CHAPTER 5
Optimal Grouping Criteria

This chapter addresses the question of optimal grouping, and quantifies the amount of efficiency lost through the use of grouped data rather than variables data. As mentioned, in many circumstances the group limits used to classify units are predetermined. For example at Eaton Yale Inc. (Hamilton, Ontario) creating step-gauges specifically for an application is not practical due to the large number of different products produced. As a result, at Eaton, they use existing generic step-gauges that are incremented in thousands of an inch. However, in some situations, it is possible and practical to set application specific group limits. Clearly, all group limit designs will not be equally informative. As a result, it is possible to determine the optimal grouping criteria for particular applications.

Section 5.1 discusses gauge limit design for one-sided acceptance sampling plans to detect mean shifts. The results agree with past work (Beja and Ladany, 1974), and extend the analysis to the more general multiple group case. The results giving optimal group limits to detect one-sided mean shifts in a normal distribution presented in Tables 5.1 and 5.2 also appear in Steiner et al. (1993A). Section 5.2 considers the further extension to two-sided tests. The analysis from Section 5.1 can also be used to determine optimal group limits in the two-sided test case. The resulting optimal group limits are applicable for both the two sets of weights approach of Section 3.2.1 and the MLE approach of Section 3.2.2. Group limit design for Shewhart control charts is addressed in Section 5.3. The optimal group limits found are applicable in all four control chart design approaches from Section 3.3. Optimal group limits to detect mean and standard deviation shifts in a normal process are determined, as well as optimal limits to detect shape and scale shifts of a Weibull process. Note that Table 5.5 gives the optimal group limits to detect mean shifts from a normal process and also appears in Steiner et al. (1993B). For the case of Shewhart control charts, we also find optimal limits for simultaneous detection of parameter shifts. This is of interest since often a process may become “out of control” through either a mean or standard deviation shift. It is very interesting to note that the results in Section 5.3 appear elsewhere in the literature in a totally different context. In Balakrishnan and Cohen (1991), Chapter 7, the results in Table 5.10 appear in a slightly different format. The problem considered by Balakrishnan and Cohen (1991) is finding the best linear unbiased estimate (BLUE) by selecting specific order statistics. For more details and additional references see their book. In the problem to find the BLUE the objective is to minimize the estimate’s variance, whereas in the optimal grouping problem, considered here, the objective is to maximize the Fisher information available about a parameter. The equivalence of the two problem exists because by the Rao-Cramer inequality (Kendall et al. 1978) the inverse of the Fisher information is a lower bound for the variance of an unbiased estimate, and a best estimate would attain this bound. The solution to find the BLUE is given in terms of percentage points which when translated by the inverse cumulative density function give the optimal group limits. In Section 5.4 optimal group probabilities for the correlation estimation Procedure I from Chapter 4 are determined. Procedure I is the only correlation estimation procedure presented where group probabilities can be set a priori, however, the results also provide a guide for the other procedures from Chapter 4.

5.1 One-Sided Acceptance Sampling Plans

Acceptance sampling problems are often given in terms of acceptable and rejectable proportions of non-conforming or out-of-specification units. As a result, we shall restrict our attention in this section to the classical case where we are interested in detecting mean shifts when the standard deviation is known. For one-sided mean shift acceptance sampling plans with single-step gauges, Beja and Ladany (1974), Sykes (1981), and Evans and Thyregod (1985) have shown that when the error risks are equal, the optimal group limit to detect mean shifts from μ_a to μ_r should be placed at $(\mu_a + \mu_r)/2$. Beja and Ladany (1974) also suggested that the optimal group limits for a two-step gauge should be placed symmetrically about $(\mu_a + \mu_r)/2$. Using this rule of thumb, a one dimensional search for the optimal group limits is possible. Note however that this solution will only be optimal if the error risks are equal. Our results for optimal group limits to detect one-sided mean shifts agree with these past results, and extend the analysis to the general *k*-step gauge and to the case when error rates are not necessarily equal.

Suppose we wish to design an acceptance sampling plan that will detect a one-sided mean shift of a certain magnitude from a normal distribution with specified error rates. In this situation, the weight based methods presented in Section 3.1 will give the optimal testing procedure. If the error rates are small and/or the magnitude of the shift to be detected is small, then the required sample size will be large enough so that $\bar{z}$ (the average weight from Section 3.1) is approximately normally distributed. In this case, the optimal group limits are determined by minimizing expression (3.4), the required sample size, subject to the constraint that the group limits remain ordered. Formally, let $\mathbf{t}$ be the k dimensional vector of standardized group limits. Then the multi-dimensional minimization problem is

$$\text{minimize } \{n(\mathbf{t}) + m(\mathbf{t})\}$$

where

$$m(\mathbf{t}) \quad = \quad \begin{cases} M & \text{if the } t\text{'s are not ordered,} \\ & \text{i.e., if } t_j > t_{j+1}, \text{ for any } j = 1, \dots k \\ 0 & \text{otherwise} \end{cases}$$

M is a large number and $n(\mathbf{t})$ is expression (3.4). Note that the solution to the above minimization problem depends on the size of mean shift we wish to detect, and on the chart's error rates. This optimization problem can be easily solved by the Nelder Mead multi-dimensional Simplex Algorithm (Press et al., 1988). The Nelder Mead algorithm is quite efficient in most circumstances, however if efficiency is of great concern, it would be better to use the Fletcher-Reeves algorithm (Press et al., 1988). The Fletcher-Reeves algorithm is more efficient because it uses not only the function, but also its gradient. The calculation of the gradient of equation (3.4) with respect to the unknown t_j's is not unduly difficult, but looks rather complex. See Appendix D for the calculation of the gradient. The amount of work necessary to find the optimal group limits in the special case when the error rates α and β are equal is reduced since the optimal group limits must be symmetric about $(\mu_a + \mu_r)/2$. Using this fact reduces in half the number of variables to consider.

The optimal standardized group limits for selected error rates and mean shifts of a half, one, and one and a half sigma unit for a standard normal process are given in Tables 5.1 and 5.2. To determine the actual group limits to use in a specific example, the standardized group limits presented in the tables must be translated. In the case of a $N(\mu, \sigma)$ process, simply multiple each of the group limits given in the tables by σ and add μ.

Table 5.1: Optimal Group Limits and Weights
Standard Normal Distribution, $\alpha = \beta$.
The calculation of n assumes $\alpha = \beta = 0.001$

μ_1	k	n	λ	i =	1	2	3	4	5	6	7
0.5	1	235.5	0	t_i	0.25						
				z_i	–0.4001	0.4001					
	2	186.0	0	t_i	–0.3417	0.8417					
				z_i	–0.6052	0	0.6052				
	3	179.6	0	t_i	–0.6925	0.2500	0.925				
				z_i	–0.74	–0.2188	0.2188	0.74			
	4	164.6	0	t_i	–0.9384	0.1139	0.6139	1.4384			
				z_i	–0.8395	–0.3667	0	0.3667	0.8395		
	5	161.1	0	t_i	–1.1254	–0.3743	0.2500	0.8743	1.6254		
				z_i	–0.9172	–0.4771	–0.1511	0.1511	0.4771	0.9172	
	6	158.9	0	t_i	–1.2749	–0.5751	–0.0142	0.5142	1.0751	1.7749	
				z_i	–0.9804	–0.5642	–0.2653	0	0.2653	0.5642	0.9804
1	1	55.6	0	t_i	0.5000						
				z_i	–0.8070	0.8070					
	2	44.4	0	t_i	–0.0424	1.0424					
				z_i	–1.1789	0	1.1789				
	3	41.2	0	t_i	–0.3428	0.5000	1.3428				
				z_i	–1.4062	–0.3972	0.3972	1.4062			
	4	0.0	0	t_i	–0.5373	0.1813	0.8187	1.5373			
				z_i	–1.5600	–0.6495	0	0.6495	1.5600		
	5	9.2	0	t_i	–0.6723	–0.0357	0.5000	1.0357	1.6723		
				z_i	–1.6692	–0.8257	–0.2615	0.2615	0.8257	1.6692	
	6	8.8	0	t_i	–0.7697	–0.1941	0.2767	0.7233	1.1941	1.7697	
				z_i	–1.7492	–0.9553	–0.4503	0	0.4503	0.9553	1.7492
1.5	1	22.4	0	t_i	0.7500						
				z_i	–1.2275	1.2275					
	2	18.1	0	t_i	0.2661	1.2339					
				z_i	–1.7172	0	1.7172				
	3	17.0	0	t_i	0.0273	0.7500	1.4727				
				z_i	–1.9817	–0.5190	0.5190	1.9817			
	4	16.6	0	t_i	–0.1068	0.4829	1.0171	1.6068			
				z_i	–2.1358	–0.8189	0	0.8189	2.1358		
	5	16.4	0	t_i	–0.1867	0.3132	0.7500	1.1868	1.6867		
				z_i	–2.2293	–1.0090	–0.3225	0.3225	1.0090	2.2293	
	6	16.3	0	t_i	–0.2365	0.1971	0.5706	0.9294	1.3029	1.7365	
				z_i	–2.2882	–1.1366	–0.5429	0	0.5429	1.1366	2.2882

Table 5.2: Optimal Group Limits and Weights
Standard Normal Distribution, $\alpha = 0.001$, $\beta = 0.005$

