Solutions to the Special K Problems, 2008

1: Find the value of $\min\limits_{|y|\le 1}\max\limits_{|x|\le 1}(x^2+xy)$ and the value of $\max\limits_{|x|\le 1}\min\limits_{|y|\le 1}(x^2+xy)$.

Solution: For fixed y, let $f(x)=x^2+xy=\left(x+\frac{y}{2}\right)^2-\frac{y^2}{4}$. The graph of $f(x)$ is a parabola which is concave up. For $-1\le x\le 1$, the maximum value of $f(x)$ occurs at one of the endpoints. Since $f(-1)=1-y$ and $f(1)=1+y$, we have

$$\max_{|x|\le 1}(x^2+xy)=\begin{cases}1+y\ , \text{ if } y\ge 0\\ 1-y\ , \text{ if } y\le 0\,,\end{cases}\qquad \text{and so } \min_{|y|\le 1}\max_{|x|\le 1}(x^2+xy)=1\,.$$

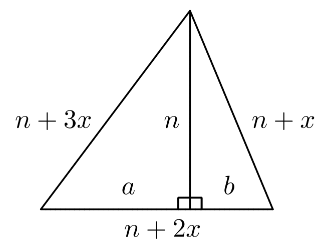
For fixed x, let $g(y)=x^2+xy$. The graph of $g(y)$ is a line (it has positive slope when $x>0$, zero slope when $x=0$, and negative slope when $x<0$). For $-1\le y\le 1$, the minimum value of $g(y)$ occurs at the left endpoint when $x\ge 0$ and the right endpoint when $x<0$. Since $g(-1)=x^2-x$ and $g(1)=x^2+x$,

$$\min_{|y|\le 1}(x^2+xy)=\begin{cases}x^2-x\ , \text{ if } x\ge 0\\ x^2+x\ , \text{ if } x\le 0\,,\end{cases}\qquad \text{and so } \max_{|x|\le 1}\min_{|y|\le 1}(x^2+xy)=0\,.$$

2: Let $f(x)=x^3-5x+1$ and let $g(x)=\dfrac{x-1}{x^2}$. Find the number of x-intercepts on the graph of $y=f(g(x))$.

Solution: Note that $f(-3)=-27+18+1<0$, $f(0)=1>0$, $f\left(\frac{1}{4}\right)=\frac{1}{64}-\frac{5}{4}+1=\frac{1-80+64}{64}<0$ and $f(3)=27-18+1>0$. By the intermediate Value Theorem, $f(x)$ has a root $\alpha\in(-3,0)$, a root $\beta\in\left(0,\frac{1}{4}\right)$ and a root $\gamma\in\left(\frac{1}{4},3\right)$. Since $f(x)$ is a cubic, these are the only three roots. The x-intercepts of $y=f(g(x))$ occur at each point $(x,0)$ such that $f(g(x))=0$, that is such that $g(x)=\alpha$, β or γ. Note that for $x\ne 0$, $a\ne 0$ we have $g(x)=a \iff \frac{x-1}{x^2}=a \iff x-1=ax^2 \iff ax^2-x=0$. The discriminate is $1-4a$, so the equation $g(x)=a$ has two solutions when $0\ne a<\frac{1}{4}$ and no solutions when $a>\frac{1}{4}$. Since $0\ne\alpha,\beta<\frac{1}{4}$ and $\gamma>\frac{1}{4}$, there are two values of x such that $g(x)=\alpha$ and two values such that $g(x)=\beta$ and no values such that $g(x)=\gamma$. Thus the graph of $y=f(g(x))$ has exactly 4 x-intercepts.

3: Determine the number of triangles which have the form shown below, where n is a positive integer and x is a real number with $0 \le x \le 1$.



Solution: Suppose a triangle has the given form, and let a and b be as shown above. Since $a+b=n+2x$ we have

$$a^2+2ab+b^2=n^2+4nx+4x^2 \quad (1)$$

Also, by Pythagoras' Theorem, we have

$$a^2+n^2=(n+3x)^2=n^2+6nx+9x^2 \qquad \text{so } a^2=6nx+9x^2 \quad (2)$$
$$b^2+n^2=(n+x)^2=n^2+2nx+x^2 \qquad \text{so } b^2=2nx+x^2 \quad (3)$$

