Week 4: Assorted Problems

1: Let $n, m \in \mathbf{Z}^+$ with $m \leq 10^n$. Show that the sum of the digits of the number $(10^n - 1)m$ is equal to $9n$.

2: Find the number of binary sequences of length 12 in which no 3 consecutive terms are equal.

3: Show that for all $n \in \mathbf{Z}^+$, there exist $a, b \in \mathbf{Z}$ such that $2n + 1 = a^2$ and $3n + 1 = b^2$ if and only if there exist $r, s \in \mathbf{Z}$ such that $n + 1 = r^2 + (r + 1)^2 = s^2 + 2(s + 1)^2$.

4: A disc of radius 1 is initially centred at $(1, 0)$. The disc rolls, without slipping, once around the inside of the circle of radius 2 centered at $(0, 0)$. Find the length of, and the area inside, the curve followed by the point on the disc which is initially at position $\left(\frac{1}{2}, 0\right)$.

5: Let $f_1, f_2, \cdots, f_n : \mathbf{R} \to \mathbf{R}$ be differentiable. Show that if $\{f_1, f_2, \cdots, f_n\}$ is linearly independent over $\mathbf{R}$ then $\dim\left(\mathrm{Span}_{\mathbf{R}}\{f_1{}', f_2{}', \cdots, f_n{}'\}\right) \geq n - 1$.

6: Find $\displaystyle\int_{x=0}^{1} \int_{y=\sqrt{x-x^2}}^{\sqrt{1-x^2}} y\, e^{x^4+2x^2y^2+y^4}\, dy\, dx.$

7: Let $n \in \mathbf{Z}^+$ and let $A \in M_n(\mathbf{R})$. Suppose that $4A^4 + I = 0$. Prove that $\mathrm{trace}(A) \in \mathbf{Z}$.

8: Let $*$ be an associative operation on a finite set S. Show that there is an element $a \in S$ such that $a * a = a$.

9: Let R be a ring with 1 and let $a, b \in R$. Show that if $1 + ab$ is invertible then so is $1 + ba$.

10: (a) Show that for every $A \in M_2(\mathbf{C})$ there exists $X \in M_2(\mathbf{C})$ such that $X^3 = A^2$.
(b) Show that there exists $A \in M_3(\mathbf{C})$ such that for all $X \in M_3(\mathbf{C})$ we have $X^3 \neq A^2$.

11: Let $3 \leq n \in \mathbf{Z}$. Let $a_0 = b_0 = n$ and $a_{k+1} = n^{a_k}$ and $b_{k+1} = b_k!$ for $k \geq 0$. Show that for all $k \geq 2$ we have $b_k < a_k < b_{k+1}$.

12: Let $f : [0, 1] \to \mathbf{R}$ be $\mathcal{C}^2$ and increasing. For $n \in \mathbf{Z}^+$, let U_n and L_n be the upper and lower Riemann sums given by $U_n = \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k}{n}\right)$ and $L_n = \frac{1}{n} \sum_{k=1}^{n} f\left(\frac{k-1}{n}\right)$. Show that for large $n \in \mathbf{Z}^+$ we have

$$\tfrac{1}{3}(2L_n + U_n) \leq \int_0^1 f \leq \tfrac{1}{3}(L_n + 2U_n).$$