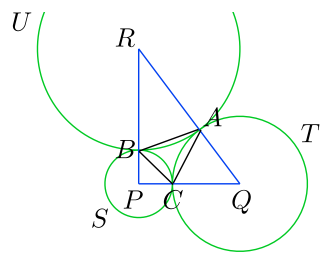
Solutions to the Special K Problems, 2014

1: Three circles, of radii 1, 2 and 3, are tangent in pairs at the points A, B and C. Find the area of triangle ABC.

Solution: Let S, T and U be the circles of radii 1, 2 and 3 respectively. Let P, Q and R be the centres of S, T and U. Let A be the point of intersection of T and U, let B be the point of intersection of U and S, and let C be the point of intersection of S and T. Since a radius of a circle interests with a tangent at right angles, we see that the points A, B and C lie on the edges of triangle PQR. Since S, T and U have radii 1, 2 and 3, we see that the triangle PQ has sides of the length $|PQ| = 1 + 2 = 3$, $|QR| = 2 + 3 = 5$ and $|RP| = 3 + 1 = 4$. Since $(3, 4, 5)$ is a Pythagorean triple, the triangle PQR is a right angled triangle with its right angle at P. The angles at Q and R are given by $\sin Q = \frac{4}{5}$ and $\sin R = \frac{3}{5}$. Using the one half base times height formula for areas of triangles, the area of triangle ABC is

$$\begin{aligned} |ABC| &= |PQR| - |PCB| - |QAC| - |RBA| \\ &= \tfrac{1}{2} \cdot 3 \cdot 4 - \tfrac{1}{2} \cdot 1 \cdot 1 - \tfrac{1}{2} \cdot 2 \cdot 2 \sin Q - \tfrac{1}{2} \cdot 3 \cdot 3 \sin R \\ &= 6 - \tfrac{1}{2} - \tfrac{8}{5} - \tfrac{27}{10} = \tfrac{6}{5}. \end{aligned}$$



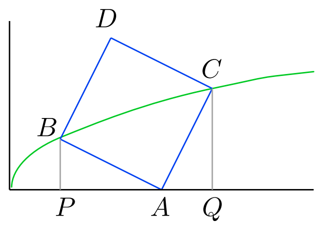
2: Find the number of ways to represent 10! as a sum of consecutive positive integers.

Solution: More generally, let $n \in \mathbf{Z}$ with $n \geq 2$. Suppose that $n = k + (k+1_+(k+2) + \cdots + l = \frac{(l-k+1)(k+l)}{2}$ where $k, l \in \mathbf{Z}^+$ with $k \leq l$. Let $u = l - k + 1$ and $v = k + l$. Note that $u, v \in \mathbf{Z}^+$ with $u < v$ and that u and v have opp[osite parity. Conversely, given $u, v \in \mathbf{Z}^+$ of opposite parity with $1 \leq u < v$ and $uv = 2n$ we can let $k = \frac{v-u+1}{2}$ and $l = \frac{v+u-1}{2}$ to get $n = \frac{(l-k+1)(k+l)}{2} = k + (k+1) + \cdots + l$. Thus the number of ways to represent n as a sum of consecutive positive integers is equal to the number of pairs of positive integers (u, v) such that u and v have opposite parity , $u < v$ and $uv = 2n$. This, in turn is equal to the number of pairs of positive integers (a, b) such that a is even and b is odd and $ab = 2n$ (indeed given u and v we can take a to be the even element in $\{u, v\}$ and b to be the odd element in $\{u, v\}$, and conversely given a and v we can take $u = \min\{a, b\}$ and $v = \max\{a, b\}$). When $n = 2^m p$ with $m \geq 0$ and p odd, the element a can be any number of the form $a = 2^{m+1} d$ with $d \big| p$ and then b is given by $b = \frac{n}{a} = \frac{p}{d}$. Thus when $n = 2^m p$, the number of ways tho represent n as a sum of consecutive positive integers is equal to $\tau(p)$, the number of divisors of p. In particular, when $n = 10! = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7^1$, the required number of ways is equal to $\tau(3^4 \cdot 5^2 \cdot 7^1) = (4+1)(2+1)(1+1) = 30$.

