SPECIAL K
Saturday 27 October 2007
9 a.m. to 12 noon

1. The notation $\underline{a}89\underline{b}$ means the four-digit (base 10) integer whose thousands digit is a, whose hundreds digit is 8, whose tens digit is 9, and whose units digit is b.
   Determine all pairs of non-zero digits a and b such that $\underline{a}89\underline{b} - 5904 = \underline{b}98\underline{a}$.

2. Lino bought a new eraser for this year's Putnam Contest. (Unfortunately, Lino's Putnam eligibility expired several years ago.) His new eraser is in the shape of a rectangular prism. He calculated the lengths of the diagonals of the faces to be $3\sqrt{5}$, $\sqrt{58}$ and $\sqrt{85}$. What is the volume of Lino's eraser?

3. Determine, with justification, all primes p for which the system of equations

$$\begin{aligned} 3x^2 - 4x^3 &\equiv 0 \pmod{p} \\ 2y + 4y^3 &\equiv 0 \pmod{p} \\ x^4 - x^3 + y^4 - y^2 &\equiv 0 \pmod{p} \end{aligned}$$

   has more than one solution (x, y) in $\mathbb{Z}_p \times \mathbb{Z}_p$ (that is, more than one solution (x, y) where each of x and y are considered modulo p).

4. Suppose that n is a positive integer.
   Let $S(n)$ be the sum of the base 10 digits of n.
   Define $S^1(n) = S(n)$ and $S^{j+1}(n) = S(S^j(n))$ for every positive integer j.
   Define $k(n) = \min\{k \in \mathbb{Z}^+ \mid S^k(n) < 10\}$.
   Define $l(n) = \max\{k(m) \mid m \in \mathbb{Z}^+, m \leq n\}$.
   Determine, with justification, $l(10^{10^{100}})$.

5. Suppose N is a positive integer. Define $P_N(x) = \sum_{n=0}^{N} \frac{x^n}{n!}$.

   (a) Prove that $P_N(x)$ has no repeated roots (real or complex).

   (b) Prove that if N is odd, then $P_N(x)$ has exactly one real root and if N is even, $P_N(x)$ has no real roots.

6. Determine, with justification, all (a, b) with $a, b \in \mathbb{Z}$, $a, b \geq 0$ such that $T = 1 + 4^a + 4^b$ is a perfect square.

BIG E
Saturday 27 October 2007
9 a.m. to 12 noon

1. Lino bought a new eraser for this year's Putnam Contest. (Unfortunately, Lino's Putnam eligibility expired several years ago.) His new eraser is in the shape of a rectangular prism. He calculated the lengths of the diagonals of the faces to be $3\sqrt{5}$, $\sqrt{58}$ and $\sqrt{85}$. What is the volume of Lino's eraser?

2. Let $V = M_{2007\times 2007}(\mathbb{R})$ be the vector space of all 2007×2007 matrices with real entries. Let $S = \{A \in V \mid AB = BA\}$, where B is the 2007×2007 matrix

$$B = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 & 0 \\ & \vdots & & \vdots & & \vdots & \\ 0 & 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

that is, with $b_{i,i+1} = 1$ for $i = 1, 2, \ldots, 2006$ and $b_{i,j} = 0$ otherwise.
Prove that S is a subspace of V and determine its dimension.

3. Point A on the x-axis with $-2 \le x \le 2$ is chosen at random. Point B in the cylinder $-1 \le x \le 1$, $y^2 + z^2 \le 1$ is also chosen at random. Determine the probability that $AB \le 1$.

4. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a continuous function. Prove that f is a bijection if and only if the following conditions are equivalent for any sequence $\{x_n\}$ of real numbers:

   (i) $\{x_n\}$ converges

   (ii) $\{f(x_n)\}$ converges

5. Suppose that n is a positive integer. Prove that if $4 \nmid n+2$ and $3 \nmid n+2$ (that is, neither 3 nor 4 divides $n+2$), then $(n+1)(n+2)$ divides $\binom{2n}{n}$.

6. Define $P(1) = P(2) = 1$ and $P(n) = P(P(n-1)) + P(n - P(n-1))$ for $n \ge 3$.
Prove that $P(2n) \le 2P(n)$ for all positive integers n.