μ_1	k	n	λ	i =	1	2	3	4	5	6	7
0.5	1	198.1	0.0070	t_i	0.2889						
				z_i	–0.3878	0.4125					
	2	156.3	0.0090	t_i	–0.2954	0.8870					
				z_i	–0.5880	0.0204	0.6220				
	3	144.0	0.0099	t_i	–0.6405	0.3009	1.2428				
				z_i	–0.7197	–0.1949	0.2423	0.7603			
	4	138.4	0.0103	t_i	–0.8812	0.0587	0.6685	1.4929			
				z_i	–0.8162	–0.3402	0.0263	0.3925	0.8620		
	5	135.4	0.0106	t_i	–1..0634	–0.3151	0.3079	0.9321	1.6839		
				z_i	–0.8913	–0.4483	–0.1227	0.1791	0.5048	0.9418	
	6	133.6	0.0107	t_i	–1.2084	–0.5121	–0.0468	0.5746	1.1359	1.8372	
				z_i	–0.9521	–0.5333	–0.2351	0.0297	0.2948	0.5936	1.0070
1	1	46.6	0.0256	t_i	0.5725						
				z_i	–0.7618	0.8533					
	2	37.3	0.0333	t_i	0.0459	1.1288					
				z_i	–1.1147	0.0792	1.2429				
	3	34.6	0.0367	t_i	–0.2387	0.5968	1.4438				
				z_i	–1.3259	–0.3028	0.4901	1.4854			
	4	33.5	0.0385	t_i	–0.4178	0.2891	0.9254	1.6528			
				z_i	–1.4649	–0.5414	0.1037	0.7552	1.6533		
	5	32.9	0.0395	t_i	–0.5384	0.0827	0.6141	1.1525	1.8020		
				z_i	–1.5608	–0.7049	–0.1481	0.3741	0.9436	1.7760	
	6	32.6	0.0402	t_i	–0.6230	–0.0658	0.3983	0.8443	1.3207	1.9127	
				z_i	–1.6290	–0.8229	–0.3278	0.1193	0.5716	1.0847	1.8684
1.5	1	18.8	0.0498	t_i	0.8471						
				z_i	–1.1378	1.3202					
	2	15.2	0.0654	t_i	0.3034	1.3495					
				z_i	–1.5924	0.1616	1.8456				
	3	14.3	0.0717	t_i	0.1636	0.8762	1.6076				
				z_i	–1.8291	–0.3309	0.7057	2.1367			
	4	13.9	0.1082	t_i	0.0443	0.6198	1.1540	1.7579			
				z_i	–1.9625	–0.6099	0.2005	1.0273	2..3134		
	5	13.7	0.0762	t_i	–0.0250	0.4587	0.8917	1.3331	1.8502		
				z_i	–2.0414	–0.7843	–0.1104	0.5349	1.2347	2.4245	
	6	13.6	0.0771	t_i	–0.0676	0.3493	0.7170	1.0763	1.4567	1.9090	
				z_i	–2.0903	–0.9006	–0.3216	0.2176	0.7654	1.3757	2.4958

Table 5.3: Optimal Group Limits and Weights
Standard Weibull Distribution, $\alpha = \beta$
The calculation of n assumes $\alpha = \beta = 0.001$

μ_1	k	n	λ	i =	1	2	3	4	5	6	7
0.5	1	110.1	0.0068	t_i	0.5934						
				z_i	–0.8764	0.3872					
	2	90.5	0.0104	t_i	0.3044	0.9060					
				z_i	–1.2995	–0.2719	0.5120				
	3	85.0	0.0122	t_i	0.1964	0.5372	1.1057				
				z_i	–1.5657	–0.6410	–0.0069	0.5714			
	4	82.7	0.0132	t_i	0.1442	0.3743	0.7126	1.2454			
				z_i	–1.7478	–0.8848	–0.3300	0.1464	0.6044		
	5	81.5	0.0138	t_i	0.1152	0.2855	0.5218	0.8492	1.3484		
				z_i	–1.8772	–1.0579	–0.5563	–0.1383	0.2464	0.6245	
	6	80.8	0.0143	t_i	0.0978	0.2313	0.4101	0.6441	0.9587	1.4273	
				z_i	–1.9711	–1.1859	–0.7246	–0.3467	–0.0069	0.3164	0.6376
1	1	1.4	0.0369	t_i	0.8591						
				z_i	–1.5449	0.7278					
	2	6.2	0.0508	t_i	0.5466	1.2247					
				z_i	–2.1412	–0.3949	0.9482				
	3	4.9	0.0569	t_i	0.4308	0.8507	1.4481				
				z_i	–2.4468	–0.9031	0.0622	1.0550			
	4	4.4	0.0599	t_i	0.3770	0.6815	1.0614	1.6015			
				z_i	–2.6151	–1.1947	–0.3962	0.3244	1.1157		
	5	4.1	0.0616	t_i	0.3485	0.5874	0.8677	1.2176	1.7137		
				z_i	–2.7131	–1.3779	–0.6907	0.0877	0.4948	1.1536	
	6	4.0	0.0627	t_i	0.3320	0.5285	0.7508	1.0125	1.3386	1.7992	
				z_i	–2.7735	–1.5007	–0.8960	–0.3704	0.1222	0.6141	1.1788
1.5	1	14.9	0.089	t_i	1.1408						
				z_i	–2.0899	1.0529					
	2	12.6	0.1127	t_i	0.8224	1.5417					
				z_i	–2.7530	–0.4487	1.3437				
	3	12.0	0.1205	t_i	0.7106	1.1712	1.7656				
				z_i	–3.0436	–1.0135	0.1300	1.4802			
	4	11.8	0.1235	t_i	0.6616	1.0014	1.3934	1.9076			
				z_i	–3.1843	–1.3137	–0.3990	0.4497	1.5561		
	5	11.7	0.1249	t_i	0.6368	0.9059	1.2014	1.5481	2.0046		
				z_i	–3.2592	–1.4934	–0.7263	–0.0325	0.6510	1.6028	
	6	11.7	0.1255	t_i	0.6228	0.8454	1.0832	1.3488	1.6619	2.0742	
				z_i	–3.3027	–1.6105	–0.9493	–0.3549	0.2124	0.7884	1.6337

Table 5.4: Optimal Group Limits and Weights
Standard Weibull Distribution, $\alpha = 0.001$, $\beta = 0.005$

μ_1	k	n	λ	i =	1	2	3	4	5	6	7
0.5	1	94.5	0.0208	t_i	0.6224						
				z_i	–0.8456	0.4006					
	2	78.1	0.0277	t_i	0.3281	0.9425					
				z_i	–1.2530	–0.2324	0.5240				
	3	73.6	0..0308	t_i	0.2179	0.5704	1.1441				
				z_i	–1.5034	–0.5868	0.0291	0.5811			
	4	71.7	0.0324	t_i	0.1648	0.4056	0.7505	1.2834			
				z_i	–1.6695	–0.8169	–0.2803	0.1786	0.6122		
	5	70.7	0.0333	t_i	0.1355	0.3157	0.5586	0.8894	1.3850		
				z_i	–1.7837	–0.9769	–0.4951	–0.0930	0.2752	0.6308	
	6	70.2	0.0340	t_i	0.1180	0.2608	0.4460	0.6840	0.9998	1.4618	
				z_i	–1.8638	–1.0927	–0.6531	–0.2911	0.0345	0.3422	0.6427
1	1	27.3	0.0758	t_i	0.9155						
				z_i	–1.4598	0.7655					
	2	23.0	0.0994	t_i	0.5991	1.2957					
				z_i	–2.0218	–0.2825	0.9845				
	3	21.8	0.1090	t_i	0.4838	0.9166	1.5263				
				z_i	–2.2987	–0.7617	0.1671	1.0872			
	4	21.4	0.1137	t_i	0.4311	0.7457	1.1341	1.6825			
				z_i	–2.4459	–1.0315	–0.2685	0.4213	1.1436		
	5	21.2	0.1162	t_i	0.4035	0.6507	0.9376	1.2938	1.7948		
				z_i	–2.5295	–1.1986	–0.5467	0.0294	0.5841	1.1776	
	6	21.1	0.1177	t_i	0.3877	0.5912	0.8190	1.0857	1.4164	1.8789	
				z_i	–2.5800	–1.3096	–0.7395	–0.2394	0.2305	0.6965	1.1994
1.5	1	13.1	0.1512	t_i	1.2196						
				z_i	–1.9534	1.1143					
	2	11.1	0.1913	t_i	0.8994	1.6413					
				z_i	–2.5730	–0.2637	1.4069				
	3	10.6	0.2051	t_i	0.7897	1.2643	1.8807				
				z_i	–2.8341	–0.7956	0.3119	1.5424			
	4	10.4	0.2110	t_i	0.7422	1.0926	1.4980	2.0361			
				z_i	–2.9576	–1.0745	–0.1931	0.6312	1.6170		
	5	10.3	0.2139	t_i	0.7182	0.9961	1.3017	1.6625	2.1451		
				z_i	–3.0226	–1.2402	–0.5041	0.1685	0.8341	1.6628	
	6	10.3	0.2155	t_i	0.7047	0.9349	1.1810	1.4571	1.7854	2.2260	
				z_i	–3.0601	–1.3480	–0.7150	–0.1402	0.4119	0.9742	1.6931

Tables 5.3 and 5.4 present the optimal group limits and weights for a standard Weibull process where the mean and standard deviation are equal to unity. Comparing the optimal limits for normal and Weibull group limits shows that the Weibull limits are not symmetric even when the error rates are equal. To compare the optimal limit in Tables 5.1 and 5.2 to the optimal Weibull process limits it is necessary to add unity to the normal process limits. In general, the optimal Weibull limits are shifted to lower values when compared to the optimal limits for a normal process.

