Solutions to the Special K Problems, 2015

1: Let $x_0 = -1$, $x_1 = 3$ and $x_n = 2x_{n-1} + x_{n-2}$ for $n \geq 2$. Find the product $x_{n-2}x_{n-1}x_n$ where n is the largest integer with $n \geq 2$ for which x_{n-2}, x_{n-1} and x_n are all prime.

Solution: Note that $x_2 = 5$, $x_3 = 13$ and $x_4 = 31$, which are all prime. Since $x_0 < x_1$ and whenever $x_{n-2} < x_{n-1}$ we also have $x_n = 2x_{n-1} + x_{n-2} = x_{n-1} + (x_{n-1} - x_{n-2}) > x_{n-1}$, it follows, by induction, that the sequence $\{x_n\}$ is increasing. Since $x_2 = 5 = 0 \bmod 5$ and whenever $x_n = 0 \bmod 5$ we also have

$$x_{n+3} = 2x_{n+2} + x_{n+1} = 2\big(2x_{n+1} + x_n\big) + x_{n+1} = 5x_{n+1} + 2x_n = 0 \bmod 5,$$

it follows, by induction, that $x_{2+3k} = 0 \bmod 5$ for all $k \geq 0$. For all $k \geq 1$, since x_{2+3k} is a multiple of 5 and $x_{2+3k} > x_2 = 5$, it follows that x_{2+3k} is composite. Thus the largest integer n for which x_{n-2}, x_{n-1} and x_n are all prime is $n = 4$, and we have $x_{n-2}x_{n-1}x_n = 5 \cdot 13 \cdot 31 = 2015$.

2: Let $f : [0,1] \to [0,1]$ be increasing and convex with $f(0) = 0$ and $f(1) = 1$ (f is *convex* means that for all $0 \leq a < b \leq 1$, the line segment from $(a, f(a))$ to $(b, f(b))$ lies on or above the graph of $y = f(x)$ for $a \leq x \leq b$). Show that $f(x)f^{-1}(x) \leq x^2$ for all $x \in [0,1]$.

Solution: We claim that $f(a)f^{-1}(a) \leq a^2$ for all $a \in [0,1]$. Since $f(0) = 0$ we have $f^{-1}(0) = 0$ so the claim is true when $a = 0$. Let $a \in (0,1]$ and note that $f(a) \in (0,1]$. Since f is convex, the graph of f lies on or below the line from $(0,0)$ to $(1,1)$, which has equation $y = x$, and so we have $f(t) \leq t$ for all $t \in [0,1]$. In particular, letting $t = f^{-1}(a)$ we obtain $a \leq f^{-1}(a)$. Since f is convex, the graph of f lies on or below the line from $(0,0)$ to $\big(f^{-1}(a), a\big)$, which has equation $y = \frac{a}{f^{-1}(a)}\,x$, and so we have $f(t) \leq \frac{a}{f^{-1}(a)}\,t$ for all $t \in [0, f^{-1}(a)]$. In particular, letting $t = a$ we obtain $f(a) \leq \frac{a}{f^{-1}(a)}\,a$ and so $f(a)f^{-1}(a) \leq a^2$, as claimed.

3: For a positive integer n, let $\tau(n)$ be the number of positive divisors of n and let $\sigma(n)$ be the sum of the positive divisors of n. Show that for all integers $n \geq 2$ we have $\dfrac{\sigma(n)}{\tau(n)} \leq \dfrac{n+1}{2}$ with equality if and only if n is prime.

Solution: When n is prime, the positive divisors of n are 1 and n so we have $\tau(n) = 2$ and $\sigma(n) = n+1$ so that $\frac{\sigma(n)}{\tau(n)} = \frac{n+1}{2}$. Suppose that n is composite. Then n has at least one divisor d with $1 < d < n$ and so $\tau(n) \geq 3$. For each divisor d of n with $1 < d < n$, we also have $1 < \frac{n}{d} < n$ so that $\frac{n}{d} \geq 2$ and hence $d \leq \frac{n}{2}$. Let $\ell = \tau(n) \geq 3$ and let the positive divisors of n be $1 = d_1 < d_2 < \cdots < d_{\ell-1} < d_\ell = n$. Then

$$\sigma(n) = n + 1 + d_2 + \cdots + d_{l-1} \leq n + 1 + (\ell-2)\tfrac{n}{2} < n + 1 + (\ell-2)\tfrac{n+1}{2} = \ell\,\tfrac{n+1}{2} = \tau(n)\,\tfrac{n+1}{2}$$

and so $\frac{\sigma(n)}{\tau(n)} < \frac{n+1}{2}$.