3: Given $a \geq 1$, find the area of the square with one vertex at $(a, 0)$, one vertex above the curve $y = \sqrt{x}$, and the other two vertices on the curve $y = \sqrt{x}$.

Solution: Let $A = (a, 0)$. Let $B = (b, \sqrt{b})$ and $C = (c, \sqrt{c})$ be the two points of the square which lie on the curve $y = \sqrt{x}$ with $b < c$, and let D be the other vertex of the square. Note that BC must be a diagonal of the square in order for D to lie above $y = \sqrt{x}$. Let $P = (b, 0)$ and $Q = (c, 0)$. Since $BACD$ is a square, the line segments AC and AB have the same length and are perpendicular, and so the triangles AQC and BPA are congruent. Thus we have $|AQ| = |BP|$ and $|QC| = |PA|$, that is $c - a = \sqrt{b}$ (1) and $a - b = \sqrt{c}$ (2). Adding equations (1) and (2) gives $c + b = \sqrt{b} + \sqrt{c}$ and so we have $\sqrt{c} - \sqrt{b} = 1$. Subtracting equation (1) from equation (2) gives $2a - (b + c) = \sqrt{c} - \sqrt{b} = 1$ and so we have $b + c = 2a - 1$. Thus the area of the square is

$$|BACD| = |AC|^2 = |AQ|^2 + |QC|^2 = |BP|^2 + |QC|^2 = b + c = 2a - 1.$$



4: Let n be an odd integer with $n > 3$. Let k be the smallest positive integer such that $kn + 1$ is a square, and let l be the smallest positive integer such that ln is a square. Show that n is prime if and only if $n < \min\{4k, 4l\}$.

Solution: Suppose first that n is prime, say $n = p$ with $p > 3$. It is clear that $l = p$. We claim that $k = p - 2$. Let $t \in \mathbf{Z}^+$ with $tp + 1$ equal to a square, say $tp + 1 = x^2$. Then $tp = x^2 - 1 = (x + 1)(x - 1)$. Since p is prime, either $p | x + 1$ or $p | x - 1$. If $p | x + 1$ then $p \leq x + 1$ and so $t = \frac{(x+1)(x-1)}{p} \geq x - 1 \geq p - 2$. If $p | x - 1$ then $p \leq x - 1$ and $t \geq x + 1 \geq p + 2$. In either case, we have $t \geq p - 2$. If $t = p - 2$ then $tp + 1 = (p - 2)p + 1 = p^2 - 2p + 1 = (p - 1)^2$ which is a square. Thus $k = \min\{t \in \mathbf{Z}^+ \big| tp + 1 \text{ is a square}\} = p - 2$ as claimed. Since $k = p - 2$ and $l = p$ and $n = p > 3$ we have $\min\{4k, 4l\} = 4k = 4p - 8 = p + (3p - 8) > p = n$.

Now suppose that n is composite. We claim that if n has a repeated prime factor then $n > 4l$ and if n has at least two distinct prime factors then $n > 4k$, so that in either case we have $n > \min\{4k, 4l\}$. Suppose first that n has a repeated prime factor, say $n = p^2 s$, where p is prime and $s \in \mathbf{Z}^+$. Note that since n is odd, $p \neq 2$ so $p \geq 3$. Also note that $l \leq s$ since $sn = p^2 s^2$ which is a square, Thus $n = p^2 s \geq 9l > 4l$, as claimed.

Finally, suppose that n has at least two distinct prime factors. Write n as $n = ab$ with a and b odd, $1 < a < n$ and $1 < b < n$ and $\gcd(a, b) = 1$. We wish to find a small value of t so that $tn + 1$ is a square. By the Chinese Remainder Theorem, there is a unique $x \in \mathbf{Z}$ with $1 < x < n$ such that $x = 1 \bmod a$ and $x = -1 \bmod b$ hence $x^2 = 1 \bmod n$. Note that for $y = n - x$ we have $y = -1 \bmod a$ and $y = 1 \bmod b$ so $y^2 = 1 \bmod n$. For $z = \min\{x, y\}$, we have $1 < z < \frac{n}{2}$ and $z^2 = 1 \bmod n$, say $z^2 = 1 + tn$. Then $tn + 1 = z^2$, which is a square, and $t = \frac{z^2 - 1}{n} < \frac{(n/2)^2}{n} = \frac{n}{4}$. Thus $k = \min\{t \in \mathbf{Z}^+ \big| tn + 1 \text{ is a square}\} < \frac{n}{4}$ and so $n > 4k$, as claimed.