5.2 Two-Sided Acceptance Sampling Plans

This section derives optimal group limits for the two-sided acceptance sampling plans of Section 3.2. For two-sided mean shift acceptance sampling plans or acceptance control charts I know of no previous work on optimal gauge design. Section 3.2 considers the hypothesis tests where the difference between μ_a^+ and μ_a^- is large in terms of sigma units, and thus the two-sided hypothesis test can be thought of as equivalent to two one-sided tests. As a result, the optimal group limits for two-sided mean shift detection will be in two clusters, half near μ_a^+ and the other half near μ_a^-. Since the two clusters of group limits are so far apart, the lower cluster of limits has very little effect on the ability to detect mean shifts to μ_r^+, and vice versa. As a result, the optimal group limits for the two sided test can be determined accurately from the analysis done for the one-sided tests in Section 5.1. Since adding a group limit near $\left(\mu_a^+ + \mu_a^-\right)/2$ will also have little effect, it is recommended that an even number of group limits be chosen. Half of the group limits will be determined by considering the lower hypothesis test, and the other half determined based on the upper hypothesis test. In other words, for the normal process use Tables 5.1 and 5.2, and for a Weibull process use Tables 5.3 and 5.4.

For example, say we wish to design an acceptance sampling plan or acceptance control chart based on 6 group limits that is to detect mean shifts of one sigma unit when μ_a^+ = 10, μ_a^- = 5 σ = 0.6, (thus μ_r^+ = 10.6, μ_r^- = 4.4), α = 0.001, β = 0.005, and the process is approximately normal. Then, from Table 5.2, the optimal standardized group limits for 3 step-gauge are

–0.2387, 0.5968, 1.4438. Thus the optimal group limits for this two-sided problem are –0.2387*0.6+5 = 4.86, 0.5968*0.6+5 = 5.36, 5.87, –0.2387*0.6+10 = 9.86, 10.36, 10.87. Notice that there are two distinct groups of gauge limits.

5.3 Shewhart Control Charts

The design of step-gauges for Shewhart type control charts is motivated by Stevens (1948). Stevens proposed designing two-step gauges so that they maximize the expected Fisher information about the null hypothesis. Shewhart charts attempt to detect whenever the process is no longer stable at the target value (or null hypothesis). As a result, the problem of determining the best group limits for control, may be thought of as equivalent to designing a step-gauge to best estimate the parameter of interest when the null hypothesis holds. This implies that the grouping criteria that maximizes the expected Fisher information at the null hypothesis should be used. Stevens considers only the case of detecting mean and standard deviation shifts of a normal process with a two-step gauge. This section extends the methodology, first to the general multiple group case, and second to the Weibull process. In addition, the process is often monitored for both mean and standard deviation shifts using data from the same step-gauge. However, the optimal step-group limits are not the same for these two purposes, so the issue of a compromise gauge design for simultaneous parameter shift detection is also considered.

As mentioned above, we are interested in assessing the expected Fisher information in a sample of size n. The information about the parameter θ in the sample of data from a k-step gauge is given by:

$$\mathrm{I}(\theta \mid \mathbf{Q}) = \left(\frac{\partial \ln(L(\theta \mid \mathbf{Q}))}{\partial \theta} \right)^2$$

$$= \left(\sum_{j=1}^{k+1} \frac{Q_j}{\pi_j(\theta)} \frac{d\pi_j(\theta)}{d\theta} \right)^2 . \qquad (5.1)$$

However, the log-likelihood for a sample of size n will be formed by the sum of n log-likelihoods, each of identical expectation (Edwards, 1972). As a result, it is equivalent, for our purposes, to consider the expected information in a single observation. The expected information in a sample of size one at θ, $E\big(I(\theta)\big)$, may be obtained by conditioning on the group into which the observation is classified. In particular, if $\mathbf{e}_1$, $\mathbf{e}_2$, ..., $\mathbf{e}_{k+1}$ denote the unit vectors of length (k+1), then

$$
\begin{aligned}
E\big(I(\theta)\big) &= \sum_{j=1}^{k+1} I(\theta \mid \mathbf{e}_j)\pi_j(\theta) \\
&= \sum_{j=1}^{k+1} \frac{1}{\pi_j(\theta)}\left(\frac{d\pi_j(\theta)}{d\theta}\right)^2 . \qquad (5.2)
\end{aligned}
$$

The parameter θ can represent any parameter of interest. The optimal group limits to detect mean or standard deviation shifts of a normal distribution are determined in Section 5.3.1. In Section 5.3.2 the optimal group limits for Weibull shape and scale parameter shifts are given. In each case, the group probability function $\pi_j(\theta)$, is adapted to the particular parameter and distribution of interest.

5.3.1 Normal Process

For the normal distribution the group probabilities are

$$
\pi_j(\mu,\sigma) = \int_{t_{j-1}}^{t_j} \phi(y)dy = \int_{t_{j-1}}^{t_j} \frac{1}{\sqrt{2\pi}\sigma}\exp\left(-\frac{(y-\mu)^2}{2\sigma^2}\right)dy .
$$

The standardized group limits t_j are utilized since ultimately interest lies in finding the information about the standard normal. Assume, without loss of generality, σ = 1. Then,

$$
\phi(y) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(y-\mu)^2}{2}\right),
$$

and

$$
\frac{d\pi_j(\mu)}{d\mu} = \phi\big(t_{j-1}\big) - \phi\big(t_j\big),
$$

and the expected information about μ from a single observation is

$$E\left(I(\mu)\right) \quad = \quad \frac{\phi(t_1;\mu)^2}{\pi_1(\mu)} + \sum_{j=2}^{k} \frac{\left(\phi(t_{j-1};\mu) - \phi(t_j;\mu)\right)^2}{\pi_j(\mu)} + \frac{\phi(t_k;\mu)^2}{\pi_{k+1}(\mu)}. \qquad (5.3)$$

The group limit design problem is to find the standardized group limits t_j's, that maximize this expected information. Without loss of generality, the calculations in the table assume μ = 0. Finding the best location for the physical group limits x_j's, from the optimal standardized limits is straightforward. If the process produces units that match a normal distribution with mean μ and standard deviation σ then $x_j = \sigma t_j + \mu$.

The function $E\left(I(\mu)\right)$ is not concave. If the first (k – 1) group limits are fixed and the k^{th} group limit is allowed to become arbitrarily large, the expected information asymptotically approaches a minimum. Extensive experimentation suggests, however, that the expected information function is unimodal. As a result, this non-linear optimization problem may be solved using either the Nelder-Mead multi-dimensional simplex method or the Fletcher Reeves algorithm (Press, et al., 1988). The Fletcher Reeves algorithm is more efficient but requires the gradient of (5.3) with respect to **t**. The gradient of (5.3) is given in Appendix E. Moreover, the optimal group limits are symmetric about zero, thus the number of variables in the problem can be reduced by one half.

Table 5.5 (see also Kulldorff, 1961) gives the optimal group limits and the efficiency of a k-step gauge relative to exact measurement. Efficiency is defined as the ratio of the statistical information available using groups to the information available using variables. Clearly, the use of more than two or three groups significantly increases the efficiency of an observation. Table 5.5 shows that more information about μ is available in ten five-group optimally gauged observations, than is available in nine exact measurements. If exact measurement is uneconomical, then a properly designed gauge is an excellent alternative.

Table 5.5: Optimal Group Limits to Detect Mean Shifts
assume that when process is "in control" $\mu = 0$

k	Efficiency	t_1	t_2	t_3	t_4	t_5	t_6
1	0.6366	0.0					
2	0.8098	–0.6120	0.6120				
3	0.8825	–0.9817	0.0	0.9817			
4	0.9201	–1.244	–0.3824	0.3824	1.244		
5	0.9420	–1.4468	–0.6589	0.0	0.6589	1.4468	
6	0.9560	–1.6108	–0.8744	–0.2803	0.2803	0.8744	1.6108

The expected Fisher information approach can also be used to design group limits when the objective is to detect shifts in the standard deviation of a process. Assuming Y to be normally distributed with $\mu = 0$ gives

$$\phi(y;\sigma) = \frac{1}{\sqrt{2\pi}\sigma}\exp\left(\frac{-y^2}{2\sigma^2}\right)$$

$$\frac{d\pi_j(\sigma)}{d\sigma} = \frac{1}{\sigma}\left(t_{j-1}\phi\left(t_{j-1};\sigma\right)-t_j\phi\left(t_j;\sigma\right)\right).$$

Thus the expected information about σ, for a single observation, is written:

$$E\left(I(\sigma)\right) = \frac{t_1^2\phi(t_1;\sigma)^2}{\sigma\,\pi_1(\sigma)}+\sum_{j=2}^{k}\frac{\left(t_{j-1}\phi(t_{j-1};\sigma)-t_j\phi(t_j;\sigma)\right)^2}{\sigma\,\pi_j(\sigma)}+\frac{t_k^2\phi(t_k;\sigma)^2}{\sigma\,\pi_{k+1}(\sigma)}. \quad (5.4)$$

Again we wish to find the group limits t_j's, that maximize this expected information. Without loss of generality, assume $\sigma = 1$. If the expected information is to be maximized when the process standard deviation is σ, multiple the group limits found for the case where $\sigma = 1$ by the desired σ value. Since extensive empirical study suggests this function is unimodal, the maximization problem can also be efficiently solved by the Nelder Mead or Fletcher Reeves algorithm. The gradient of $E\left(I(\sigma)\right)$ is given in Appendix E. The results are shown in Table 5.6. Note that for an even number of groups, the middle step gauge placement has arbitrary sign, and is not zero as in the mean shift case. This is because, to detect standard deviation shifts, a group limit placed at $t = 0$ will provide no additional information.