4: Triangle ABC has a right angle at B. The angle bisector at A meets BC at D and the angle bisector at C meets AB at E. Given that $AD = 9$ and $CE = 8\sqrt{2}$, find AC.

Solution: Let $\alpha = \angle BAD = \angle CAD$ and let $\gamma = \angle BCE = \angle ACE$. Since $\pi = \angle A + \angle B + \angle C = 2\alpha + \frac{\pi}{2} + 2\gamma$, we have $\alpha + \gamma = \frac{\pi}{4}$. Note that

$$AB = AD\cos\alpha = 9\cos\alpha \quad\text{and}\quad BC = CE\cos\gamma = 8\sqrt{2}\cos\left(\alpha - \tfrac{\pi}{4}\right) = 8(\cos\alpha - \sin\alpha).$$

and that $\tan 2\alpha = \frac{BC}{AB}$ so we have

$$\frac{2\tan\alpha}{1-\tan^2\alpha} = \tan 2\alpha = \frac{BC}{AB} = \frac{8(\cos\alpha + \sin\alpha)}{9\cos\alpha} = \frac{8(1+\tan\alpha)}{9}$$
$$9\tan\alpha = 4(1+\tan\alpha)(1-\tan^2\alpha) = 4(1 + \tan\alpha - \tan^2\alpha - \tan^3\alpha)$$
$$0 = 4\tan^3\alpha + 4\tan^2\alpha + 5\tan\alpha - 4 = (2\tan\alpha - 1)(2\tan^2\alpha + 3\tan\alpha + 4)$$

and so $\tan\alpha = \frac{1}{2}$. Since $\tan\alpha = \frac{1}{2}$ we have $\cos\alpha = \frac{2}{\sqrt{5}}$ and $\cos 2\alpha = 2\cos^2\alpha - 1 = \frac{2\cdot 4}{5} - 1 = \frac{3}{5}$ and so

$$AC = \frac{AB}{\cos 2\alpha} = \frac{9\cos\alpha}{\cos 2\alpha} = \frac{9 \cdot \frac{2}{\sqrt{5}}}{\frac{3}{5}} = 6\sqrt{5}.$$

5: Let $f_1(x) = x^2 - 1$ and let $f_{n+1}(x) = f_1\big(f_n(x)\big)$ for $n \geq 1$. For each positive integer n, find the number of distinct real roots of the polynomial $f_n(x)$.

Solution: We shall show that $f_n(x)$ has exactly $n+1$ distinct real roots. We use induction to prove the stronger claim that for all $n \geq 1$ and all $c \in \mathbf{R}$, the equation $f_n(x) = c$ has exactly 2 (distinct real) solutions when $c > 0$, exactly $n+1$ solutions when $c = 0$, exactly n solutions when $c = -1$, and no solutions when $c < -1$. The claim is true when $n = 1$ because we have $f_1(x) = c \Longleftrightarrow x^2 - 1 = c \Longleftrightarrow x^2 = 1 + c$ and this equation has 2 solutions when $c > -1$, 1 solution when $c = -1$ and no solutions when $c < -1$. Let $n \in \mathbf{Z}$ with $n \geq 1$. Suppose, inductively, that the equation $f_n(x) = c$ has exactly 2 solutions when $c > 0$, exactly $n+1$ solutions when $c = 0$, exactly n solutions when $c = -1$ and no solutions when $c < -1$. We have $f_{n+1}(x) = f_1\big(f_n(x)\big) = f_n(x)^2 - 1$ and so

$$f_{n+1}(x) = c \Longleftrightarrow f_n(x)^2 - 1 = c \Longleftrightarrow f_n(x)^2 = 1 + c.$$