5: A **zigzag** is a set of the form $Z = \{ta+(1-t)b \big| 0 \leq t \leq 1\} \cup \{a+tu | t \geq 0\} \cup \{b-tu | t \geq 0\}$ for some $a, b, u \in \mathbf{R}^2$ with $u \neq 0$ (Z is the union of the line segment between a and b with a ray at a in the direction of u and a ray at b in the direction $-u$). Given a positive integer n, find the maximum number of regions into which n zigzags divide the plane.

Solution: Let a_n be the maximum number of regions into which n zigzags divide the plane. Since 1 zigzag divides the plane into 2 regions, we have $a_1 = 2$. Suppose that we have chosen a configuration of n zigzags which divides the plane into a_n regions. Suppose that we add one more zigzag, say Z, to the configuration, and that Z intersects the existing zigzags in a total of p points. These p points divide Z into $p+1$ parts, and each of these parts divides one of the existing regions into two regions. Thus we increase the number of regions by $p+1$ obtaining a total of a_n+p+1 regions. When we add the zigzag Z, each of its 3 parts (namely its one line segment and its two rays) intersects each of the $3n$ parts of the existing n zigzags at most once, so the maximum number of points of intersection is $p = 3 \cdot 3n = 9n$. Thus we have $a_{n+1} \leq a_n + 9n + 1$.

We claim that is possible to choose a configuration of $n+1$ zigzags so that the maximum possible number of points of intersection is attained. To do this, first choose a configuration of n lines $L_0, L_1, \cdots, L_n$ so that each pair of lines intersect and no three lines intersect (for example, we could choose L_k to be the line $y = kx - k^2$). Choose $\epsilon > 0$ to be smaller than half the distance from any line L_k to any point of intersection of two other lines. For each k, let L_k' be a line parallel to L_k separated from L_k by a distance of ϵ. Then each of L_k and L_k' will intersect all of the other $2n$ lines. For each k, choose a rectangle $a_k b_k c_k d_k$ with $a_k, b_k \in L_k$ and $c_k, d_k \in L_k'$ which is sufficiently large that for every $j \neq k$ the lines L_j and L_j' both intersect L_k between a_k and b_k and both intersect L_k' between c_k and d_k, and consequently both intersect the diagonal $a_k c_k$ of the rectangle. We then let Z_k be the zigzag formed by the ray from a_k through b_k, the ray from c_k through d_k and the diagonal $a_k c_k$. In this way we obtain a configuration of zigzags $Z_0, Z_1, \cdots, Z_n$ with the proper that each of the three parts of Z_k intersects each of the three parts of Z_l whenever $k \neq l$.

We have shown that sequence a_n is given recursively by $a_1 = 2$ and $a_{n+1} = a_n + 9n + 1$. Thus

$$\begin{aligned} a_n &= 2 + (9 \cdot 1 + 1) + (9 \cdot 2 + 1) + (9 \cdot 3 + 1) + \cdots + (9 \cdot (n-1) + 1) \\ &= 2 + 9(1 + 2 + 3 + \cdots + (n-1)) + (n-1) \\ &= 9 \cdot \tfrac{n(n-1)}{2} + n + 1 \\ &= \frac{9n^2 - 7n + 2}{2}. \end{aligned}$$

6: Let $\{a_n\}$ be a sequence of real numbers with the property that for every $r \in \mathbf{R}$ with $r > 1$, we have $\lim\limits_{k \to \infty} a_{\lfloor r^k \rfloor} = 0$. Show that $\lim\limits_{n \to \infty} a_n = 0$.

