properties of the connection matrix. In general, we do not need to assume symmetry of the connection matrix, an assumption often made in neural network models (but see [12]). The literature on these models is substantial (almost formidable); we cite only references directly relevant for our purposes and refer the reader to the recent survey book [10].

In describing a network configuration, it is convenient to work with the signed directed graph of its connection matrix. The *directed graph* (digraph) of a real $n \times$

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n matrix $A = [A_{ij}]$ consists of a set of vertices $\{1, 2, \ldots, n\}$ and a set of directed edges, with edge set $\{(j, i) | A_{ij} \neq 0\}$. Here, each vertex corresponds to a neuron, and note that a nonzero A_{ij} is represented as an arrow from j to i and is thus the appropriate direction for our purposes (although the opposite direction is more common in the matrix literature). A *cycle* of length k in this digraph is a set of edges $(i_1, i_2), (i_2, i_3), \ldots, (i_k, i_1)$ with vertices $i_1, i_2, \ldots, i_k$ distinct. The *signed digraph* of A, $D(A)$, is obtained from the digraph by attaching to each edge (j, i) the sign of A_{ij}. The cycle is positive (resp., negative) according as the product $A_{i_2 i_1}, A_{i_3 i_2}, \ldots, A_{i_1 i_k}$ is positive (resp., negative). As in [17], [18], a network is called *frustrated* if the signed digraph of its connection matrix has a negative cycle: a negative loop is an example of a frustrated network.

These graph-theoretic ideas are used in § 3 to prove our main result, namely, that frustration is essential for delay-induced instability leading to oscillation, a result suspected for symmetric networks in [18]. However, not all frustrated networks exhibit delay-induced oscillation. To discuss them further, we use a "converse" of the Perron–Frobenius theorem [6]. Finally in § 4, we consider nonlinear aspects of a frustrated system to illustrate possible complicated behaviour which can result from delay-induced instability.

2. The model equations. Hopfield's circuit equations [11] with no signals from outside but with time delay as in [18] are

$$C_i u_i'(t) = -\frac{1}{R_i} u_i(t) + \sum_{j=1}^{n} T_{ij} f_j(u_j(t - \tau_j)), \quad i = 1, \ldots, n. \tag{2.1}$$

These give the time evolution of $u_i(t)$, the voltage on the input of neuron i with input capacitance C_i. The (real) connection matrix $T = [T_{ij}]$ has $T_{ii} = 0$, and for $i \neq j$, $T_{ij} = R_{ij}$ (resp., $-R_{ij}$) when the noninverting (resp., inverting) output of neuron j is connected to the input of neuron i through a resistance R_{ij}. Thus $R_i = (\sum_j |T_{ij}|)^{-1}$ is the parallel resistance at the input of neuron i. The matrix T is assumed to be irreducible. The transfer function $f_i(u_i) \in C^1$ is sigmoidal, strictly increasing, odd, with $\lim_{u_i \to \pm\infty} f_i(u_i) = \pm 1$, and $f_i'(0) \geq f_i'(u_i)$. The time delay $\tau_i \geq 0$ is incorporated to account for the finite processing time (updating and propagating inside the unit) in neuron i.

All neurons are assumed to be identical; thus $C_i = C$, $\tau_i = \tau$ and $f_i = f$ for all i. In this case, time can be nondimensionalized by setting $\tilde{t} = t/RC$, $\tilde{\tau} = \tau/RC$, where R is a constant resistance, say, $R = \sum_j R_j/n$. As noted in [17], it is the quantity τ/RC that is important for stability.

Letting $\tilde{u}_i(\tilde{t}) = u_i(t)$ and dropping $\tilde{}$, (2.1) becomes

$$u_i'(t) = -\frac{R u_i(t)}{R_i} + \sum_{j=1}^{n} J_{ij} f(u_j(t - \tau)), \quad i = 1, \ldots, n, \tag{2.2}$$

where $J_{ii} = 0$, $J_{ij} = RT_{ij}$, $i \neq j$, with $\sum_j |J_{ij}| = R/R_i$.