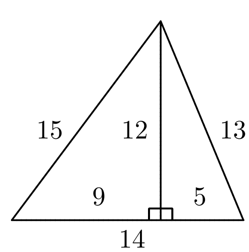
Subtracting equations (2) and (3) from (1) gives

$$2ab=n^2-4nx-6x^2\,.$$

Square both sides and use equations (2) and (3) to get

$$\begin{aligned}4(6nx+9x^2)(2nx+x^2)&=(n^2-4nx-6x^2)^2\\ 4(12n^2x^2+24nx^3+9x^3)&=(n^4-8n^3x+4n^2x^2+48nx^3+36x^4)\\ 48nx^3+44n^2x^2+8n^3x-n^4&=0\\ n(2x+n)^2(12x-n)&=0\,.\end{aligned}$$

Since $x\ge 0$ and $n>0$, we must have $12x=n$. Since $0\le x$ so that $0\le 12x=n\le 12$ and n is a positive integer, this gives 12 possibilities, namely $n=1,2,\cdots,12$ with $x=\frac{n}{12}$. Finally note that each of these 12 possibilities yields a triangle which is similar to the one shown below, scaled by the factor x.



There is a shorter solution which uses Heron's formula. The area of the triangle is given by $A=\frac{1}{2}(n+2x)(n)$. By Heron's Formula, the area is also given by $A=\sqrt{\left(\frac{3}{2}n+3x\right)\left(\frac{1}{2}n+2x\right)\left(\frac{1}{2}x+x\right)\left(\frac{1}{2}n\right)}$. Equate these expressions for A and square both sides to get $\frac{1}{4}(n+2x)^2n^2=\frac{3}{16}(n+2x)(n+4x)(n+2x)n$, that is $4n=3(n+4x)$, so we must have $n=12x$, as above.

4: A bathroom floor is tiled by regular hexagons of the same size. Points A, B and C are vertices of hexagons in the tiling. Prove that $\angle ABC \neq 45°$.

Solution: Each hexagonal tile can be cut into 6 equilateral triangles, so we generalize slightly and allow A, B and C to be vertices of these triangles. Choose coordinates so that B is at $(0,0)$ and so that an edge of a triangle has one end at B and the other at $(2,0)$. Then all vertices of all triangles are at points of the form $(a, b\sqrt{3})$ with $a, b \in \mathbf{Z}$. Say $A = (a, b\sqrt{3})$ and $C = (c, d\sqrt{3})$ with $a, b, c, d \in \mathbf{Z}$. Let $\theta = \angle ABC$. Then we have $\sin\theta = \dfrac{|(a, b\sqrt{3}, 0) \times (c, d\sqrt{3}, 0)|}{|(a, b\sqrt{3}, 0)||(c, d\sqrt{3}, 0)|}$ and $\cos\theta = \dfrac{(a, b\sqrt{3}) \bullet (c, d\sqrt{3})}{|(a, b\sqrt{3})||(c, d\sqrt{3})|}$ so

$$\tan\theta = \frac{|(a, b\sqrt{3}, 0) \times (c, d\sqrt{3}, 0)|}{(a, b\sqrt{3}) \bullet (c, d\sqrt{3})} = \frac{(ad - bc)\sqrt{3}}{ac + 3bd} .$$

Since $\tan\theta$ is a rational multiple of $\sqrt{3}$, it cannot be equal to 1, and so $\theta \neq \frac{\pi}{4}$.

5: Show that the product $P(n) = \displaystyle\prod_{k=1}^{n-1} \frac{k^{2k}}{k^{n+1}}$ is an integer whenever n is prime.

Solution: Note that $P(2) = 1$ which is an integer. Suppose that n is an odd prime, and write $n = 2l + 1$. Then

$$\begin{aligned}
P(n) = P(2l+1) &= \prod_{k=1}^{2l} \frac{k^{2k}}{k^{2n+2}} = \frac{1^2}{1^{2l+2}} \cdot \frac{2^4}{2^{2l+2}} \cdot \frac{3^6}{3^{2l+2}} \cdots \frac{(2l)^{4l}}{(2l)^{2l+2}} \\
&= \left(\frac{1}{1^{l+1}} \cdot \frac{2^2}{2^{l+1}} \cdot \frac{3^3}{3^{l+1}} \cdots \frac{l^l}{l^{l+1}} \cdot \frac{(l+1)^{l+1}}{(l+1)^{l+1}} \cdot \frac{(l+2)^{l+2}}{(l+2)^{l+1}} \cdots \frac{(2l)^{2l}}{(2l)^{l+1}} \right)^2 \\
&= \left(\frac{(l+2)(l+3)^2(l+4)^3 \cdots (2l)^{l-1}}{1^l \cdot 2^{l-1} \cdot 3^{l-2} \cdots l^1} \right)^2 \\
&= \left(\frac{(l+2)(l+3)(l+4)\cdots(2l)}{1 \cdot 2 \cdot 3 \cdot 4 \cdots l} \cdot \frac{(l+3)(l+4)\cdots(2l)}{1 \cdot 2 \cdot 3 \cdots (l-1)} \cdot \frac{(l+4)\cdots(2l)}{1 \cdot 2 \cdots (l-2)} \cdots \frac{(2l)}{1 \cdot 2} \right)^2 \\
&= \left(\frac{1}{2l+1}\binom{2l+1}{l} \cdot \frac{1}{2l+1}\binom{2l+1}{l-1} \cdot \frac{1}{2l+1}\binom{2l+1}{l-2} \cdots \frac{1}{2l+1}\binom{2l+1}{2} \right)^2 \\
&= \left(\prod_{k=2}^{(n-1)/2} \frac{1}{n}\binom{n}{k} \right)^2 ,
\end{aligned}$$