Table 5.6: Optimal Group Limits to Detect Sigma Shifts
assume that when process is "in control" $\sigma = 1$

k	Efficiency	t_1	t_2	t_3	t_4	t_5	t_6
1	0.3042	±1.5758					
2	0.6522	–1.4825	1.4825				
3	0.7074	–1.4520	1.1855	2.0249			
or 3	0.7074	–2.0249	–1.1855	1.4520			
4	0.8244	–1.9956	–1.1401	1.1401	1.9956		
5	0.8588	–1.9827	–1.1193	0.9837	1.6189	2.3267	
or 5	0.8588	–2.3269	–1.6190	–0.9837	1.1190	1.9821	
6	0.8943	–2.3130	–1.6002	–0.9558	0.9558	1.6002	2.3130

The group limits that maximize the expected information about σ are not the same as those that maximize the expected information about μ. As a result, since charts are often used to monitor for mean shifts and standard deviation shifts simultaneously a compromise gauge design is considered. Often the detection of mean shifts is given priority. For this reason, the proposed methodology for the grouping design allows some flexibility in the amount of emphasis given to detecting mean and standard deviation shifts. The analysis proceeds by using the weighted sum of efficiency ratings for mean and standard deviation estimation as the optimization criteria. In other words, maximize

$$Eff(\mu,\sigma;d) \quad = \quad d\ Eff(\mu) + (1-d)Eff(\sigma),$$

where d is the weight, $Eff(\mu)$ is the efficiency of mean estimation, and $Eff(\sigma)$ is the efficiency of standard deviation estimation. Unfortunately, this new combined efficiency criteria is a bimodal function, and therefore the optimization procedure may yield a local maximum. There is a boundary at around $d = 0.35$ where the improved mean estimate yielded by using a group limit at $t = 0$ is outweighed by the better standard deviation estimate obtained by staggering the group limits around t = 0. Empirical results suggest that the global maximum can be found using two different specific starting guesses for the group limits. The best group limits with the middle limit at t = 0 and the best group limits staggered about t = 0 are found and the solution with the largest $Eff(\mu,\sigma;d)$ value is chosen. Tables 5.7 presents the optimal 3-group gauge limits for different weights found in this manner. Table 5.8 gives the optimal compromise group designs for various number of group limits when the mean estimation is given greater weight (d = 0.7).

Table 5.7: Suggested 3-Group Limits to Detect Mean and Sigma Shifts
assume that if a process is "in control" μ = 0 and σ = 1

weight d	Efficiency μ	Efficiency σ	t_1	t_2	t_3
0.1	0.6543	0.7335	–1.3974	1.0861	1.9584
0.2	0.7030	0.7248	–1.3384	0.9559	1.8742
0.3	0.7587	0.7060	–1.2835	0.7752	1.7625
0.4	0.8523	0.6481	–1.3906	0	1.3906
0.5	0.8569	0.6446	–1.3577	0	1.3577
0.6	0.8623	0.6379	–1.3117	0	1.3117
0.7	0.8685	0.6262	–1.2529	0	1.2529
0.8	0.8749	0.6066	–1.1779	0	1.1779
0.9	0.8803	0.5757	–1.0859	0	1.0859

Table 5.8: Optimal Group Limits to Detect Mean and Sigma Shifts
d = 0.7, assume that if process is "in control" μ = 0 and σ = 1

k	Efficiency μ	Efficiency σ	t_1	t_2	t_3	t_4	t_5	t_6
1	0.6366	0	0					
2	0.7822	0.4664	–0.8487	0.8487				
3	0.8685	0.6262	–1.2529	0	1.2529			
4	0.9082	0.7384	–1.5500	–0.5295	0.5295	1.5500		
5	0.9333	0.8039	–1.7703	–0.8768	0	0.8768	1.7703	
6	0.9489	0.8486	–1.9481	–1.1366	–0.3889	0.3889	1.1366	1.9481

5.3.2 Weibull Process

When designing a step-gauge for Shewhart control charts when the underlying distribution is Weibull the methodology presented in Section 5.3.1 is used. However, since the Weibull is defined in terms of shape and scale parameters a and b, the optimal step-gauge design to detect shifts in a and b independently are first considered. Later the optimal group design for detecting shifts in both parameters simultaneously are discussed. The group limit designs for simultaneous parameter detection would be appropriate if, for example, we are interested in detecting mean and/or standard deviation shifts of a Weibull process.

If the process produces parts that are best modeled by a Weibull distribution, the group probabilities can be written:

$$\pi_j(a,b) \quad = \quad \exp\left(-\left(\frac{t_{j-1}}{b}\right)^a\right) - \exp\left(-\left(\frac{t_j}{b}\right)^a\right)$$

The standard Weibull is defined to have unit shape and scale parameters, $a = b = 1$.

With interest focused on the shape parameter of the Weibull distribution, and assuming, without loss of generality, that $b = 1$, we can write:

$$\pi_j(a) \quad = \quad \exp\left(-t_{j-1}^a\right) - \exp\left(-t_j^a\right),$$

and thus, $$\frac{d\pi_j(a)}{da} \quad = \quad t_j^a \ln\left(t_j\right)\exp\left(-t_j^a\right) - t_{j-1}^a \ln\left(t_{j-1}\right)\exp\left(-t_{j-1}^a\right).$$

We can assume $b = 1$ because, as will be explained later, the optimal limits found for the standard Weibull are easily translated to any value of b. From the above expressions, the expected information about the shape parameter, from a single observation, is:

$$E\left(I(a)\right) \quad = \quad \sum_{j=1}^{k+1} \frac{\left(t_j^a \ln\left(t_j\right)\exp\left(-t_j^a\right) - t_{j-1}^a \ln\left(t_{j-1}\right)\exp\left(-t_{j-1}^a\right)\right)^2}{\exp\left(-t_{j-1}^a\right) - \exp\left(-t_j^a\right)} \qquad (5.5)$$

This equation can be maximized using either the Nelder Mead multi-dimensional simplex algorithm, or more efficiently by Fletcher Reeves algorithm (see Appendix D for details on the gradients of $E(I(a))$ and $E(I(b))$). Table 5.9 presents the results of the maximization problem. The efficiency rating is calculated relative to the amount of information available in variables data.

When the scale parameter of the Weibull is of interest a similar analysis is possible. Assuming, without loss of generality, that a = 1, gives the group probabilities

$$\pi_j(b) \quad = \quad \exp\left(-\frac{t_{j-1}}{b}\right) - \exp\left(-\frac{t_j}{b}\right),$$

and therefore, $$\frac{d\pi_j(b)}{db} \quad = \quad \frac{t_{j-1}}{b^2}\exp\left(-\frac{t_{j-1}}{b}\right) - \frac{t_j}{b^2}\exp\left(-\frac{t_j}{b}\right).$$

So $$E(I(b)) \quad = \quad \sum_{j=1}^{k+1} \frac{\left(\frac{t_{j-1}}{b^2}\exp\left(-\frac{t_{j-1}}{b}\right) - \frac{t_j}{b^2}\exp\left(-\frac{t_j}{b}\right)\right)^2}{\exp\left(-t_{j-1}/b\right) - \exp\left(-t_j/b\right)}. \qquad (5.6)$$

The group limits that maximize expression (5.6) above are given in Table 5.10.

Table 5.9: Optimal Group Limits to Detect Shape Shifts
assume that if the process is "in control" a = 1

k	Efficiency	t_1	t_2	t_3	t_4	t_5	t_6
1	0.2801	0.1189					
2	0.6557	0.1418	3.2891				
3	0.7527	0.1505	2.6936	4.5643			
4	0.8285	0.0516	0.2486	2.6173	4.4970		
5	0.8692	0.0534	0.2580	2.3339	3.6005	5.3934	
6	0.8990	0.0245	0.1154	0.3257	2.2921	3.5641	5.3593

Table 5.10: Optimal Group Limits to Detect Scale Shifts
assume that if the process is "in control" b = 1

k	Efficiency	t_1	t_2	t_3	t_4	t_5	t_6
1	0.6476	1.5936					
2	0.8203	1.0176	2.6112				
3	0.8910	0.7540	1.7716	3.3652			
4	0.9269	0.6004	1.3545	2.3720	3.9656		
5	0.9476	0.4999	1.0998	1.8538	2.8714	4.4650	
6	0.9606	0.4276	0.9269	1.5273	2.2813	3.2989	4.8925

Often we are interested in both parameters simultaneously. This is the case when interest lies in the mean and/or the standard deviation of a Weibull process. For that reason, the optimal step-gauge design is determined for shifts in both parameters. As the relationship between shape and scale parameters and the corresponding mean and standard deviation is complex, an optimization criteria based on the average efficiency is proposed. Table 5.11 presents the results, showing the step-gauge design that has the highest average efficiency to detect scale and shape parameter shifts. For this compromise solution, the non-linear optimization procedure may give only a local maximum. Extensive empirical study suggests that we get different local maxima of the average of (5.5) and (5.6) depending on how many of the initial gauge limits start on either side of the mean value of the standardize Weibull, i.e. t = 1. Consequently, for a k-step gauge there are (k+1) local maximums.