If, as is done in most investigations, it is also assumed that each neuron has equal input resistance, then $R = R_i$ for all i, and (2.2) becomes

$$u_i'(t) = -u_i(t) + \sum_{j=1}^{n} J_{ij} f(u_j(t - \tau)), \quad i = 1, \ldots, n. \tag{2.3}$$

The normalisation condition is

$$\sum_{j=1}^{n} |J_{ij}| = 1, \quad i = 1, \dots, n. \tag{2.4}$$

Thus in the normalised connection matrix, each row has the sum of the absolute values of entries equal to one. As a neural network model, independent of a physical realization, (2.3) holds *without* the normalisation condition. Matrix J is the connection matrix, with nonzero J_{ij} positive (resp., negative) according as edge (j, i) is excitatory (resp., inhibitory). We work mainly with this more general connection matrix, that is, (2.3) with J assumed to be irreducible and with all main diagonal entries zero. For certain results, we specify that J also satisfies the above normalisation condition (2.4).

If initial conditions for the functions u_i are specified on the interval $[-\tau, 0]$, then basic existence-and-uniqueness theorems [5, § 26], [9, §§ 2.2, 2.3] ensure that a unique solution to (2.3) exists for all positive times.

Equilibrium solutions of (2.3), denoted by u_i^*, are given by the solutions to the transcendental system of equations $u_i^* = \sum_j J_{ij} f(u_j^*)$. As $f(0) = 0$, the null solution $u_i^* = 0$ for all i is always an equilibrium. For any stationary solution u_i^*, its linear stability is determined by the equations

$$x_i'(t) = -x_i(t) + \sum_{j=1}^{n} J_{ij} f'(u_j^*) x_j(t - \tau), \quad i = 1, \dots, n, \tag{2.5}$$

where J is the irreducible connection matrix with no self-connection. Thus, setting $x_i = c_i e^{zt}$, the linear stability of the solution u_i^* is governed by the matrix equation

$$zI = -I + e^{-z\tau} F, \tag{2.6}$$

where the matrix F has entries $F_{ij} = J_{ij} f'(u_j^*)$ and I is the $n \times n$ identity matrix. In the special case of the null solution, when $u_i^* = 0$ for all values of i, then $f'(u_j^*) = f'(0) \equiv \beta > 0$, which is the gain of the transfer function, and the matrix equation has a nontrivial solution (at least one c_i not zero) exactly when

$$\det[zI + I - e^{-z\tau} \beta J] = 0. \tag{2.7}$$

The null solution of (2.5) is asymptotically stable exactly when $\mathrm{Re}(z) < 0$ for every root of (2.7). This equation is considered in [2], [18], where use is made of the fact that it uncouples to give n scalar equations

$$z = -1 + \beta \lambda_j e^{-z\tau} \tag{2.8}$$

for each eigenvalue $\lambda_j \in \sigma(J)$, where $\sigma(J)$ denotes the spectrum of the matrix J.

Consider (2.8) in the limiting cases of small and large delay. The null solution of (2.5) is asymptotically stable for $\tau = 0$ if and only if $s(J) < 1/\beta$, where $s(J) = \max\{\mathrm{Re}(\lambda_j) : \lambda_j \in \sigma(J)\}$ is the *spectral abscissa* of J, and it is asymptotically stable for all values of the delay τ if and only if $\rho(J) < 1/\beta$, where $\rho(J) = \max\{|\lambda_j| : \lambda_j \in \sigma(J)\}$ is the *spectral radius* of the matrix J.

