Week 2: Assorted Problems

1: Let n and b be positive integers with $b \geq 6$. Suppose that, in base b, we have $n = 1254_b$ and $2n = 2541_b$. Find n and b (in base 10).

2: Let $x, y, z \in \mathbf{R}$. Suppose that $x + y + z = 1$, $x^2 + y^2 + z^2 = 2$ and $x^3 + y^3 + z^3 = 3$. Find $x^4 + y^4 + z^4$.

3: Let $U = \big\{(x, y) \in \mathbf{R}^2 \big| x^2 + y^2 < 1\,,\ y \neq x\big\}$ and let $f(x, y) = \ln(1 - x^2 - y^2) - \dfrac{1}{(y - x)^2}$. Find the maximum value of $f(x, y)$ for $(x, y) \in U$.

4: Find all positive integers $n < 200$ such that $n^2 + (n + 1)^2$ is a square.

5: Four points A, B, C and D are chosen at random on the unit circle $x^2 + y^2 = 1$. Find the probability that the line segments AB and CD intersect.

6: Let ABC be a triangle with the sides opposite A, B and C of length a, b and c. For $0 \leq x \leq c$, let $P = P(x)$ be the point on the line segment AB whose distance from A is equal to x, and let $\theta = \theta(x)$ be the angle $\theta = \angle APC$. Show that $\displaystyle\int_0^c \cos\theta(x)\; dx = a - b$.

7: Let $a_1 = 1$ and for $n \geq 2$ let $a_n = \sum\limits_{k=1}^{n-1} a_k a_{n-k}$. Find $\sum\limits_{n=1}^{\infty} a_n \left(\frac{2}{9}\right)^n$.

8: Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $a, b, c, d \in \mathbf{R}$ with $a^2 + b^2 + c^2 + d^2 < \frac{1}{5}$. Show that $I + A$ is invertible.

9: Let $A \in M_n(\mathbf{R})$ with $A^3 = A + I$. Show that $\det A > 0$.

10: Find $\displaystyle\lim_{n\to\infty} \sqrt{1 + 2\sqrt{1 + 3\sqrt{1 + 4\sqrt{\cdots + (n-1)\sqrt{1 + n}}}}}\,.$

11: Let $a, b \in \mathbf{Z}^+$. Suppose that a and b are not both multiples of 10. Show that there are infinitely many palindromes (in base 10) of the form $a + kb$ with $k \in \mathbf{Z}^+$.

12: (a) Show that $\displaystyle\int_0^\infty \ln\left(1 + \tfrac{1}{x^2}\right) \tan^{-1} x\; dx = \tfrac{\pi^2}{6}$.

(b) Let $I_n = \displaystyle\int_0^\infty \frac{\tan^{-1} x}{(1 + x^2)^n}\, dx$. Show that $\sum\limits_{n=1}^{\infty} \frac{1}{n} I_n = \frac{\pi^2}{6}$.