Week 5: Assorted Problems

1: Find a positive integer n whose decimal representation is of the form $n = aabbc$, where $a, b, c \in \{1, 2, \cdots, 9\}$ with $b = c + 1$, such that the number n^2 is a 9-digit number which contains each of the digits $1, 2, \cdots, 9$ exactly once.

2: Show that there does not exist a 5-digit perfect square with distinct digits of the same parity.

3: Let $f(x)$ be a polynomial such that $3\frac{d}{dx}\big(xf(x)\big) = 2f(x) + f(x+1)$. Show that f is constant.

4: Let $n \in \mathbf{Z}^+$. For each $k \in \mathbf{Z}^+$ let $S_k = 1^k + 2^k + 3^k + \cdots + n^k$. Show that $S_5 + S_7 = 2{S_3}^2$.

5: Find the maximum and minimum possible values of $\dfrac{n(n+2)(n+3)(n+7)(n+16)}{\operatorname{lcm}\big(n, n+2, n+3, n+7, n+16\big)}$ where $n \in \mathbf{Z}^+$.

6: An $(n-1)$-sphere of radius 1 is centred at each of the $n+1$ vertices of a regular n-simplex in $\mathbf{R}^n$ with edges of length 2 (so each of the $n+1$ spheres is externally tangent to every other sphere). Find the radius of the $(n-1)$-sphere centred at the centre of the simplex which is externally tangent to all of the unit spheres.

7: Let $\{a_n\}_{n\geq 1}$ be a sequence of positive real numbers and let $s_n = \sum\limits_{k=1}^{n} a_k$. Suppose that $\sum\limits_{n=1}^{\infty} a_n = \infty$. Prove that $\sum\limits_{n=1}^{\infty} \dfrac{a_n}{s_n} = \infty$.

8: Find all $z, w \in \mathbf{C}$ with $z \neq w$ such that $z^5 + z = w^5 + w$ and $z^5 + z^2 = w^5 + w^2$.

9: Let $f(x)$ be a polynomial of degree n. Show that

$$\sum_{k=0}^{n} \frac{f^{(k)}(0)}{(k+1)!}\, x^{k+1} = \sum_{k=0}^{n} (-1)^k \frac{f^{(k)}(x)}{(k+1)!}\, x^{k+1}\,.$$

10: Let $f : \mathbf{R} \to \mathbf{R}$ be differentiable with $f(0) = 0$ and $f'(x) = f(x)^2 - x^2$ for all $x \in \mathbf{R}$. Prove that $\lim\limits_{x\to\infty} f'(x)$ exists and find its value.

11: Let $0 < a_n \in \mathbf{R}$ for $n \geq 1$ with $\lim\limits_{n\to\infty} a_n = 0$. Prove that $\lim\limits_{n\to\infty} \dfrac{1}{n} \sum\limits_{k=1}^{n} \ln\left(a_n + \frac{k}{n}\right)$ exists and find its value.

12: Let $a_n = \sum\limits_{k=1}^{n} \frac{1}{k!}$. Find $\sum\limits_{n=k}^{\infty} \dbinom{n}{k}(e - a_n)$. Hint: consider $\displaystyle\int_0^1 (1-x)^n e^x \, dx$.