Solution: Suppose, for a contradiction, that $\lim\limits_{n \to \infty} a_n \neq 0$. Choose $\epsilon > 0$ so that for every $l \in \mathbf{Z}^+$ there exists $n \geq l$ such that $|a_n| > \epsilon$. We shall construct a sequence of indices $n_1 < n_2 < n_3 \cdots$ with each $|a_{n_j}| > \epsilon$ and a sequence of closed bounded intervals $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ and a sequence of exponents $k_1 > k_2 > k_3 > \cdots$ such that for every $r \in I_j$ we have $\lfloor r^{k_j} \rfloor = n_j$, and then we shall use the existence of these sequences to obtain a contradiction.

Choose an index $n_1 > 1$ such that $|a_{n_1}| > \epsilon$. Let $I_1 = \left[n_1, n_1 + \frac{1}{2}\right]$ and $k_1 = 1$ and note that for all $r \in I_1$ we have $\lfloor r^{k_1} \rfloor = \lfloor r \rfloor = n_1$. Suppose that we have constructed indices $n_1 < n_2 < \cdots < n_{j-1}$ with $|a_{n_i}| > \epsilon$ and closed bounded intervals $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_{j-1}$ and exponents $k_1 < k_2 < \cdots < k_{j-1}$ such that for all $r \in I_i$ we have $\lfloor r^{k_i} \rfloor = n_i$. Say $I_{j-1} = [a, b]$ where $1 < n_1 \leq a < b \leq n_1 + \frac{1}{2}$. Since $1 < a < b$ we have $\left(\frac{b}{a}\right)^k \to \infty$ as $k \to \infty$. Choose m large enough that for all $k \geq m$ we have $\left(\frac{b}{a}\right)^k \geq a + \frac{1}{2}$. Then for $k \geq m$ we have $b^k \geq a^{k+1} + \frac{1}{2}a^k \geq a^{k+1} + \frac{1}{2}$ so that the intervals $\left[a^k, b^k - \frac{1}{2}\right]$ and $\left[a^{k+1}, b^{k+1} - \frac{1}{2}\right]$ overlap, and we have $\bigcup\limits_{k=m}^{\infty} \left[a^k, b^k - \frac{1}{2}\right] = [a^m, \infty)$. Choose n_j so that $n_j > n_{j-1}$, $n_j \geq a^m$ and $|a_{n_j}| > \epsilon$. Choose k_j so that $n_j \in \left[a^{k_j}, b^{k_j} - \frac{1}{2}\right]$ and note that $\left[n_j, n_j + \frac{1}{2}\right] \subseteq \left[a^{k_j}, b^{k_j}\right]$. Let $I_j = \left[{n_j}^{1/k_j}, (n_j + \frac{1}{2})^{1/k_j}\right]$ and note that for all $r \in I_j$ we have $r^{k_j} \in \left[n_j, n_j + \frac{1}{2}\right]$ so that $\lfloor r^{k_j} \rfloor = n_j$.

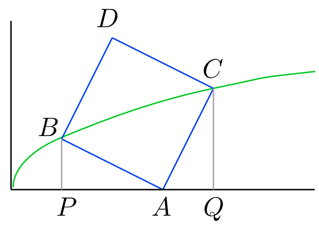
We can now obtain the desired contradiction as follows. Since the nested intervals $I_1 \supset I_2 \supset I_3 \supset \cdots$ are nonempty, closed and bounded, their intersection is nonempty. Choose $r \in \bigcap\limits_{j=1}^{\infty} I_j$. For each index j, since $r \in I_j$ we have $\lfloor r^{k_j} \rfloor = n_j$. Since $\lim\limits_{k \to \infty} a_{\lfloor r^k \rfloor} = 0$ it follows that $\lim\limits_{j \to \infty} a_{\lfloor r^{k_j} \rfloor} = 0$ hence $\lim\limits_{j \to \infty} a_{n_j} = 0$. But this is impossible since $|a_{n_j}| > \epsilon$ for all j.

Solutions to the Big E Problems, 2014

1: Given $a \geq 1$, find the area of the square with one vertex at $(a,0)$, one vertex above the curve $y=\sqrt{x}$, and the other two vertices on the curve $y=\sqrt{x}$.

