Week 3: Assorted Problems

1: Find every prime number which has an even number of digits and is palindromic.

2: Show that there does not exist a 4-digit palindromic square.

3: Let ℓ, m and n be the slopes of the 3 sides of an equilateral triangle in $\mathbf{R}^2$. Show that $\ell m + mn + n\ell = -3$.

4: In triangle ABC, let D, E and F be the points on the sides BC, CA and AB such that AD, BE and CF are the internal angle bisectors at A, B and C. Show that

$$\frac{\cos(A/2)}{AD} + \frac{\cos(B/2)}{BE} + \frac{\cos(C/2)}{CF} = \frac{1}{BC} + \frac{1}{CA} + \frac{1}{AB}.$$

5: Show that given any group of people, it is possible to separate the people into two rooms in such a way that for every person in the group, at least half of that person's friends are in the other room.

6: Evaluate $\dfrac{1 + \frac{1}{2^3} + \frac{1}{3^3} + \frac{1}{4^3} + \cdots}{1 - \frac{1}{2^3} + \frac{1}{3^3} - \frac{1}{4^3} + \cdots}$.

7: Find $\displaystyle\int_0^{\pi/3} \frac{dx}{5 - 4\cos x}$.

8: A large floor is tiled with unit squares. A small square with sides of length ℓ is tossed, at random, onto the floor. Find the probability that the square lands entirely within one of the unit square tiles.

9: Let $a_1, a_2, \cdots, a_n$ be distinct real numbers, and let $f(x) = \prod_{k=1}^{n} (x - a_k)$. For $1 \leq k \leq n$, let $g_k(x) = \dfrac{f(x)}{f'(a_k)(x - a_k)}$. Show that $f'(x) = \sum_{k=1}^{n} f'(a_k) g_k(x) = \sum_{k=1}^{n} f'(a_k) g_k(x)^2$.

10: Let $f : [0,1] \to \mathbf{R}$ be continuous. Define $f_n : [0,1] \to \mathbf{R}$ recursively by $f_0(x) = f(x)$ and $f_{n+1}(x) = \displaystyle\int_0^x f_n(t)\, dt$ for $n \geq 0$. Suppose that $f_n(1) = 0$ for all $n \geq 0$. Show that $f(x) = 0$ for all x.

11: Let A, B and C be nonempty sets in $\mathbf{R}^n$. Suppose that A is bounded, C is closed and convex, and $A + B \subseteq A + C$. Show that $B \subseteq C$.

12: Let $2 \leq n \in \mathbf{Z}$ and let $A, B, C, D \in M_n(\mathbf{C})$. Suppose that $AC - BD = I$ and $AD + BC = O$.
(a) Show that $CA - DB = I$ and $DA + CB = O$.
(b) Show that $\det(AC) \geq 0$ and $(-1)^n \det(BD) \geq 0$.