Table 5.11: Optimal Group Limits to Detect Shape and Scale Shifts
assume that when the process is "in control" $a = b = 1$

k	Average Efficiency	t_1	t_2	t_3	t_4	t_5	t_6
1	0.4096	2.4552					
2	0.6351	0.2591	2.5473				
3	0.7613	0.2148	1.7140	3.6038			
4	0.8257	0.1181	0.6377	2.0668	3.9147		
5	0.8736	0.0814	0.4054	1.4950	2.7639	4.5538	
6	0.9021	0.0626	0.3027	1.0291	2.0483	3.2591	5.0264

The empirical results also suggest that to obtain the global optimal, as presented in Table 5.11, a starting solution must have an equal number of group limits on either side of t =1. For the case of a odd number of group limits, the extra group limit is better placed on the upper side of the mean. So, for example, the best gauge placement using only k = 3 gauge limits, would be at 0.215, 1.715, and 3.6, and this grouping criterion has an average efficiency of 76% for detecting scale and shape parameter shifts.

The optimal group limits presented in Tables 5.9-5.11 all show the best group limits for detecting shifts in the standard Weibull when $a = b = 1$. Fortunately, these optimal limits can easily be rescaled for the general Weibull. To translate the optimal group limits from the case $a = b = 1$, use the formula:

$$x_i = b t_i^a \qquad i = 1, \ldots, k+1$$

where t_i represent the standardized group limits presented in Tables 5.9-5.11, and x_i are the rescaled group limits.

5.4 Destructive Testing Procedure I

This section addresses the question of optimal group probabilities to estimate the correlation using Procedure I of Section 4.1. Note that in Chapter 4 the group divisions were defined in terms of group probabilities rather than group limits. However, group probabilities can easily be translated to group limits through the use of the inverse cumulative density function of the normal distribution, i.e. $t_1 = \Phi^{-1}(p)$. Given an actual correlation ρ_{ab} and the sample size *n*, the values of p_a and p_b that minimize the predicted standard deviation given in equation (4.6) can be found. Due to the well behaved nature of the function that approximates the standard deviation of ρ^*_{ab} Nelder Mead multidimensional simplex method (Press et al. 1988) is appropriate. The results showing the pair of p_a and p_b values that minimize the approximate standard deviation for ρ^*_{ab} for actual ρ_{ab} between -0.95 and 0.95 are plotted in Figure 5.1.

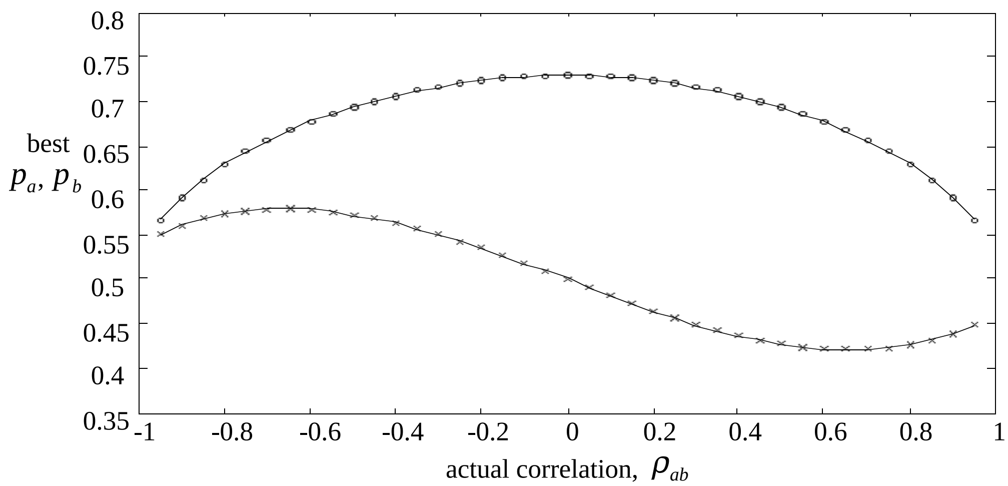

Figure 5.1: Optimal p_a and p_b Values to Estimate ρ_{ab}
'o' = best p_a, 'x' = best p_b

The plots for optimal p_a and p_b values appear approximately quadratic and cubic in nature respectively, and can be very closely approximated by fitting polynomial regression lines. This gives

$$\text{best } p_a = 0.733 - 0.165\rho_{ab}^2$$
$$\text{best } p_b = 0.499 - 0.184\rho_{ab} + 0.146\rho_{ab}^3 \qquad (5.7)$$

Determining the optimal values for p_a and p_b requires a prior estimate for the correlation ρ_{ab}. However, due to the relative insensitivity of the standard deviation of ρ^*_{ab} near the optimal p_a and p_b values (explored in the Section 4.1.1), choosing good p_a and p_b values can be done even with little idea of the actual ρ_{ab}. With little prior information regarding the correlation level, proof-load levels close to p_a = 0.65 and p_b = 0.45 are recommended. These proof-load levels provide correlation estimates with close to optimal standard deviation values for any true correlation level. Procedure I utilizes a single proof-load in each mode, thus in each dimension there is a single group limit. Based on the above suggested group probabilities, the best group limits are –0.126 and 0.385 for strength mode A and B respectively. These group limits are given in terms of a standardized normal process.

CHAPTER 6
Summary, Conclusion and Possible Extensions

This thesis develops quality control and improvement techniques based on grouped data. Grouped data commonly occur in industry when exact measurements are either prohibitively expensive or impossible. The methodology presented allows the creation of acceptance sampling plans, acceptance control charts, and Shewhart control charts based on grouped data. In addition, a number of correlation estimation procedures are derived that are applicable when data is grouped due to destructive testing.

In Chapter 1 the concept of grouped data is introduced, and an outline of the three major areas of application is presented. The three application areas are acceptance sampling plans, control charts and correlation estimation under destructive tests. Chapter 1 also provides a detailed literature survey of previous research relating to the use of grouped data in quality control and improvement problems. In Chapter 2, much of the notation used in the thesis is defined, and algorithms, derived from existing work in the literature, for the calculation of maximum likelihood parameter estimates from grouped data are presented. In addition, Chapter 2 shows that existing *ad hoc* quality control techniques often used in industry for grouped data are inadequate. Chapter 3 turns to the derivation of one-sided and two-sided acceptance sampling plans, acceptance control charts and Shewhart control charts based on grouped data. The solution methodology is based on the asymptotic properties of the chosen test statistic. In each case, different solution strategies are compared and contrasted. In addition, Chapter 3 discusses in detail the design of such plans and charts when utilizing small sample sizes. Chapter 4 presents four different procedures that use proof-loading to estimate the correlation between destructively measured strength properties. Unlike existing techniques, all the procedures involves grouping units in two modes and require no precise measurements. The first two procedures are adaptations of existing techniques that use a single proof-load in each mode and give only estimates of the correlation. The second two procedures are further extensions that utilize two proof-loads in each mode. The resulting additional information allows the estimation of the two individual means and standard deviations as well as the correlation. Chapter 5 addresses the issue of optimal grouping criteria. The best way to group observations depends on the application, but the optimal groupings for acceptance sampling plans, Shewhart control charts for normal or Weibull processes, and correlation estimation under destructive testing are derived through optimization techniques.

The ultimate goal of Statistical Process Control (SPC) is increased quality in manufactured products or services provided. For the most part, SPC techniques have been developed for two types of data: variables data and dichotomous data. However, existing quality control tools will not work well if they are inappropriate for the situation. For example, variables-based SPC techniques are commonly applied in an *ad hoc* way to grouped data and may lead to misleading results and incorrect decisions. Grouping data into two or more groups is a natural compromise between variables data and dichotomous data. Grouped data are common in industry and occur when precise measurement is expensive but gauging articles into groups is feasible. In this thesis the SPC methodology presented is developed and designed for grouped data. The derived sampling plans, control charts and correlation estimation procedures for grouped data are quite competitive, in terms of required sample size, to variables based methods. The slight loss in efficiency is often more than compensated by lower data collection costs, since grouping data may be easier and cheaper. In addition, the resulting charts and plans are easily implemented in a shop floor environment.

In a more general context, the presented methodology provides a framework to deal with grouped data in the areas of parameter estimation and hypothesis testing. The methods have been described in the context of quality improvement, but they are more widely applicable. The solution approaches suggested are very adaptable. The likelihood approach is appropriate for grouped data from any underlying distribution, and with any parameter of interest, since, due to the data grouping, the appropriate distribution is always multinomial. Thus, for any underlying distribution of the quality characteristic only the group probabilities change leaving the proposed design methodology unchanged. As a result, the techniques are also applicable when the underlying distribution of the quality characteristic is non-normal. For example, the proposed techniques have application in the service industry where service times are often modeled as exponentials. The weights-based methods are also very appealing due to their simplicity. In the weights-based methods each unit is assigned a weight based on the group into which it is classified, and the average weight of a sample is used as the test statistic. Thus, the weights-based methods are easily implemented since these calculations can be done without sophisticated measuring devices or computers.