Solution: Let $A=(a,0)$. Let $B=(b,\sqrt{b})$ and $C=(c,\sqrt{c})$ be the two points of the square which lie on the curve $y=\sqrt{x}$ with $b<c$, and let D be the other vertex of the square. Note that BC must be a diagonal of the square in order for D to lie above $y=\sqrt{x}$. Let $P=(b,0)$ and $Q=(c,0)$. Since $BACD$ is a square, the line segments AC and AB have the same length and are perpendicular, and so the triangles AQC and BPA are congruent. Thus we have $|AQ|=|BP|$ and $|QC|=|PA|$, that is $c-a=\sqrt{b}$ (1) and $a-b=\sqrt{c}$ (2). Adding equations (1) and (2) gives $c+b=\sqrt{b}+\sqrt{c}$ and so we have $\sqrt{c}-\sqrt{b}=1$. Subtracting equation (1) from equation (2) gives $2a-(b+c)=\sqrt{c}-\sqrt{b}=1$ and so we have $b+c=2a-1$. Thus the area of the square is

$$|BACD|=|AC|^2=|AQ|^2+|QC|^2=|BP|^2+|QC|^2=b+c=2a-1.$$



2: There are n closed (non-degenerate) line segments in $\mathbf{R}^3$. The sum of the lengths of the line segments is equal to 2014. Show that there is a plane in $\mathbf{R}^3$, which is disjoint from all of the line segments, such that the distance from the plane to the origin is less that 600.

Solution: Let the i^{th} line segment be $L_i=\{p_i+tu_i|0\leq t\leq 1\}$ where $p_i=(a_i,b_i,c_i)\in\mathbf{R}^3$ and $0\neq u_i=(x_i,y_i,z_i)\in\mathbf{R}^3$. The length of L_i is equal to $|u_i|=\sqrt{{x_i}^2+{y_i}^2+{z_i}^2}$ and we have $\sum\limits_{i=1}^n|u_i|=2014$. Suppose, for a contradiction, that every plane in $\mathbf{R}^3$ whose distance from the origin is less than 600 intersects with one of the line segments L_i. Then in particular, for $c\in(-600,600)$, each of the planes $x=c$, $y=c$ and $z=c$ intersects one of the segments L_i.

Let $A_i=\{a_i+tx_i\big|0\leq t\leq 1\}$ be the orthogonal projection of L_i onto the x-axis. The length of A_i is equal to $|x_i|$. Note that the interval $(-600,600)$ must be contained in the union $\bigcup\limits_{i=1}^n A_i$ because if $c\notin A_i$ for any i then the plane $x=c$ would not intersect any L_i. It follows that $1200\leq\sum\limits_{i=1}^n|x_i|$. Similarly, we must have $1200\leq\sum|y_i|$ and $1200\leq\sum\limits_{i=1}^n|z_i|$. By the Cauchy Schwarz Inequality, for $x,y,z\in\mathbf{R}$ we have

$$|x|+|y|+|z|=\Big|(1,1,1)\bullet(|x|,|y|,|z|)\Big|\leq\big|(1,1,1)\big|\,\big|(|x|,|y|,|z|)\big|=\sqrt{3}\sqrt{x^2+y^2+z^2}$$

and so

$$3600\leq\sum_{i=1}^n\big(|x_i|+|y_i|+|z_i|\big)\leq\sqrt{3}\sum_{i=1}^n\sqrt{{x_i}^2+{y_i}^2+{z_i}^2}=\sqrt{3}\sum_{i=1}^n|u_i|^2=2014\sqrt{3}.$$

But $2014\sqrt{3}<2014\cdot\frac{7}{4}=\frac{1007\cdot7}{2}=\frac{7049}{2}<\frac{7050}{2}=3525<3600$.

3: Let n be an odd integer with $n > 3$. Let k be the smallest positive integer such that $kn + 1$ is a square, and let l be the smallest positive integer such that ln is a square. Show that n is prime if and only if $n < \min\{4k, 4l\}$.