For a fixed value of τ, the stability region in the complex plane of the parameter λ_j lies inside a teardrop-shaped region (symmetric about the real axis), giving the half plane in the limit as $\tau \to 0^+$ and the circle of radius $1/\beta$ as $\tau \to \infty$ [2], [18]. For a finite value of τ, the teardrop crosses the positive real axis at $1/\beta$ (independent of τ) and crosses the negative real axis at a point to the left of $-1/\beta$, given implicitly as the root of $\lambda_j = -(\omega^2+1)^{1/2}/\beta$ with $\tan(\omega\tau) = -\omega$ and $\frac{\pi}{2} < \omega\tau < \pi$; see Fig. 2.1. This teardrop decreases in size as τ increases, giving the possibility of delay-induced instability. In many neural network models, the connection matrix J is assumed to be symmetric, in which case each element of $\sigma(J)$ is real; only the real axis in Fig. 2.1 needs to be studied in this case.

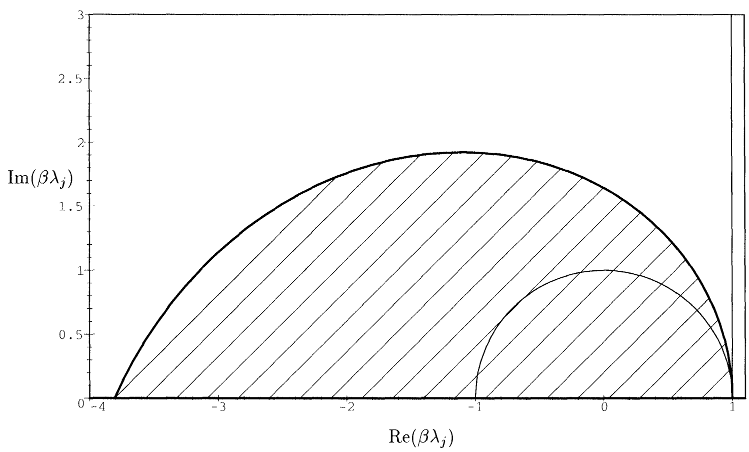

FIG. 2.1. *Values in the complex plane of the real and imaginary parts of the parameter $\beta\lambda_j$ for which* (2.8) *has no root with nonnegative real part: only half the region* (*which is symmetric with respect to the x coordinate axis*) *is shown for $\tau = .5$. Boundaries for the limiting cases $\tau = \infty$* (*interior of the unit circle*) *and $\tau = 0$ are also represented.*

3. Delay-induced instability. As seen in the last section, it is possible for the delay τ to destabilize an equilibrium which is asymptotically stable in the absence of time delay.

A necessary and sufficient condition for the occurrence of delay-induced instability of the null solution of (2.5) in terms of $\sigma(J)$ can be stated from the previous discussion [2, Cor. 2.8] as

$$s(J) < \frac{1}{\beta} < \rho(J). \tag{3.1}$$

By contrast, if $s(J) > 1/\beta$, then the null solution is unstable for all τ; if $1/\beta > \rho(J)$, then the null solution is asymptotically stable for all τ; see § 2.

We use the following definition [13]. Matrix A has the *Perron property* if $\rho(A) \in \sigma(A)$; that is, $\rho(A) = s(A)$. Using (3.1), we have the following result.

THEOREM 1. *Delay-induced instability for the null solution of* (2.5) *is impossible if and only if the connection matrix J has the Perron property.*

Using the qualitative definitions of [13], the signed digraph $D(J)$ *requires* (resp., *allows*) the Perron property if every (resp., some) matrix A with $D(A) = D(J)$ has the Perron property. This leads to the following qualitative characterization.

THEOREM 2. *Consider the network linearized as in* (2.5) *with connection matrix* J *having signed digraph* $D(J)$. *Then the following statements are equivalent.*

i. *The network is not frustrated.*

ii. $D(J)$ *requires the Perron property.*

iii. *Delay-induced instability of the null solution of* (2.3) *is impossible.*

Proof. If the connection matrix has all cycles positive (such a matrix is called *cyclically nonnegative* in [6]), then the network is not frustrated. The equivalence of i. and ii. is proved in [6, Thm. 1.1]. Assume ii.; then iii. follows from (3.1). The converse also holds, since $\rho(A) \geq s(A)$ for any matrix A. ☐

This result shows that frustration is essential for delay-induced instability, confirming the numerical observation in [18]. Note that the connection matrix is not assumed to be symmetric; rather it is the Perron property that is important for this result. For example, any nonnegative matrix has this property. The contrapositive of this theorem is illustrated by the following.