A number of interesting extensions to this work are possible and are currently being pursued. Likelihood methods can be extended to sequential sampling methods through the sequential probability ratio test. Sequential procedures have the advantage of requiring, on average, smaller sample sizes to achieve the same operating characteristics as the fixed sample size solutions. Sequential testing procedures, however, have several drawbacks. The sample size required to reach a decision is not known a priori, and units must be considered one at a time. This extension would be of particular interest since it would allow the design of cumulative sum (CUSUM) charts. CUSUM charts are currently very popular in the literature since they are easy to use and are better at detecting small parameter shifts than Shewhart charts. Another possible extension is to develop hypothesis testing procedures for the correlation under destructive testing. A methodology similar to the one presented in Chapter 3 would probably also be applicable in this case. A third extension involves the application of grouped data to experimental design. The ultimate goal of both experimental design and control charts is quality improvement. However, control charts passively monitor the output of a process until an "out of control" signal is obtained, and then investigate and remove the cause of the problem. Designed experiments, in contrast, provide a more active statistical tool for achieving quality improvement. With designed experiments a number of possible input conditions are tried, and through a statistical analysis the probable optimal combination of process inputs is determined. Inputs consist of such things as raw materials, temperature and machine settings. Often the correct combination of inputs results in a reduction in the "natural variation" of a process. This thus leads directly to more consistent and higher quality outputs.

These proposed extensions, together with the work from this thesis, would provide practitioners with a fairly complete quality control and improvement system designed for grouped data. The thesis material makes the first important steps in this direction, and provides a methodology that can be extended to the other areas. In conclusion, the presented SPC techniques based on grouped data will be a valuable addition to a quality practitioner's repertoire of quality control and improvement techniques, and provide a methodology for parameter estimation and hypothesis testing based on grouped data in general.

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APPENDICES

Appendix A: Notation

This appendix summarizes most of the notation and acronyms used in this thesis.

Latin alphabet

a	Weibull shape parameter
b	Weibull scale parameter
c	arbitrary constant
CLT	central limit theorem
f_N	probability density function (p.d.f.) of the normal distribution
F_N	cumulative density function (c.d.f) of the normal distribution
f_W	p.d.f of the Weibull distribution
F_W	c.d.f. of the Weibull distribution
GLR	generalized likelihood ratio
H_0	null hypothesis
H_1	alternate hypothesis
H_{-1}	alternative hypothesis in the downward direction
i	unit index
j	group index
LACL	lower acceptance control limit
LCL	lower control limit
m	multiple of sigma units used to set control limits for Shewhart charts
MLE	maximum likelihood estimate
n	sample size
OC	operating characteristic
p_a	probability of failure under proof-load in mode A.
p_a^*	maximum likelihood estimate of p_a
PL_a	proof-load level in mode A.
Q	vector of sample grouping
Q_i	observed number of units in group i
t_i	location of standardized gauge limits
UACL	upper acceptance control limit
UCL	upper control limit
w	weights associated with H_{-1} vs. H_1
X_i	location of physical gauge limits
Y	random variable representing value of quality characteristic
z	weights associated with H_0 vs. H_1 test

Greek Alphabet

α	type I error rate
α'	actual type I error rate
β	type II error rate
β'	actual type II error rate
γ^*	critical GLR value
ϕ	probability density function
Φ	cumulative distribution function of the standard normal distribution
λ	critical likelihood ratio value
λ_α	adjustment of λ for sample size increase based on false alarm equation
λ_β	adjustment of λ for sample size increase based on power equation
μ	mean of the normal distribution
μ_0	stable mean value
μ_1	alternative mean value greater than μ_0
μ_{-1}	alternative mean value less than μ_0
μ_a	acceptable mean value
μ_r	rejectable mean value
μ_w	expected average w weight
μ_z	expected average z weight
π_j	probability of falling into group j
ρ_{ab}	correlation between strength modes A and B
ρ_{ab}^*	MLE for ρ_{ab}
θ	parameter of interest
θ_a	acceptable parameter value
θ_r	rejectable parameter value
θ_0	target parameter value
θ_1	parameter value of interest greater than θ_0
θ_{-1}	parameter value of interest less than θ_0
σ	standard deviation of the normal distribution
σ_w	standard deviation of the w weights
σ_z	standard deviation of the z weights

Appendix B: Interpretation of Weights

The group weight for group *j*, as expressed by equation (3.16), can be rescaled to be approximately equal the expected value of an observation that falls into group *j* given $\mu = \mu_0$. Since the weights can be rescaled, it is possible, without loss of generality, to restrict attention to the case when $\mu_0 = 0$ and $\sigma = 1$. For the normalized problem, μ_1 represents the size of mean shift we wish to detect given as a multiple of σ.

First, find the expected group value given $\mu = \mu_0$.

$$E\left(y \mid y \in j^{th} \text{ group}\right) = \frac{\int_{t_{j-1}}^{t_j} x\phi(x)dx}{\int_{t_{j-1}}^{t_j} \phi(x)dx}$$

$$= \frac{\phi\left(t_{j-1}\right) - \phi\left(t_j\right)}{Q\left(t_{j-1}, t_j\right)}$$

where

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-x^2/2\right), \text{ the p.d.f. of the standard normal}$$

$$Q(x,y) = \int_{s=x}^{y} \phi(s)ds,$$

and the t_j's are the standardized gauge limits.

Now consider the Taylor expansion in μ_1 of the weight assigned to all units that fall into group *j*, see equation (3.16) (assume $\mu_1 = -\mu_{-1}$).

$$\text{weight for group } j = \ln\left[\frac{\int_{t_{j-1}}^{t_j} \exp\left(-\frac{1}{2}(x-\mu_1)^2\right)}{\int_{t_{j-1}}^{t_j} \exp\left(-\frac{1}{2}(x+\mu_1)^2\right)}\right]$$

$$= \quad \frac{2\left(\frac{1}{\sqrt{2\pi}}\exp\left(-t_{j-1}^2/2\right)-\frac{1}{\sqrt{2\pi}}\exp\left(-t_j^2/2\right)\right)}{\frac{1}{\sqrt{2\pi}}\int_0^{t_{j-1}}\exp\left(-x^2\right)dx-\frac{1}{\sqrt{2\pi}}\int_0^{t_j}\exp\left(-x^2\right)dx}\,\mu_1 + O\left(\mu_1^3\right)$$

$$= \quad 2\frac{\phi\left(t_{j-1}\right)-\phi\left(t_j\right)}{Q\left(t_{j-1},t_j\right)}\,\mu_1 + O\left(\mu_1^3\right)$$

$$= \quad 2\mu_1 E\left(y \,\middle|\, y\in j^{th}\text{ group}\right) + O\left(\mu_1^3\right)$$

For reasonable values of t_{j-1} and t_j (-5 < t < 5) the coefficients for all terms in the above expansion of order higher than three is very small and decreasing as the order becomes higher. For typical values of μ_1, such as 1 or 2, these higher terms can be ignored, and the group weight for group *j* is approximately equal to $2\mu_1$ times the expected value of an observation that falls into group *j* given $\mu = \mu_0$.

Appendix C: Expected Value of Proof-load MLEs

This appendix contains the details of the proof that the MLEs given by equations (4.7) and (4.11) are all unbiased estimates.

Showing that the MLEs given by equations (4.7) are unbiased requires the following intermediate results:

$$
\begin{aligned}
E(n_a) &= np_a, & E(n_b) &= n(p_b - p_{a\cap b}) \\
E(m_a) &= m(p_a - p_{a\cap b}), & E(m_b) &= mp_b \\
E(n_a \mid n_a + n_b) &= \frac{p_a(n_a + n_b)}{p_a + p_b - p_{a\cap b}} \\
E(m_b \mid m_a + m_b) &= \frac{p_b(m_a + m_b)}{p_a + p_b - p_{a\cap b}} \\
E(n_a m_b) &= E(n_a)\,E(m_b) \\
E(m_a n_b) &= E(m_a)\,E(n_b)
\end{aligned}
$$

Using these expressions gives

$$
\begin{aligned}
\mathrm{E}\left(\frac{n_a(m_a + m_b)}{n_a + n_b} \mid n_a + n_b\right) &= \frac{\mathrm{E}(n_a \mid n_a + n_b)}{n_a + n_b} E(m_a + m_b) \\
&= \frac{\dfrac{p_a}{p_a + p_b - p_{a\cap b}}(n_a + n_b)}{n_a + n_b}\left(m(p_a - p_{a\cap b}) + mp_b\right) \\
&= mp_a
\end{aligned}
$$

Therefore, since this result does not depend on $n_a + n_b$

$$
\mathrm{E}\left(\frac{n_a(m_a + m_b)}{n_a + n_b}\right) = mp_a
$$

Similarly, $\mathrm{E}\left(\frac{m_b(n_a + n_b)}{m_a + m_b}\right) = np_b$

Also, we have

$$\mathrm{E}\left(\frac{m_b n_a - m_a n_b}{m_a + m_b} \mid m_a + m_b\right) = \frac{\mathrm{E}(n_a)\mathrm{E}(m_b \mid m_a + m_b) - \mathrm{E}(n_b)\mathrm{E}(m_a \mid m_a + m_b)}{m_a + m_b}$$

$$= \frac{\dfrac{np_a p_b (m_a + m_b)}{p_a + p_b - p_{a\cap b}} - \dfrac{n(p_b - p_{a\cap b})(p_a - p_{a\cap b})(m_a + m_b)}{p_a + p_b - p_{a\cap b}}}{m_a + m_b}$$

$$= np_{a\cap b}$$

Therefore, $\mathrm{E}\left(\frac{m_b n_a - m_a n_b}{m_a + m_b}\right) = np_{a\cap b}$

and $\mathrm{E}\left(\frac{m_b n_a - m_a n_b}{n_a + n_b}\right) = mp_{a\cap b}$

Thus, considering the MLEs in expressions (4.7),

$$p_a^* = \frac{n_a(n_a + n_b + m_a + m_b)}{(n+m)(n_a + n_b)} = \frac{1}{n+m}\left(n_a + \frac{n_a(m_a + m_b)}{n_a + n_b}\right)$$