When $c > 0$, we have $f_{n+1}(x) = c \Longleftrightarrow f_n(x) = \pm\sqrt{1+c}$; the equation $f_n(x) = \sqrt{1+c}$ has exactly 2 solutions (since $\sqrt{1+c} > 0$) and the equation $f_n(x) = -\sqrt{1+c}$ has no solutions (since $-\sqrt{1+c} < -1$) and so the equation $f_{n+1}(x) = c$ has exactly 2 solutions. When $c = 0$, we have $f_{n+1}(x) = c \Longleftrightarrow f_n(x) = \pm\sqrt{1+c} = \pm 1$; the equation $f_n(x) = 1$ has exactly 2 solutions and the equation $f_n(x) = -1$ has exactly n solutions and so the equation $f_{n+1}(x) = c$ has exactly $n+2$ solutions. When $c = -1$ we have $f_{n+1}(x) = c \Longleftrightarrow f_n(x)^2 = 1 + c = 0 \Longleftrightarrow f_n(x) = 0$ and this equation has exactly $n+1$ solutions. When $c < -1$ we have $f_{n+1}(x) = c \Longleftrightarrow f_n(x)^2 = 1 + c$ and this equation has no solutions (since $1 + c < 0$).

6: Let $\mathbf{Z}^+$ be the set of natural numbers. Let S be a set of subsets of $\mathbf{Z}^+$ and let $n \in \mathbf{Z}^+$. Suppose that for all distinct sets $A, B \in S$, the intersection $A \cap B$ has at most n elements. Show that S is finite or countable.

Solution: Let $\mathbf{N}$ be the set of natural numbers. Define a map $F : S \to \mathbf{N}^{n+1}$ as follows. Given $A \in S$, if $A = \emptyset$ then we define $\phi(A) = (0, 0, \cdots, 0) \in \mathbf{N}^{n+1}$, if A is finite with $A = \{a_1, a_2, \cdots, a_k\}$ where $a_1 < a_2 < \cdots < a_k$ and $k \leq n$ then we define $F(A) = (a_1, a_2, \cdots, a_k, 0, 0, \cdots, 0)$, if A is finite with $A = \{a_1, a_2, \cdots, a_k\}$ where $a_1 < a_2 < \cdots < a_k$ and $k \geq n+1$ then we define $F(A) = (a_1, a_2, \cdots, a_{n+1})$, and if A is countable with $A = \{a_1, a_2, a_3, \cdots\}$ where $a_1 < a_2 < a_3 < \cdots$ then we define $F(A) = (a_1, a_2, \cdots, a_{n+1})$. Because two distinct sets A and B in S have at most n points in common, it follows that the map F is injective. Since F is injective and $\mathbf{N}^{n+1}$ is countable, it follows that S is finite or countable.

BIG E
Saturday November 7, 2015
10:00 am - 1:00 pm

1: Let $x_0 = 1$ and $x_1 = 2$, and for $n \geq 1$ let $x_{2n} = x_{2n-1} + 2x_{2n-2}$ and $x_{2n+1} = 2x_{2n} - 3x_{2n-1}$. Find a closed form formula for x_{2n} and x_{2n+1}.

Solution: Note that $x_{2n+1} = 2x_{2n} - 3x_{2n-1} = 2(x_{2n-1} + 2x_{2n-2}) - 3x_{2n-1} = -x_{2n-1} + 4x_{2n-2}$ so we have

$$\begin{pmatrix} x_{2n} \\ x_{2n+1} \end{pmatrix} = \begin{pmatrix} 2x_{2n-2} + x_{2n-1} \\ 4x_{2n-2} - x_{2n-1} \end{pmatrix} = A \begin{pmatrix} x_{2n-2} \\ x_{2n-1} \end{pmatrix} \quad \text{where} \quad A = \begin{pmatrix} 2 & 1 \\ 4 & -1 \end{pmatrix}.$$

It follows, inductively, that

$$\begin{pmatrix} x_{2n} \\ x_{2n+1} \end{pmatrix} = A \begin{pmatrix} x_{2n-2} \\ x_{2n-1} \end{pmatrix} = A^2 \begin{pmatrix} x_{2n-4} \\ x_{2n-3} \end{pmatrix} = \cdots = A^n \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}.$$