Example 3. Let the signed digraph of the $n \times n$ connection matrix J be a directed cycle of length n (a ring) with an odd number of negative edges, giving rise to a frustrated network. Let the product of entries of J on the cycle be $-k$, $k > 0$. The eigenvalues of J are the nth roots of $-k$. Thus, delay-induced instability for the null solution of (2.3) is possible exactly when $\sec(\pi/n) > \beta k^{1/n} > 1$. Note that $D(J)$ does not allow (and so does not require) the Perron property.

If the signed digraph of the connection matrix allows but does not require the Perron property (in which case the matrix is frustrated), then delay-induced instability of the null solution of (2.3) is possible for some magnitudes of the connection strengths. This is illustrated by the following example.

Example 4. Let

$$J = \begin{pmatrix} 0 & 1 & 0 \\ -a & 0 & 1-a \\ 0 & 1 & 0 \end{pmatrix}, \tag{3.2}$$

where $0 < a < 1$, be the connection matrix of a frustrated three-neuron system. Here J satisfies the normalisation condition (2.4), the network is frustrated, and $D(J)$ allows but does not require the Perron property. For general a, $\sigma(J) = \{0, \pm\sqrt{1-2a}\}$. When $a = 0.5$, $\sigma(J) = \{0\}$; thus J has the Perron property for this value of a, and the null solution of (2.3) is linearly asymptotically stable for all delays. When $a = .25$, $\sigma(J) = \{0, \pm\sqrt{2}/2\}$, and thus for all τ, when $\beta < \sqrt{2}$, the null solution of (2.3) is linearly asymptotically stable, and when $\beta > \sqrt{2}$, it is unstable (see Fig. 2.1). By contrast, when $a = .75$, $\sigma(J) = \{0, \pm i\sqrt{2}/2\}$; thus J does not have the Perron property, and delay-induced instability of the null solution of (2.3) occurs for $\beta > \sqrt{2}$ at some value of $\tau > 0$.

A network in which every cycle in the connection matrix is negative is called *fully frustrated* [18]. The example below gives a fully frustrated network.

Example 5. Let the signed digraph of the $n \times n$ network connection matrix J consist of a single neuron (labeled 1) connected by negative 2-cycles to each of $n-1$ other neurons (labelled $2, \ldots, n$). The product of each 2-cycle, namely $J_{1k}J_{k1}$, $k = 2, \ldots, n$, is negative and $\sigma(J) = \{0, \pm i \sum_{k=2}^{n} |J_{1k}J_{k1}|\}$. Thus $D(J)$ does not allow the Perron property, and so any network with this connection matrix exhibits delay-induced instability for $\beta > 1/\sum_{k=2}^{n} |J_{1k}J_{k1}|$. In fact, $D(J)$ requires all pure imaginary eigenvalues [7].

When the connection matrix J is normalised as in (2.4), we have further quantitative results for a network that is not frustrated.

THEOREM 6. *Assume that the network described by* (2.3), (2.4) *is not frustrated. Then if* $\beta < 1$, *the null solution of* (2.3) *is linearly asymptotically stable for all delays; if* $\beta > 1$, *it is unstable for all delays.*

Proof. If the network is not frustrated, then, by Theorem 2, delay-induced instability is impossible. Furthermore, the connection matrix J has all cycles positive. Thus, there exists a signature matrix $S(= S^{-1})$ so that SJS is nonnegative [6]. Hence $\sigma(J) = \sigma(SJS)$, and SJS is row stochastic. Thus $\rho(J) = s(J) = 1$. The result follows from statements in § 2; see also Fig. 2.1. □

Example 7. As an example of Theorem 6, consider an all-excitatory connection matrix, with $J_{ij} = 1/(n-1)$, $i \neq j$; thus J is a symmetric circulant matrix (see [18], [20]). For this example, Wu [20] proves for the nonlinear equation (2.3) that when $\beta < 1$, the null solution is a global attractor, whereas when $\beta > 1$, two nonzero asymptotically stable equilibria appear.