Therefore, $\mathrm{E}(p_a^*) = \frac{1}{n+m}(np_a + mp_a) = p_a$

Similarly, $p_b^* = \frac{1}{n+m}\left(m_b + \frac{m_b(n_a + n_b)}{m_a + m_b}\right)$

and therefore $\mathrm{E}(p_b^*) = p_b$

Also, $p_{a\cap b}^* = \frac{(m_b n_a - m_a n_b)(n_a + n_b + m_a + m_b)}{(n+m)(n_a + n_b)(m_a + m_b)}$

$$p_{a\cap b}^* = \frac{1}{n+m}\left(\frac{(m_b n_a - m_a n_b)}{m_a + m_b} + \frac{(m_b n_a - m_a n_b)}{n_a + n_b}\right)$$

$$\therefore \quad \mathrm{E}(p_{a\cap b}^*) = \frac{1}{n+m}(np_{a\cap b} + mp_{a\cap b}) = p_{a\cap b}$$

Therefore, the MLEs given in equations (4.7) are all unbiased.

The MLEs given for Procedure III by equations (4.11) can be shown to be unbiased in a similar manner. To illustrate the technique it is shown that $\mathrm{E}\left(p^*_{a2}\right) = p_{a2}$ and $\mathrm{E}\left(p^*_{a2\cap b2}\right) = p_{a2\cap b2}$, the unbiasedness of the MLEs p^*_{a1}, p^*_{b1}, p^*_{b2}, $p^*_{a1\cap b2}$ and $p^*_{a2\cap b1}$ follow directly.

The following intermediate results are required:

$$
\begin{aligned}
E(n_{a2}) &= np_{a2}, \qquad E(n_{b2}) = n(p_{b2} - p_{a2\cap b2}) \\
E(m_{a2}) &= m(p_{a2} - p_{a2\cap b2}), \; E(m_{b2}) = mp_{b2} \\
E(n_{a2} \mid n_{a2} + n_{b2}) &= \frac{p_{a2}(n_{a2} + n_{b2})}{p_{a2} + p_{b2} - p_{a2\cap b2}} \\
E(m_{b2} \mid m_{a2} + m_{b2}) &= \frac{p_{b2}(m_{a2} + m_{b2})}{p_{a2} + p_{b2} - p_{a2\cap b2}} \\
E(n_{a2}m_{b2}) &= E(n_{a2})\,E(m_{b2}) \\
E(m_{a2}n_{b2}) &= E(m_{a2})\,E(n_{b2})
\end{aligned}
$$

Using these expressions gives

$$
\begin{aligned}
\mathrm{E}\left(\frac{n_{a2}(m_{a2} + m_{b2})}{n_{a2} + n_{b2}} \mid n_{a2} + n_{b2}\right) &= \frac{\mathrm{E}(n_{a2} \mid n_{a2} + n_{b2})}{n_{a2} + n_{b2}} E(m_{a2} + m_{b2}) \\
&= \frac{\frac{p_{a2}}{p_{a2} + p_{b2} - p_{a2\cap b2}}(n_{a2} + n_{b2})}{n_{a2} + n_{b2}} \left(m(p_{a2} - p_{a2\cap b2}) + mp_{b2}\right) \\
&= mp_{a2}
\end{aligned}
$$

Therefore, since this result does not depend on $n_{a2} + n_{b2}$

$$
\mathrm{E}\left(\frac{n_{a2}(m_{a2} + m_{b2})}{n_{a2} + n_{b2}}\right) = mp_{a2}
$$

Also,

$$
\mathrm{E}\left(\frac{m_{b2}n_{a2} - m_{a2}n_{b2}}{m_{a2} + m_{b2}} \mid m_{a2} + m_{b2}\right) = \frac{\mathrm{E}(n_{a2})\mathrm{E}(m_{b2} \mid m_{a2} + m_{b2}) - \mathrm{E}(n_{b2})\mathrm{E}(m_{a2} \mid m_{a2} + m_{b2})}{m_{a2} + m_{b2}}
$$

$$= \frac{\dfrac{np_{a2}p_{b2}\left(m_{a2}+m_{b2}\right)}{p_{a2}+p_{b2}-p_{a2\cap b2}} - \dfrac{n\left(p_{b2}-p_{a2\cap b2}\right)\left(p_{a2}-p_{a2\cap b2}\right)\left(m_{a2}+m_{b2}\right)}{p_{a2}+p_{b2}-p_{a2\cap b2}}}{m_{a2}+m_{b2}}$$

$$= np_{a2\cap b2}$$

Therefore, $\mathrm{E}\left(\dfrac{m_{b2}n_{a2}-m_{a2}n_{b2}}{m_{a2}+m_{b2}}\right) = np_{a2\cap b2}$

and $\mathrm{E}\left(\dfrac{m_{b2}n_{a2}-m_{a2}n_{b2}}{n_{a2}+n_{b2}}\right) = mp_{a\cap b}$

Thus $p^*_{a2} = \dfrac{n_{a2}\left(n_{a2}+n_{b2}+m_{a2}+m_{b2}\right)}{(n+m)\left(n_{a2}+n_{b2}\right)} = \dfrac{1}{n+m}\left(n_{a2}+\dfrac{n_{a2}\left(m_{a2}+m_{b2}\right)}{n_{a2}+n_{b2}}\right)$

Therefore, $\mathrm{E}\left(p^*_{a2}\right) = \dfrac{1}{n+m}\left(np_{a2}+mp_{a2}\right) = p_{a2}$

Also, $p^*_{a2\cap b2} = \dfrac{\left(m_{b2}n_{a2}-m_{a2}n_{b2}\right)\left(n_{a2}+n_{b2}+m_{a2}+m_{b2}\right)}{(n+m)\left(n_{a2}+n_{b2}\right)\left(m_{a2}+m_{b2}\right)}$

$$p^*_{a2\cap b2} = \frac{1}{n+m}\left(\frac{\left(m_{b2}n_{a2}-m_{a2}n_{b2}\right)}{m_{a2}+m_{b2}}+\frac{\left(m_{b2}n_{a2}-m_{a2}n_{b2}\right)}{n_{a2}+n_{b2}}\right)$$

$$\therefore \quad \mathrm{E}\left(p^*_{a2\cap b2}\right) = \frac{1}{n+m}\left(np_{a2\cap b2}+mp_{a2\cap b2}\right) = p_{a2\cap b2}$$

Therefore, the MLEs p^*_{a2} and $p^*_{a2\cap b2}$ given in equations (4.11) are unbiased.

Appendix D: Gradient of Sample Size Formula

In addressing the question of optimal gauge limits in the case of a one-sided acceptance sampling plan it is necessary to find the gauge limits that minimize the required sample size. Although this minimization problems can be solved directly through the use of the Nelder-Mead multidimensional simplex algorithm more efficient techniques exist that utilize the gradient of the function to be minimized. As a result, this appendix shows the calculation of the gradient.

To determine the optimal gauge limits to detect shifts in the mean of a normal distribution, the sample size, as given in equation (3.4), must be minimized . For given α, β, and acceptable and rejectable mean values μ_a and μ_r, equation (3.4) is only a function of the standardized gauge limits **t**. From Section 3.1, equation (3.4) is

$$n = \left(\frac{\Phi^{-1}(1-\beta)\sigma_z(\mu_r) - \Phi^{-1}(\alpha)\sigma_z(\mu_a)}{\mu_z(\mu_r) - \mu_z(\mu_a)} \right)^2 .$$

Taking the derivative with respect to the gauge limit t_i gives:

$$\frac{\partial n}{\partial t_i} = 2\left(\frac{\Phi^{-1}(\alpha)\sigma_z(\mu_a) - \Phi^{-1}(1-\beta)\sigma_z(\mu_r)}{\mu_z(\mu_a) - \mu_z(\mu_r)} \right)$$

$$\left(\frac{\Phi^{-1}(\alpha)\,\partial\sigma_z(\mu_a)/\partial t_i - \Phi^{-1}(1-\beta)\,\partial\sigma_z(\mu_r)/\partial t_i}{\mu_z(\mu_a) - \mu_z(\mu_r)} \right)$$

$$\left(\frac{\Phi^{-1}(\alpha)\sigma_z(\mu_a) - \Phi^{-1}(1-\beta)\sigma_z(\mu_r)}{\left(\mu_z(\mu_a) - \mu_z(\mu_r)\right)^2} \right)\left(\frac{\partial \mu_z(\mu_a)}{\partial t_i} - \frac{\partial \mu_z(\mu_r)}{\partial t_i} \right)$$

where $$\frac{\partial \mu_z(\mu)}{\partial t_j} = \phi(t_j,\mu)(z_j - z_{j+1}) + \frac{\pi_j(\mu)}{\pi_j(\mu_r)}\left(\phi(t_j,\mu_r) - \frac{\pi_j(\mu_r)}{\pi_j(\mu_a)}\phi(t_j,\mu_a) \right)$$

$$- \frac{\pi_{j+1}(\mu)}{\pi_{j+1}(\mu_r)}\left(\phi(t_j,\mu_r) - \frac{\pi_{j+1}(\mu_r)}{\pi_{j+1}(\mu_a)}\phi(t_j,\mu_a) \right)$$

and $$\frac{\partial\sigma_z(\mu)}{\partial t_j} = \sigma_z(\mu)^{-1}\left(-2\mu_z(\mu)\frac{\partial\mu_z(\mu)}{\partial t_j} + \frac{\partial\pi_j(\mu)}{\partial t_j}z_j^2 + \frac{\partial\pi_{j+1}(\mu)}{\partial t_j}z_{j+1}^2 + 2\pi_j(\mu)z_j\frac{\partial z_j}{\partial t_j} + 2\pi_{j+1}(\mu)z_{j+1}\frac{\partial z_{j+1}}{\partial t_j}\right)$$