Let us diagonalize A. The characteristic polynomial is

$$f_A(x) = \det(A - xI) = \det \begin{pmatrix} 2-x & 1 \\ 4 & -1-x \end{pmatrix} = x^2 - x - 6 = (x-3)(x+2)$$

so the eigenvalues are $\lambda = 3$ and $\mu = -2$. We have

$$A - 3I = \begin{pmatrix} -1 & 1 \\ 4 & -4 \end{pmatrix} \sim \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad A + 2I = \begin{pmatrix} 4 & 1 \\ 4 & 1 \end{pmatrix} \sim \begin{pmatrix} 4 & 1 \\ 0 & 0 \end{pmatrix}$$

and so $u = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector for 3 and $v = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$ is an eigenvector for -2. Thus $P^{-1}AP = D$ where

$$P = (u, v) = \begin{pmatrix} 1 & -1 \\ 1 & 4 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}.$$

Thus we have

$$\begin{aligned} \begin{pmatrix} x_{2n} \\ x_{2n+1} \end{pmatrix} &= A^n \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = (PDP^{-1})^n \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = PD^nP^{-1} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -1 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 3^n & 0 \\ 0 & (-2)^n \end{pmatrix} \cdot \tfrac{1}{5} \begin{pmatrix} 4 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ &= \tfrac{1}{5} \begin{pmatrix} 3^n & -(-2)^n \\ 3^n & 4(-2)^n \end{pmatrix} \begin{pmatrix} 6 \\ 1 \end{pmatrix} = \tfrac{1}{5} \begin{pmatrix} 6 \cdot 3^n - (-2)^n \\ 6 \cdot 3^n + 4(-2)^n \end{pmatrix}. \end{aligned}$$

2: Let n be a positive integer. Find the smallest positive integer d such that $d = \det(A)$ for some $n \times n$ matrix whose entries all lie in $\{\pm 1\}$.

Solution: We shall show that the smallest such integer d is equal to 2^{n-1}. We claim that for every $n \times n$ matrix A with entries in $\{\pm 1\}$, the determinant of A is a multiple of 2^{n-1}. Let A be any $n \times n$ matrix with entries in $\{\pm 1\}$. Perform the row operations $R_k \mapsto R_k + R_1$ for each $k > 1$ to obtain a matrix B. The first row of B has entries in $\{\pm 1\}$, and all other rows of B have entries in $\{-2, 0, 2\}$. Perform the row operations $R_k \mapsto \frac{1}{2} R_k$ for each $k > 1$ to obtain a matrix C with entries in $\{-1, 0, 1\}$. Then we have $\det(A) = \det(B) = 2^{n-1} \det(C)$ and so $\det(A)$ is a multiple of 2^{n-1}, as claimed.

For each $n \in \mathbf{Z}^+$, let A_n be the $n \times n$ matrix with entries $a_{k,l} = 1$ when $k \leq l$ and $a_{kl} = -1$ when $k > l$. Note that $\det(A_1) = \det(1) = 1 = 2^0$. Let $n \geq 2$ and suppose, inductively that $\det(A_{n-1}) = 2^{n-2}$. Note that when we perform the row operation $R_1 \mapsto R_1 - R_2$ on the matrix A_n we obtain the matrix

$$B_n = \begin{pmatrix} 2 & 0 & \cdots & 0 \\ -1 & & & \\ \vdots & & A_{n-1} & \\ -1 & & & \end{pmatrix}.$$

Thus we have $\det(A_n) = \det(B_n) = 2 \det(A_{n-1}) = 2^{n-1}$. By induction, we have $\det(A_n) = 2^{n-1}$ for all $n \geq 1$. Thus $d = 2^{n-1}$ is the smallest positive integer with the desired property.

3: Let $0 < a_n \in \mathbf{R}$ for all integers $n \geq 1$. Let $b_1 = 1$, and let $b_{n+1} = b_n + \dfrac{a_n}{b_n}$ for all $n \geq 1$. Show that $\sum a_n$ converges if and only if $\{b_n\}$ converges.

