Week 6: Assorted Problems

1: Find two non-congruent right-angled triangles of the same area, both with integral side lengths.

2: A square piece of paper has corners at positions A, B, C and D, listed in counterclockwise order. The paper is folded once, making a crease from a point P on AB to a point Q on CD. After the fold, the corners at B and C are not moved, the corner at A is moved to position A' which lies on BC, and the corner at D is moved to position D' which lies outside the square $ABCD$. Let R be the point of intersection of $A'D'$ with CD. Show that the perimeter of the triangle $A'CR$ is half the perimeter of the square $ABCD$.

3: Show that the real number $x = 1 + \sqrt{2 + \sqrt{1 + \sqrt{2 + \cdots}}}$ is irrational.

4: Find $\lim\limits_{n\to\infty} \left(\dfrac{\left(1+\frac{1}{n}\right)^n}{e} \right)^n$.

5: Evaluate the sum $\sum\limits_{k=n}^{2n} \binom{k}{n} \frac{1}{2^k}$.

6: Let $*$ be an associative binary operation on a set S. Suppose that for all $a \in S$ there exist $b, c \in S$ such that $(ba)a = a$ and $a(ac) = a$. Show that for every $a \in S$ there exist $r, s \in S$ such that $ar = ra = a$ and $as = sa = r$.

7: Find every continuous function $f : \mathbf{R} \to \mathbf{R}$ with $f(1) = 2$ such that $f(x\cos\theta) f(x\sin\theta) = f(x)$ for all $x, \theta \in \mathbf{R}$.

8: Let $f : \mathbf{R} \to \mathbf{R}$ be $\mathcal{C}^2$. Suppose that $\lim\limits_{x\to\infty} \left(x^2 f''(x) + 4x f'(x) + 2f(x) \right) = 1$. Show that $\lim\limits_{x\to\infty} f(x) = \frac{1}{2}$ and $\lim\limits_{x\to\infty} x f'(x) = 0$.

9: Show that if $f : [0,1] \to \mathbf{R}$ is a continuous function such that $x f(y) + y f(x) \le 1$ for all $x, y \in [0,1]$ then we have $\displaystyle\int_0^1 f(x)\,dx \le \frac{\pi}{4}$, and find an example of such a function for which equality holds.

10: Find all pairs (a, b) of positive integers such that $a + b$ and $ab + 1$ are both powers of 2.

11: Let $a > 0$ and let $f : [0, a] \to [0, \infty)$ be $\mathcal{C}^1$ with $f(0) = 0$, $f(a) = 1$ and $\displaystyle\int_0^1 f(x)\,dx = 1$. Show that $\displaystyle\int_0^a \sqrt{f(x)^2 + f'(x)^2}\,dx \ge \sqrt{2}$.

12: Let $A \in M_n(\mathbf{C})$ with $A \ne cI$ for any $c \in \mathbf{C}$. Show that A is similar to a matrix which has at most one non-zero entry along its main diagonal.