This follows since, as given in Section 3.1,

$$\mu_z(\mu) = \sum_{j+1}^{k+1}\pi_j(\mu)\ln\left(\frac{\pi_j(\mu_a)}{\pi_j(\mu_r)}\right),$$

$$\sigma_z(\mu) = \sqrt{\sum_{j=1}^{k+1}\pi_j(\mu)\ln\left(\frac{\pi_j(\mu_a)}{\pi_j(\mu_r)}\right) - \mu_z^2(\mu)}$$

and

$$\pi_j(\mu) = \int_{t_{j-1}}^{t_j}\phi(y;\mu)dy,$$

$$\phi(t,\mu) = \frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}(t-\mu)^2\right)$$

$$\frac{\partial\pi_j(\mu)}{\partial t_j} = \phi(t_j,\mu)$$

$$\frac{\partial\pi_{j+1}(\mu)}{\partial t_j} = -\phi(t_j,\mu)$$

Appendix E: Normal Information Gradient

The optimal gauge limits for Shewhart control charts are the limits that maximize the expected statistical information available in a sample. The maximization can be done most quickly if also provided with gradients. The gradient of the expected information for the mean and standard deviation of a normal distribution is derived below. For Shewhart type charts to detect mean shifts it is necessary to maximize (5.3) which for given α, β, μ_0 and μ_1 is a function only of the gauge limits **t**. The equation (5.3) is repeated below:

$$E\left(I(\mu)\right) \;=\; \frac{\phi(t_1;\mu)^2}{\pi_1(\mu)} + \sum_{j=2}^{k} \frac{\left(\phi(t_{j-1};\mu) - \phi(t_j;\mu)\right)^2}{\pi_j(\mu)} + \frac{\phi(t_k;\mu)^2}{\pi_{k+1}(\mu)}$$

Then since,

$$\pi_j(\mu) \;=\; \int_{t_{j-1}}^{t_j} \phi(y;\mu)dy,$$

$$\phi(x,\mu) \;=\; \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(x-\mu)^2\right) \qquad \frac{\partial\theta}{\partial x} \;=\; \frac{\mu - x}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(x-\mu)^2\right)$$

$$\frac{\partial \pi_j}{\partial t_j} \;=\; \phi\left(t_j,\mu\right) \qquad \frac{\partial \pi_{j+1}}{\partial t_j} \;=\; -\phi\left(t_j,\mu\right)$$

Defining $t_0 = -\infty$, and $t_{k+1} = \infty$, the derivative of the expected information about the mean with respect to the group limits is:

$$\frac{\partial E\left(I(\mu)\right)}{\partial t_j} \;=\; \frac{2\left(\phi\left(t_j\right) - \phi\left(t_{j+1}\right)\right)}{\pi_{j+1}} \frac{\partial \phi\left(t_j\right)}{\partial t_j} + \frac{\left(\phi\left(t_j\right) - \phi\left(t_{j+1}\right)\right)^2}{\pi_{j+1}^2} \phi\left(t_j\right)$$
$$- \frac{2\left(\phi\left(t_{j-1}\right) - \phi\left(t_j\right)\right)}{\pi_j} \frac{\partial \phi\left(t_j\right)}{\partial t_j} - \frac{\left(\phi\left(t_{j-1}\right) - \phi\left(t_j\right)\right)^2}{\pi_j^2} \phi\left(t_j\right)$$

For Shewhart type charts to detect standard deviation shifts expression (5.4) must be maximized, which also for given α, β, μ_0 and μ_1 is a function just of the gauge limits **t**. The equation (5.4) is repeated below:

$$E\big(I(\sigma)\big) = \frac{t_1^2 \phi(t_1;\sigma)^2}{\sigma\, \pi_1(\sigma)} + \sum_{j=2}^{k} \frac{\left(t_{j-1} - t_j\right)^2 \left(\phi(t_{j-1};\sigma) - \phi(t_j;\sigma)\right)^2}{\sigma\, \pi_j(\sigma)} + \frac{t_k^2 \phi(t_k;\sigma)^2}{\sigma\, \pi_{k+1}(\sigma)}$$

Using the above equations gives:

$$\frac{\partial E\big(I(\sigma)\big)}{\partial t_j} = \frac{2\left(t_j\phi\left(t_j\right) - t_{j+1}\phi\left(t_{j+1}\right)\right)}{\pi_{j+1}(\sigma)} \frac{\partial \phi\left(t_j\right)}{\partial t_j} + \frac{\left(t_j\phi\left(t_j\right) - t_{j+1}\phi\left(t_{j+1}\right)\right)^2}{\pi_{j+1}^2(\sigma)} \phi\left(t_j\right) - \frac{2\left(t_{j-1}\phi\left(t_{j-1}\right) - t_j\phi\left(t_j\right)\right)}{\pi_j(\sigma)} \frac{\partial \phi\left(t_j\right)}{\partial t_j} - \frac{\left(t_{j-1}\phi\left(t_{j-1}\right) - t_j\phi\left(t_j\right)\right)^2}{\pi_j^2(\sigma)} \phi\left(t_j\right)$$

Appendix F: Weibull Information Gradients

The optimal gauge limits for Shewhart control charts are the limits that maximize the expected statistical information available. This appendix provides the gradients with respect to the gauge limits for the expected information of a grouped Weibull distribution.

For Shewhart type charts to detect shape parameter shifts the expected information as expressed in equation (5.5) must be maximized. Given α, β, μ_0 and μ_1 equation (5.5) is a function only of the gauge limits **t**. Equation (5.5) is repeated below:

$$E(I(a)) = \sum_{j=1}^{k+1} \frac{\left(t_j^a \ln(t_j)\exp(-t_j^a) - t_{j-1}^a \ln(t_{j-1})\exp(-t_{j-1}^a)\right)^2}{\exp(-t_{j-1}^a) - \exp(-t_j^a)}$$

Then defining

$$h_i = t_i^a \ln(t_i)\exp(-t_i^a)$$

and

$$g_i = ah_i + t_i^a \exp(-t_i^a) - ah_i t_i^a,$$

gives

$$\begin{aligned}\frac{\partial E(I(a))}{\partial t_j} = {} & \frac{2(h_i - h_{i-1})g_i}{t_i\left(\exp(t_{i-1}^a) - \exp(t_i^a)\right)} - \frac{a(h_i - h_{i-1})^2 t_i^a \exp(t_i^a)}{t_i\left(\exp(t_{i-1}^a) - \exp(t_i^a)\right)^2} \\ & - \frac{2(h_{i+1} - h_i)g_i}{t_i\left(\exp(t_i^a) - \exp(t_{i+1}^a)\right)} + \frac{a(h_{i+1} - h_i)^2 t_i^a \exp(t_i^a)}{t_i\left(\exp(t_i^a) - \exp(t_{i+1}^a)\right)^2}\end{aligned}$$

For Shewhart type charts to detect scale parameter shifts from a Weibull distribution the optimal gauge limits maximize the expected information as expressed in equation (5.6):

$$E(I(b)) = \sum_{j=1}^{k+1} \frac{\left(\frac{t_{j-1}}{b^2}\exp\left(-\frac{t_{j-1}}{b}\right) - \frac{t_j}{b^2}\exp\left(-\frac{t_j}{b}\right)\right)^2}{\exp(-t_{j-1}/b) - \exp(-t_j/b)}$$

Now define

$$d_i = \frac{t_i}{b^2}\exp\left(-\frac{t_i}{b}\right)$$

Then using the above definitions gives:

$$\begin{aligned}\frac{\partial E(I(b))}{\partial t_j} = & \frac{2(d_{i-1}-d_i)\left(\frac{t_i}{b}-1\right)d_i}{t_i\left(\exp(t_{i-1}/b)-\exp(t_i/b)\right)} - \frac{bd_i(d_{i-1}-d_i)^2}{t_i\left(\exp(t_{i-1}/b)-\exp(t_i/b)\right)^2} \\ & - \frac{2(d_i-d_{i+1})\left(\frac{t_i}{b}-1\right)d_i}{t_i\left(\exp(t_i/b)-\exp(t_{i+1}/b)\right)} + \frac{bd_i(d_i-d_{i+1})^2}{t_i\left(\exp(t_i/b)-\exp(t_{i+1}/b)\right)^2}\end{aligned}$$