4. Centre manifold computation. In this section, we present an example of a frustrated network containing three units, and we consider the nature of the bifurcations taking place when the stability of the null solution is lost. Whereas all results of the previous sections concerned linearizations of system (2.3) in the form of (2.5), we now have to consider the effects of the nonlinear terms in the original equation: these calculations indicate the nature of the Hopf bifurcations taking place when the null solution of (2.5) loses its stability through a pair of pure imaginary eigenvalues acquiring positive real parts.

Consider a network of three units governed by (2.3) with $f(u) = \beta \tanh(u)$, and connection matrix J with no self-connection ($J_{ii} = 0$, $i = 1, 2, 3$). We do not assume that the normalisation condition (2.4) is satisfied.

The stability of the equilibrium $x = 0$ in this system is regulated by (2.7), which is reducible in this case to three scalar transcendental equations. Here, however, we are able to directly analyze the cubiclike characteristic equation. Indeed, by expanding the above determinant, we obtain

$$\det[zI + I - \beta J e^{-z\tau}] = (z+1)^3 - \Gamma\beta^3 e^{-3z\tau} - \Xi\beta^2(z+1)e^{-2z\tau} = 0, \tag{4.1}$$

where the coefficients Γ and Ξ are computed from the entries of the matrix J as $\Gamma = J_{12}J_{23}J_{31} + J_{13}J_{32}J_{21}$ and $\Xi = J_{23}J_{32} + J_{13}J_{31} + J_{12}J_{21}$. This equation has recently been completely analyzed from a stability point of view [3]. For our purposes, it suffices to know the values of the coefficients Γ and Ξ for which all roots of (4.1) have negative real parts as illustrated in Fig. 4.1. The upper boundary of the stability region is the straight line $\Xi\beta^2 = 1 - \Gamma\beta^3$, where $z = 0$ is a root of (4.1), and the lower boundary is the trace of the curve for $z = i\omega$, $\omega > 0$, given in parametric form by

$$\begin{aligned} \Gamma(\omega) &= [2(1+\omega^2)(\omega\sin(\omega\tau) - \cos(\omega\tau))]/\beta^3, \\ \Xi(\omega) &= [1 + \omega^2 + 2(1-\omega^2)\cos(2\omega\tau) - 4\omega\sin(2\omega\tau)]/\beta^2. \end{aligned} \tag{4.2}$$

(See [3] for a derivation.) The nature of the bifurcations taking place as the stability of the null solution is lost at these parameter values can now be determined. For this, we need some terminology and notation.

Consider a system of n differential delay equations written, in standard notation [9], as