which is an integer, since when n is a prime and $1 \leq k < n$, n divides $\binom{n}{k}$.

6: Determine whether the sequence $\left\{\dfrac{1}{n\sin n}\right\}$ converges.

Solution: For each positive integer k we can find an integer $n_k \in \left[2\pi k + \frac{2\pi}{3}, \pi k + \frac{2\pi}{3}\right]$ (since this interval is of size $\frac{\pi}{3} > 1$). Then $n_k > 2\pi k$ and $\sin n_k \geq \frac{\sqrt{3}}{2}$, and so $n_k \sin n_k \geq \sqrt{3}\pi k$, hence $\frac{1}{n_k \sin n_k} \leq \frac{1}{\sqrt{3}\pi k} \to 0$ as $k \to \infty$. This shows that if the sequence $\left\{\frac{1}{n\sin n}\right\}$ does converge, then its limit must be zero.

For an integer k, let $\overline{k}$ denote the real number with $\overline{k} \in [0,\pi)$ such that $\overline{k} = k + \pi l$ for some integer l. Note that for any positive integer m, we can find n with $1 \leq n \leq m$ such that $\overline{n} \in \left[0, \frac{\pi}{m}\right) \cup \left[\pi - \frac{\pi}{m}, \pi\right)$; indeed if none of the m numbers $\overline{1}, \overline{2}, \overline{3}, \cdots \overline{m}$ were in the interval $\left[0 \frac{\pi}{m}\right)$, then one of the $m-1$ intervals $\left[\frac{\pi}{m}, \frac{2\pi}{m}\right), \left[\frac{2\pi}{m}, \frac{3\pi}{m}\right), \left[\frac{3\pi}{m}, \frac{4\pi}{m}\right), \cdots \left[\frac{(n-1)\pi}{m}, \frac{\pi}{m}\right)$ would contain two of the numbers $\overline{1}, \overline{2}, \overline{3}, \cdots, \overline{m}$, and if say $\overline{n_1}$ and $\overline{n_2}$ were in the same interval with $n_1 < n_2$, then we could take $n = n_2 - n_1$ and then $\overline{n} \in \left[0, \frac{\pi}{m}\right) \cup \left[\pi - \frac{\pi}{m}, \pi\right)$.

Choose $m_1 = 1$ and $n_1 = 1$. Having chosen m_k and n_k with $1 \leq n_k \leq m_k$ and $\overline{n_k} \in \left[0, \frac{\pi}{m_k}\right) \cup \left[\pi - \frac{\pi}{m_k}, \pi\right)$, choose m_{k+1} large enough so that $\frac{\pi}{m_{k+1}} < \min\{\overline{1}, \pi - \overline{1}, \overline{2}, \pi - \overline{2}, \overline{3}, \pi - \overline{3}, \cdots, \overline{m_k}, \pi - \overline{m_k}\}$, then choose n_{k+1} with $1 \leq n_{k+1} \leq m_{k=1}$ so that $\overline{n_{k+1}} \in \left[0, \frac{\pi}{m_{k+1}}\right]$. Our choice of m_{k+1} ensures that $\overline{n_{k+1}} \notin \{\overline{1}, \overline{2}, \overline{3}, \cdots, \overline{n_k}\}$ so that $n_{k+1} > n_k$. Also, we have $n_k \leq m_k$ and $|\sin n_k| = \sin \overline{n_k} \leq \sin \frac{\pi}{m_k} \leq \frac{\pi}{m_k}$, and so $\left|\frac{1}{n_k \sin n_k}\right| \geq \frac{1}{\pi}$. This implies that the limit of the sequence $\left\{\frac{1}{n\sin n}\right\}$ cannot be 0, so it diverges.

Solutions to the Big E Problems, 2008

1: Let A be a 10×10 matrix with integer entries. Suppose that 92 of those entries yield a remainder of 1 after division by 3. Show that $\det(A)$ is a multiple of 3.