Solution: Suppose first that n is prime, say $n = p$ with $p > 3$. It is clear that $l = p$. We claim that $k = p-2$. Let $t \in \mathbf{Z}^+$ with $tp + 1$ equal to a square, say $tp + 1 = x^2$. Then $tp = x^2 - 1 = (x + 1)(x - 1)$. Since p is prime, either $p|x + 1$ or $p|x - 1$. If $p|x + 1$ then $p \le x + 1$ and so $t = \frac{(x+1)(x-1)}{p} \ge x - 1 \ge p - 2$. If $p|x - 1$ then $p \le x - 1$ and $t \ge x + 1 \ge p + 2$. In either case, we have $t \ge p - 2$. If $t = p - 2$ then $tp+1 = (p-2)p+1 = p^2-2p+1 = (p-1)^2$ which is a square. Thus $k = \min\{t \in \mathbf{Z}^+ \big| tp+1 \text{ is a square}\} = p-2$ as claimed. Since $k = p-2$ and $l = p$ and $n = p > 3$ we have $\min\{4k, 4l\} = 4k = 4p-8 = p+(3p-8) > p = n$.

Now suppose that n is composite. We claim that if n has a repeated prime factor then $n > 4l$ and if n has at least two distinct prime factors then $n > 4k$, so that in either case we have $n > \min\{4k, 4l\}$. Suppose first that n has a repeated prime factor, say $n = p^2 s$, where p is prime and $s \in \mathbf{Z}^+$. Note that since n is odd, $p \ne 2$ so $p \ge 3$. Also note that $l \le s$ since $sn = p^2 s^2$ which is a square, Thus $n = p^2 s \ge 9l > 4l$, as claimed.

Finally, suppose that n has at least two distinct prime factors. Write n as $n = ab$ with a and b odd, $1 < a < n$ and $1 < b < n$ and $\gcd(a, b) = 1$. We wish to find a small value of t so that $tn + 1$ is a square. By the Chinese Remainder Theorem, there is a unique $x \in \mathbf{Z}$ with $1 < x < n$ such that $x = 1 \bmod a$ and $x = -1 \bmod b$ hence $x^2 = 1 \bmod n$. Note that for $y = n - x$ we have $y = -1 \bmod a$ and $y = 1 \bmod b$ so $y^2 = 1 \bmod n$. For $z = \min\{x, y\}$, we have $1 < z < \frac{n}{2}$ and $z^2 = 1 \bmod n$, say $z^2 = 1 + tn$. Then $tn + 1 = z^2$, which is a square, and $t = \frac{z^2-1}{n} < \frac{(n/2)^2}{n} = \frac{n}{4}$. Thus $k = \min\{t \in \mathbf{Z}^+ \big| tn+1 \text{ is a square}\} < \frac{n}{4}$ and so $n > 4k$, as claimed.

4: Let $f : [0, 1] \to \mathbf{R}$. Suppose f is continuous on $[0, 1]$ with $f(0) = f(1) = 0$ and $f(x) > 0$ for all $x \in (0, 1)$, and f'' exists and is continuous in $(0, 1)$. Show that

$$\int_0^1 \left| \frac{f''(x)}{f(x)} \right| dx > 4\,.$$

Solution: Since f is continuous on $[0, 1]$, by the Extreme Value Theorem we can choose $u \in [0, 1]$ so that $f(u) \ge f(x)$ for all $x \in [0, 1]$. Since $f(x) > 0$ for all $x \in (0, 1)$ we must have $f(u) > 0$ hence $u \ne 0$ and $u \ne 1$ and so $u \in (0, 1)$. Since f is differentiable in $(0, u)$ and continuous on $[0, u]$, by the Mean Value Theorem we can choose $a \in (0, u)$ such that $f'(a) = \frac{f(u)-f(0)}{u-0} = \frac{f(u)}{u}$. Since f is differentiable in $(u, 1)$ and continuous on $[u, 1]$, by the Mean Value Theorem we can choose $b \in (u, 1)$ such that $f'(b) = \frac{f(1)-f(u)}{1-u} = \frac{-f(u)}{1-u}$. Then

$$\begin{aligned}
\int_0^1 \left| \frac{f''(x)}{f(x)} \right| dx &\ge \int_0^1 \frac{|f''(x)|}{f(u)}\, dx = \frac{1}{f(u)} \int_0^1 |f''(x)|\, dx \ge \frac{1}{f(u)} \int_a^b |f''(x)|\, dx \\
&\ge \frac{1}{f(u)} \left| \int_a^b f''(x)\, dx \right| = \frac{1}{f(u)} \Big| f'(b) - f'(a) \Big| = \frac{1}{f(u)} \left| \frac{-f(u)}{1-u} - \frac{f(u)}{u} \right| \\
&= \frac{1}{1-u} + \frac{1}{u} = \frac{1}{u(1-u)} = \frac{1}{\frac{1}{4} - \left(u - \frac{1}{2}\right)^2} \ge \tfrac{1}{1/4} = 4.
\end{aligned}$$