$$\dot{\mathbf{x}} = L\mathbf{x}_t + \mathbf{g}(\mathbf{x}_t), \tag{4.3}$$

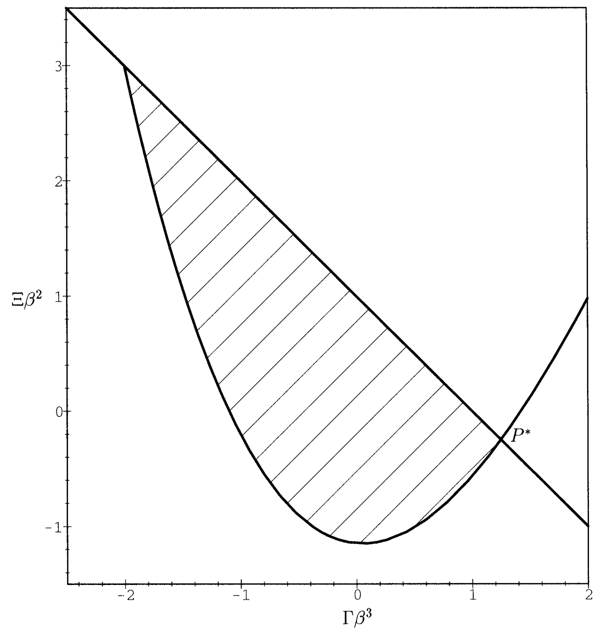

FIG. 4.1. *Linear stability region for the three-unit network with no self-connection. Three eigenvalues have zero real parts at the point P^*, including one zero eigenvalue. The case $\tau = \pi$ is illustrated.*

with $\mathbf{x}_t = \mathbf{x}(t+\theta)$, $-h \le \theta \le 0$, $C = C([-h,0],\mathbb{R}^n)$, $L : C \to \mathbb{R}^n$ a linear operator, and $\mathbf{g} \in C^r(C,\mathbb{R}^n)$, $r \ge 1$. L may be expressed in integral form as

$$L\phi = \int_{-h}^{0} [d\eta(\theta)]\phi(\theta), \tag{4.4}$$

where $\eta : [-h,0] \to \mathbb{R}^n$ is a function of bounded variation. We assume that parameters in (4.3) are such that the linear part of the equation,

$$\dot{\mathbf{x}}(t) = L\mathbf{x}_t, \tag{4.5}$$

has m eigenvalues with zero real parts, all other eigenvalues having negative real parts. It can be shown in this case that there exists in the state space C an m-dimensional invariant manifold, the centre manifold, and that long-term behaviour of solutions to the nonlinear equation is well approximated by the flow on this manifold [9].

At a point in parameter space where the linear equation (4.5) possesses m eigenvalues with zero real parts, there exists a splitting of the space $C = P \oplus Q$. Here P is an m-dimensional subspace spanned by the solutions of (4.5) corresponding to the m zero real-part eigenvalues, and P and Q are invariant under the flow associated with (4.5). Further, the centre manifold introduced above is given by

$$M_f = \{\phi \in C : \phi = \Phi\mathbf{z} + \mathbf{h}(\mathbf{z},\mathbf{g}), \mathbf{z} \text{ in a neighbourhood of zero in } \mathbb{R}^m\}.$$

The flow on this centre manifold is

$$\mathbf{x}_t = \Phi\mathbf{z}(t) + \mathbf{h}(\mathbf{z}(t),\mathbf{g}), \tag{4.6}$$

where Φ (a set of m n-dimensional column vectors represented as an $n \times m$ matrix) is a basis for P, $\mathbf{h} \in Q$, and $\mathbf{z}$ satisfies the ordinary differential equation

$$\dot{\mathbf{z}} = B\mathbf{z} + \mathbf{b}\mathbf{g}(\Phi\mathbf{z} + \mathbf{h}(\mathbf{z}, \mathbf{g})). \tag{4.7}$$

In (4.7), B is the $m \times m$ matrix of eigenvalues of (4.5) with zero real part, and $\mathbf{b}$ is determined from the solution to the equation adjoint to (4.5). Specifically, if we let Ψ (a set of m n-dimensional row vectors represented by an $m \times n$ matrix) be the basis for the invariant subspace of the adjoint problem corresponding to P, then $\mathbf{b} = \Psi(0)$. Ψ is normalised by

$$\langle \Psi, \Phi \rangle = \mathrm{I}, \tag{4.8}$$

where I is the $m \times m$ identity matrix,

$$\langle \psi, \phi \rangle = (\psi(0), \phi(0)) - \int_{-h}^{0} \int_{0}^{\theta} \psi(\xi - \theta)[d\eta(\theta)]\phi(\xi)\, d\xi \tag{4.9}$$

is the bilinear form associated with (4.5), and $(,)$ represents the usual scalar (dot) product of two vectors. If we let the elements of Ψ be linear combinations of those of Φ, i.e., $\Psi = K\Phi^T$ (K is an $m \times m$ matrix of constants), then $K = \langle \Phi^T, \Phi \rangle^{-1}$ or $\Psi = \langle \Phi^T, \Phi \rangle^{-1}\Phi$. Thus the problem of describing the long-term behaviour of solutions to the n-dimensional system of delay-differential equations (4.3) has been reduced (locally) to the problem of describing the behaviour of solutions to the m-dimensional system of ordinary differential equations (4.7).