Solution: Treat the entries of A as elements of $\mathbf{Z}_3$ (integers modulo 3). Then 92 of the entries are equal to 1. Since at most 8 entries are not equal to 1, there are at most 8 rows whose entries are not all equal to 1, and so there are at least two rows whose entries are all equal to 1. Since A has two equal rows, we have $\det A = 0$ in $\mathbf{Z}_3$. When the entries of A are treated as integers, we have $\det A \equiv 0$ modulo 3.

2: Let $f(x) = x^3 - 5x + 1$ and let $g(x) = \dfrac{x-1}{x^2}$. Find the number of x-intercepts on the graph of $y = f(g(x))$.

Solution: This is problem 2 from the Special K contest.

3: Find all twice differentiable functions $f(x)$ defined on $(0, \infty)$ such that $f'(x) > 0$ and $f\big(f'(x)\big) = -f(x)$ for all $x > 0$.

Solution: Since $f'(x) > 0$, we can substitute $f(x)$ for x in the equality $f\big(f'(x)\big) = -f(x)$ (1) to obtain $f\big(f'\big(f'(x)\big)\big) = -f\big(f'(x)\big)$ (2). Since $f'(x) > 0$ for all x we know that $f(x)$ is increasing and hence 1:1, and so equation (2) gives $f'\big(f'(x)\big) = x$ (3). On the other hand, taking the derivative on both sides of (1) gives $f'\big(f'(x)\big)f''(x) = -f'(x)$ (4). Equations (3) and (4) give $xf''(x) = -f'(x)$ (5). Thus we have

$$\frac{f''(x)}{f(x)} = -\frac{1}{x} \Longrightarrow \int \frac{f''(x)}{f(x)}\,dx = \int -\frac{1}{x}\,dx \Longrightarrow \ln\big(f'(x)\big) = -\ln x + a$$
$$\Longrightarrow f'(x) = \frac{b}{x} \Longrightarrow f(x) = b\ln x + c$$

where $a \in \mathbf{R}$, $b = e^a > 0$, and $c \in \mathbf{R}$. Also, we have

$$f\big(f'(x)\big) = -f(x) \Longrightarrow f\,(b/x) = -b\ln x - c \Longrightarrow b\ln(b/x) + c = -b\ln x - c$$
$$\Longrightarrow b\ln b - b\ln x + c = -b\ln x - c \Longrightarrow c = -\tfrac{1}{2}b\ln b\,.$$

Thus $f(x) = b\ln x - \frac{1}{2}b\ln b = b\ln\big(x/\sqrt{b}\big)$. Conversely, given any $b > 0$, the function $f(x) = b\ln\big(x/\sqrt{b}\big)$ satisfies the requirements that $f'(x) > 0$ and $f'\big(f(x)\big) = -f(x)$ for all $x > 0$.

4: Show that the product $P(n) = \displaystyle\prod_{k=1}^{n-1} \frac{k^{2k}}{k^{n+1}}$ is an integer whenever n is prime.

Solution: This is problem 5 from the Special K contest.

5: Determine whether the sequence $\left\{\dfrac{1}{n\sin n}\right\}$ converges.

Solution: This is problem 6 from the Special K contest.

6: Let $\pi > \alpha_1 > \alpha_2 > \alpha_3 > \cdots$ with each $\alpha_n > 0$. Form a polygonal path which winds clockwise starting with a line segment A_0A_1 of length 1, and adding edges A_nA_{n+1} all of length 1, with angle $A_{n-1}A_nA_{n+1}$ equal to α_n. Prove that there is a point contained in the interiors of all of these angles. (The *interior* of the angle $A_{n-1}A_nA_{n+1}$ is the intersection of the open half-plane bounded by the line $A_{n-1}A_n$ containing the point A_{n+1}, with the open half-plane bounded by the line A_nA_{n+1} containing the point A_{n-1}).

Solution: Let C_n be the open disk inside the angle $A_{n-1}A_nA_{n+1}$ which is tangent to the two edges at the midpoints. Notice that C_{n+1} is contained in C_n, and they have one common boundary point at the midpoint of A_nA_{n+1}, and that similarly C_{n+2} is contained in C_{n+1} sharing one common boundary point at the midpoint of $A_{n+1}A_{n+2}$, and so C_{n+2} is contained in C_n and shares no common boundary points. Thus the closure of C_{n+2} is contained in C_n. The limit point of the centers of the discs is contained in the closure of every disc and hence (since the closure of C_{n+2} is contained in C_n) also in the interior of every disc.

We remark that the conclusion is false if we only assume that $\pi \geq \alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \cdots$ with each $\alpha > 0$.