Week 1: Assorted Problems

1: Solve $\left|\left|\left||x-8|-4\right|-2\right|-1\right| = \frac{1}{2}$.

2: Let $a_0 = 0$ and $a_{n+1} = 1 + a_n + \sqrt{1+4a_n}$ for $n \geq 0$. Find a formula for a_n in terms of n.

3: Let A_n be the number of binary sequences $(x_1 x_2 \cdots x_{2n})$ of length $2n$ such that for $1 \leq k$ we have $(x_{2k-1}, x_{2k}) \neq (0,0)$ and $x_{2k} = x_{2k+1}$. Find A_{10}.

4: Let $S = \{x = (x_1, x_2, x_3, \cdots) \big|\ \text{each } x_k \in \mathbf{Z}^+\}$. Define functions $A, B, C, L : S \to S$ by $A(x) = a = (a_1, a_2, \cdots)$ with $a_n = \sum_{k=1}^{n} x_k$, $B(x) = b = (b_1, b_2, \cdots)$ with $b_n = x_n + 1$, $C(x) = c = (c_1, c_2, \cdots)$ with $c_1 = 1$ and $c_n = x_{n-1}$ for $n \geq 2$, and $L(x) = C(B(A(x)))$. Find $\lim_{n\to\infty} L^n(x)$ when $x = (1, 1, 1, \cdots)$.

5: Show that an 8×8 grid, with the 4 corner squares removed, cannot be tiled using L-shaped tiles which each cover 4 unit squares.

6: Show that the numbers $\sqrt{2}$, $\sqrt{3}$ and $\sqrt{4}$ cannot all be terms in the same geometric sequence.

7: Let $f : \mathbf{R} \to \mathbf{R}$. Suppose that $\left| \sum_{k=1}^{n} 3^k \big(f(x+ky) - f(x-ky) \big) \right| \leq 1$ for every $n \in \mathbf{Z}^+$ and every $x, y \in \mathbf{R}$. Show that f must be constant.

8: Evaluate $\tan(20°)\tan(40°)\tan(60°)\tan(80°)$.

9: Evaluate the sum $\sum_{n=1}^{\infty} \left(\frac{1}{3k-2} + \frac{1}{3k-1} - \frac{2}{3k} \right) = \frac{1}{1} + \frac{1}{2} - \frac{2}{3} + \frac{1}{4} + \frac{1}{5} - \frac{2}{6} + \cdots$.

10: Show that $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \sin(\log n)}{n^r}$ converges for every $r > 0$.

11: Let $A \in M_n(\mathbf{C})$ and let $a \in \mathbf{C}$ with $a \neq 0$. Suppose that $A - A^* = 2aI$.

(a) Show that $|\det A| \geq |a|^n$.

(b) Show that if $|\det A| = |a|^n$ then $A = aI$.

12: Let $f : \mathbf{R} \to (0, \infty)$ be increasing and differentiable with f' bounded and $\lim_{x\to\infty} f(x) = \infty$. Let $F(x) = \displaystyle\int_0^x f(t)\,dt$. Let $a_0 = 1$ and $a_{n+1} = a_n + \frac{1}{f(a_n)}$ for $n \geq 0$, and let $b_n = F^{-1}(n)$ for $n \geq 0$. Prove that $\lim_{n\to\infty} (a_n - b_n) = 0$.