Solution: If we suppose, inductively, that $1 = b_1 < b_2 < \cdots < b_n$ then we have $b_{n+1} = b_n + \frac{a_n}{b_n} \leq b_n + a_n < b_n$, and it follows that the sequence $\{b_n\}$ is increasing with each $b_n \geq 1$. Thus the sequence $\{b_n\}$ converges if and only if it is bounded above.

Let s_n denote the n^{th} partial sum $s_n = \sum\limits_{k=1}^{n} a_k$. Since each $a_k > 0$, the sequence of partial sums $\{s_n\}$ is increasing, and hence it converges if and only if it is bounded above.

Suppose that $\sum a_n$ converges, say $\sum\limits_{k=1}^{\infty} a_k = S$. Since the sequence of partial sums $\{s_n\}$ is increasing, we have $s_n \leq S$ for all n. Since $b_{n+1} = b_n + \frac{a_n}{b_n}$ we have $a_n = b_n(b_{n+1} - b_n)$ for all $n \geq 1$ and so

$$s_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} b_k(b_{k+1} - b_k) \geq \sum_{k=1}^{n} (b_{k+1} - b_k) = b_{n+1} - b_1 = b_{n+1} - 1.$$

Thus $b_{n+1} \leq s_n + 1 \leq S + 1$ for all $n \geq 1$ so the sequence $\{b_n\}$ is bounded above (by $S+1$), so it converges.

Conversely, suppose that $\{b_n\}$ converges, say $b_n \to B$. Since $\{b_n\}$ is increasing we have $b_n \leq B$ for all n. Since $a_n = b_n(b_{n+1} - b_n)$ we have

$$s_n = \sum_{k=1}^{n} a_k = \sum_{k=1}^{n} b_k(b_{k+1} - b_k) \leq \sum_{k=1}^{n} B(b_{k+1} - b_k) = B(b_{n+1} - 1) \leq B(B-1).$$

Since the sequence $\{s_n\}$ is increasing and bounded above (by $B(B-1)$), it converges.

4: Let G be a group. Suppose the map $\phi : G \to G$ given by $\phi(x) = x^3$ is an injective group homomorphism. Show that G is abelian.

Solution: Let $x, y \in G$. For any $a, b \in G$ we have $a^3b^3 = \phi(a)\phi(b) = \phi(ab) = (ab)^3 = a(ba)^2b$, and canceling a from the left and b from the right gives $a^2b^2 = (ba)^2$. Taking $a = x^2$ and $b = y^2$ gives $x^4y^4 = (y^2x^2)^2$ and taking $a = y$ and $b = x$ gives $y^2x^2 = (xy)^2$, and so we have $x^4y^4 = (y^2x^2)^2 = \big((xy)^2\big)^2 = (xy)^4 = x(yx)^3y$. Canceling x from the left and y from the right gives $x^3y^3 = (yx)^3$. Since the homomorphism ϕ is injective with $\phi(xy) = \phi(x)\phi(y) = x^3y^3 = (yx)^3 = \phi(yx)$, it follows that $xy = yx$.

5: For a positive integer n, let $\pi(n)$ be the product of the positive divisors of n. Show that for all positive integers n and m, if $\pi(n) = \pi(m)$ then $n = m$.

Solution: First note that for a positive integer n, each divisor d of n can be paired with the divisor $\frac{n}{d}$ so we have

$$\pi(n)^2 = \Big(\prod_{d|n} d\Big)^2 = \prod_{d|n} d \cdot \frac{n}{d} = \prod_{d|n} n = n^{\tau(n)} \quad \text{so that} \quad \pi(n) = n^{\tau(n)/2},$$

where $\tau(n)$ denotes the number of positive divisors on n. For a positive integer n and a prime p, let $e_p(n)$ denote the exponent of p in the prime factorization of n, and note that for $k \in \mathbf{Z}^+$ we have $e_p(n^k) = k\, e_p(n)$.