To complete the proof, we shall show that the first inequality in the above calculation is strict, that is

$$\int_0^1 \frac{|f''(x)|}{f(x)}\, dx > \int_0^1 \frac{|f''(x)|}{f(u)}\, dx.$$

Since both integrands are continuous in $(0, 1)$ with $\frac{|f''(x)|}{f(x)} \ge \frac{|f''(x)|}{f(c)}$ for all $x \in (0, 1)$, it suffices to show that there exists at least one point $x \in (0, 1)$ at which $\frac{|f''(x)|}{f(x)} > \frac{|f''(x)|}{f(c)}$. Equivalently, it suffices to show that there is a point $x \in (0, 1)$ such that $f''(x) \ne 0$ and $f(x) < f(c)$. Let $v = \inf\{x \in [0, 1] \big| f(x) \le f(c)\}$. Note that $0 < v \le u$ and we have $f(x) < f(c)$ for all $x \in [0, v)$, and also $f(v) = f(c)$ since f is continuous. Suppose, for a contradiction, that $f''(x) = 0$ for all $x \in (0, v)$. Since $f''(x) = 0$ for all $x \in (0, v)$ and f is continuous at 0 and v with $f(0) = 0$ and $f(v) = f(c)$, we must have $f(x) = \frac{f(c)}{v} x$ for all $x \in [0, v]$. But then $f'(v) = \frac{f(c)}{v} > 0$ which contradicts the fact that $f(v) = f(c)$ so that f has a maximum value at v.

5: Let $\{a_n\}$ be a sequence of real numbers with the property that for every $r \in \mathbf{R}$ with $r > 1$, we have $\lim_{k\to\infty} a_{\lfloor r^k \rfloor} = 0$. Show that $\lim_{n\to\infty} a_n = 0$.

Solution: Suppose, for a contradiction, that $\lim_{n\to\infty} a_n \neq 0$. Choose $\epsilon > 0$ so that for every $l \in \mathbf{Z}^+$ there exists $n \geq l$ such that $|a_n| > \epsilon$. We shall construct a sequence of indices $n_1 < n_2 < n_3 \cdots$ with each $|a_{n_j}| > \epsilon$ and a sequence of closed bounded intervals $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ and a sequence of exponents $k_1 > k_2 > k_3 > \cdots$ such that for every $r \in I_j$ we have $\lfloor r^{k_j} \rfloor = n_j$, and then we shall use the existence of these sequences to obtain a contradiction.

Choose an index $n_1 > 1$ such that $|a_{n_1}| > \epsilon$. Let $I_1 = \left[n_1, n_1 + \frac{1}{2}\right]$ and $k_1 = 1$ and note that for all $r \in I_1$ we have $\lfloor r^{k_1} \rfloor = \lfloor r \rfloor = n_1$. Suppose that we have constructed indices $n_1 < n_2 < \cdots < n_{j-1}$ with $|a_{n_i}| > \epsilon$ and closed bounded intervals $I_1 \supseteq I_2 \supseteq \cdots \supseteq I_{j-1}$ and exponents $k_1 < k_2 < \cdots < k_{j-1}$ such that for all $r \in I_i$ we have $\lfloor r^{k_i} \rfloor = n_i$. Say $I_{j-1} = [a,b]$ where $1 < n_1 \leq a < b \leq n_1 + \frac{1}{2}$. Since $1 < a < b$ we have $\left(\frac{b}{a}\right)^k \to \infty$ as $k \to \infty$. Choose m large enough that for all $k \geq m$ we have $\left(\frac{b}{a}\right)^k \geq a + \frac{1}{2}$. Then for $k \geq m$ we have $b^k \geq a^{k+1} + \frac{1}{2}a^k \geq a^{k+1} + \frac{1}{2}$ so that the intervals $\left[a^k, b^k - \frac{1}{2}\right]$ and $\left[a^{k+1}, b^{k+1} - \frac{1}{2}\right]$ overlap, and we have $\bigcup_{k=m}^{\infty} \left[a^k, b^k - \frac{1}{2}\right] = [a^m, \infty)$. Choose n_j so that $n_j > n_{j-1}$, $n_j \geq a^m$ and $|a_{n_j}| > \epsilon$. Choose k_j so that $n_j \in \left[a^{k_j}, b^{k_j} - \frac{1}{2}\right]$ and note that $\left[n_j, n_j + \frac{1}{2}\right] \subseteq \left[a^{k_j}, b^{k_j}\right]$. Let $I_j = \left[{n_j}^{1/k_j}, (n_j + \frac{1}{2})^{1/k_j}\right]$ and note that for all $r \in I_j$ we have $r^{k_j} \in \left[n_j, n_j + \frac{1}{2}\right]$ so that $\lfloor r^{k_j} \rfloor = n_j$.