Although straightforward in principle, the practical implementation of this procedure, especially in the case of a centre manifold of dimension greater than 2, is far from trivial. Thus, it was necessary to use the algebraic manipulation language Maple [19] to perform the calculations leading to the expressions below. To illustrate, consider

$$J = \begin{pmatrix} 0 & 1 & 0 \\ -1.25 & 0 & 1 \\ 1.25 & 1 & 0 \end{pmatrix} \tag{4.10}$$

as the connection matrix of a frustrated network, with $\sigma(J) = \{1, -.5 \pm i\}$. Then with $f(u) = \beta \tanh(u)$, delay-induced instability of the null solution of (2.5) occurs for $1 > \beta > 2/\sqrt{5}$, giving rise to a Hopf bifurcation. Specifically, for $\beta = .96$, this occurs at $\tau = 4.265$, $\omega = .3899$. Thus at this point, the solutions to the system of three *delay* differential equations (2.3) may be approximated by the flow on a two-dimensional centre manifold governed by a system of two *ordinary* differential equations whose normal form in polar coordinates is

$$\begin{aligned} \frac{dr}{dt} &= -0.541r^3, \\ \frac{d\theta}{dt} &= .3899. \end{aligned} \tag{4.11}$$

This allows us to conclude that the bifurcation is supercritical and delay-induced stable periodic solutions of (2.3) occur for $\tau > 4.265$.

For the same matrix J given by (4.10), if $\beta = 1$, then (2.7) possesses a zero eigenvalue and a pair of pure imaginary eigenvalues $\pm.5i$ when $\tau = \pi$; see (4.2) and

P^* on Fig. 4.1. For these values, there is a three-dimensional centre manifold for the zero solution of (2.3) at the point $(\Gamma, \Xi) = (1.25, -.25)$. After the centre manifold computation outlined above and the transformation to normal form, we obtain the system

$$
\begin{aligned}
dx/dt &= -.5y + (.0642(x^2+y^2) + .4453z^2)y - (.0304(x^2+y^2) + .3783z^2)x, \\
dy/dt &= .5x - (.0642(x^2+y^2) + .4453z^2)x - (.0304(x^2+y^2) + .3783z^2)y, \\
dz/dt &= -.1531(x^2+y^2)z - .2513z^3,
\end{aligned}
\tag{4.12}
$$

which can also be written in cylindrical coordinates, with $r^2 = x^2 + y^2$, as

$$
\begin{aligned}
dr/dt &= -0.0304r^3 - .3783rz^2, \\
d\theta/dt &= 0.5, \\
dz/dt &= -.1531zr^2 - .2513z^3.
\end{aligned}
\tag{4.13}
$$

This system occuring at the coincidence of a pitchfork and a Hopf bifurcation has been studied [14], [15] and unfolded to

$$
\begin{aligned}
dr/dt &= \alpha r - 0.0304r^3 - .3783rz^2, \\
d\theta/dt &= 0.5, \\
dz/dt &= \gamma z - .1531zr^2 - .2513z^3.
\end{aligned}
\tag{4.14}
$$