Let n and m be positive integers, and suppose that $\pi(n) = \pi(m)$. Then for all primes p we have

$$\tau(n)\, e_p(n) = e_p\big(n^{\tau(n)}\big) = e_p\big(\pi(n)^2\big) = e_p\big(\pi(m)^2\big) = e_p\big(m^{\tau(m)}\big) = \tau(m)\, e_p(m).$$

Let $c = \tau(m)/\tau(n)$. Then for all primes p we have $e_p(n) = c\, e_p(m)$. If we had $c < 1$ then we would have $e_p(n) < e_p(m)$ for all primes p which would imply that $n < m$ and $\tau(n) < \tau(m)$ hence $\pi(n) < \pi(m)$. Similarly, if we had $c > 1$ then we would have $\pi(n) > \pi(m)$. Thus we must have $c = 1$ and so $e_p(n) = e_p(m)$ for every prime p, and so $n = m$.

6: Find $\displaystyle\int_0^\infty \frac{\big|\cos(\pi x)\big|}{4x^2-1}\,dx$.

Solution: Note that the integrand is not defined when $x=\frac{1}{2}$, but it is continuous from the left and continuous from the right with

$$\lim_{x\to\frac{1}{2}^-}\frac{|\cos\pi x|}{4x^2-1}=\lim_{x\to\frac{1}{2}^-}\frac{\cos\pi x}{4x^2-1}=\lim_{x\to\frac{1}{2}^-}\frac{-\pi\sin\pi x}{8x}=-\frac{\pi}{4}$$
$$\lim_{x\to\frac{1}{2}^+}\frac{|\cos\pi x|}{4x^2-1}=\lim_{x\to\frac{1}{2}^+}\frac{-\cos\pi x}{4x^2-1}=\lim_{x\to\frac{1}{2}^+}\frac{\pi\sin\pi x}{8x}=\frac{\pi}{4}.$$

For each integer $k\geq 0$ we have

$$\int_k^{k+1}\frac{|\cos\pi x|}{4x^2-1}\,dx=\frac{1}{4}\int_k^{k+1}\frac{|\cos\pi x|}{x-\frac{1}{2}}-\frac{|\cos\pi x|}{x+\frac{1}{2}}\,dx=\frac{1}{4}\left(\int_{k-\frac{1}{2}}^{k+\frac{1}{2}}\frac{|\sin\pi t|}{t}\,dt-\int_{k+\frac{1}{2}}^{k+\frac{3}{2}}\frac{|\sin\pi t|}{t}\,dt\right)$$

and so for each integer $n\geq 1$ we have

$$\begin{aligned}\int_0^n\frac{|\cos\pi x|}{4x^2-1}\,dx&=\sum_{k=0}^{n-1}\int_k^{k+1}\frac{|\cos\pi x|}{4x^2-1}\,dx=\frac{1}{4}\sum_{k=0}^{n-1}\left(\int_{k-\frac{1}{2}}^{k+\frac{1}{2}}\frac{|\sin\pi t|}{t}\,dt-\int_{k+\frac{1}{2}}^{k+\frac{3}{2}}\frac{|\sin\pi t|}{t}\,dt\right)\\&=\frac{1}{4}\left(\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{|\sin\pi t|}{t}\,dt-\int_{n-\frac{1}{2}}^{n+\frac{1}{2}}\frac{|\sin\pi t|}{t}\,dt\right)=-\frac{1}{4}\int_{n-\frac{1}{2}}^{n+\frac{1}{2}}\frac{|\sin\pi t|}{t}\,dt\end{aligned}$$

since $\displaystyle\int_{-1/2}^{1/2}\frac{|\sin\pi t|}{t}\,dt=0$. Thus for each integer $n\geq 1$,

$$\left|\int_0^n\frac{|\cos\pi x|}{4x^2-1}\,dx\right|=\frac{1}{4}\int_{n-\frac{1}{2}}^{n+\frac{1}{2}}\frac{|\sin\pi t|}{t}\,dt\leq\frac{1}{4}\int_{n-\frac{1}{2}}^{n+\frac{1}{2}}\frac{1}{n-\frac{1}{2}}\,dt=\frac{1}{4n-2}.$$

Since $\displaystyle\left|\int_0^n\frac{|\cos\pi x|}{4x^2-1}\,dx\right|=\frac{1}{4n-2}\longrightarrow 0$ as $n\to\infty$, it follows that $\displaystyle\int_0^\infty\frac{|\cos\pi x|}{4x^2-1}\,dx=0$.