We can now obtain the desired contradiction as follows. Since the nested intervals $I_1 \supset I_2 \supset I_3 \supset \cdots$ are nonempty, closed and bounded, their intersection is nonempty. Choose $r \in \bigcap_{j=1}^{\infty} I_j$. For each index j, since $r \in I_j$ we have $\lfloor r^{k_j} \rfloor = n_j$. Since $\lim_{k\to\infty} a_{\lfloor r^k \rfloor} = 0$ it follows that $\lim_{j\to\infty} a_{\lfloor r^{k_j} \rfloor} = 0$ hence $\lim_{j\to\infty} a_{n_j} = 0$. But this is impossible since $|a_{n_j}| > \epsilon$ for all j.

6: Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be bijective. Suppose that f maps connected sets to connected sets and that f maps disconnected sets to disconnected sets. Prove that f and f^{-1} are both continuous.

Solution: Let $g = f^{-1} : \mathbf{R}^n \to \mathbf{R}^n$. Note that g sends connected sets to connected sets because for a connected set $B \subset \mathbf{R}^n$, if $g(B)$ was disconnected then $f(g(B))$ would be disconnected (since f sends disconnected sets to disconnected sets) but $f(g(B)) = B$, which is connected. Similarly, g sends disconnected sets to disconnected sets. Since f and g satisfy the same hypotheses, it suffices to show that f is continuous.

To show that f is continuous, we shall show that $f^{-1}(B)$ is open for every open ball B in $\mathbf{R}^n$. Let B be an open ball in $\mathbf{R}^n$. Let $A = f^{-1}(B) = g(B)$. We need to show that A is open, or equivalently, that $\mathbf{R}^n \setminus A$ is closed. Suppose, for a contradiction, that $\mathbf{R}^n \setminus A$ is not closed. Then $\mathbf{R}^n \setminus A$ is not equal to its closure $\overline{\mathbf{R}^n \setminus A}$. Choose $a \in \overline{\mathbf{R}^n \setminus A}$ with $a \notin \mathbf{R}^n \setminus A$, that is $a \in A$. Let $b = f(a) \in B$. Note that $\{b\} \cup \mathbf{R}^n \setminus B$ is disconnected since B is an open ball in $\mathbf{R}^n$ and $b \in B$. Since g sends disconnected sets to disconnected sets, the set $\{a\} \cup \mathbf{R}^n \setminus A = g\big(\{b\} \cup \mathbf{R}^n \setminus B\big)$ is disconnected. On the other hand, since $\mathbf{R}^n \setminus B$ is connected (because B is a ball in $\mathbf{R}^n$) and g sends connected sets to connected sets, it follows that the set $\mathbf{R}^n \setminus A = g\big(\mathbf{R}^n \setminus B\big)$ is connected, and since $a \in \overline{\mathbf{R}^n \setminus A}$ it follows that $\{a\} \cup \mathbf{R}^n \setminus A$ is connected (here we used the fact that for any set $C \subseteq \mathbf{R}^n$, if C is connected and $c \in \overline{C}$ then $\{c\} \cup C$ is connected). We have shown that the set $\{a\} \cup \mathbf{R}^n \setminus A$ is both connected and disconnected, which is impossible.