All possible behaviours for (2.3) in a neighbourhood of point P^* must be contained in system (4.14). Indeed, Fig. 4.2 shows all possible phase-portraits for this three-dimensional system. The planar representation is in the (r, z) plane as the azimuthal coordinate (θ) decouples to quadratic order: a rotation about the z-axis must be added to the portraits for a visualization of the full flow. Stationary solutions on the z-axis of the planar system correspond to equilibria in (4.14), whereas stationary solutions off this axis are associated with periodic solutions in (4.14). As indicated in Fig. 4.2, one of these periodic solutions is stable wherever one exists in (α, γ) parameter space, except for the wedge between the γ-axis and the line labeled LCP. A secondary bifurcation is possible, for values of (α, γ) along the line LCP, giving rise to a limit cycle not induced by a Hopf bifurcation. This secondary limit cycle, like the one created in the Hopf bifurcation at $\alpha = 0$ when γ is positive, is of saddle type and as such will not directly affect the observable dynamics.

5. Discussion. We have presented relations between the destabilizing influence of time lags and the properties of the matrix of connections in a neural network model, described by (2.3). These results apply to not necessarily symmetric networks, and the Perron property of the connection matrix of the associated digraph was shown to be the essence of the "frustration" property of the network: this allows us to settle the conjecture of [18] that frustration is necessary for delay-induced oscillations to be possible. This concerns the linear stability of steady states. When the latter become unstable, a Hopf bifurcation is expected to occur, and nonlinear terms have to be considered to determine the nature of the bifurcation. We have presented an example of a three-unit, asymmetric frustrated network with no self-connection and have unfolded the codimension-two bifurcation occurring in it.

Recently, Lyapunov functionals have been used [4], [8] to obtain sufficient conditions for delay independent global asymptotic stability of equilibria for neural network models given by (2.3). These results are complementary to ours.

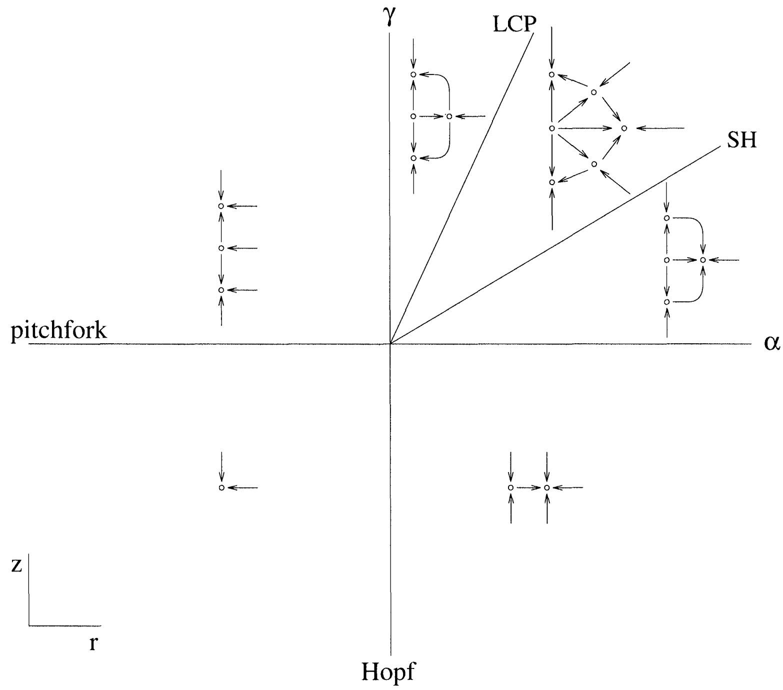

FIG. 4.2. *Unfolding of the degenerate flow near* P^*. *These are phase portraits of* (4.14) *for small values of the parameters* α *and* γ. *The equilibrium points off the* z *coordinate axis correspond to limit cycles in* (2.3). *Secondary bifurcations occur along the lines labeled LCP* (*for limit cycle pitchfork*) *and SH* (*for secondary Hopf*).

Our main results show an intimate link between properties of the connection matrix (the distribution of its eigenvalues) and local stability of an equilibrium solution of (2.3). In a sense, local stability (of the steady state) in time depends on global properties (of the network) in space.

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