Week 7: Assorted Problems

1: The digits $1, 2, \cdots, 9$ are permuted at random to form a 9-digit number n. Find the probability that n is a multiple of 11.

2: Show that for all $n \in \mathbf{Z}^+$ and $\theta \in \mathbf{R}$ we have $\sum\limits_{k=0}^{n-1} 2^k \tan(2^k\theta) = \cot\theta - 2^n \cot(2^n\theta)$.

3: Show that for all $n \in \mathbf{Z}^+$ and $\theta \in \mathbf{R}$ we have

$$\frac{\sin\theta + \sin 2\theta + \cdots + \sin n\theta}{\cos\theta + \cos 2\theta + \cdots + \cos n\theta} = \tan \tfrac{n+1}{2}\theta.$$

4: For which positive integers n does there exist an n-element set $S \subseteq \mathbf{Z}^2$ with the property that for each point $(x, y) \in S$, exactly 2 of the 4 points $(x \pm 1, y), (x, y \pm 1)$ lie in S?

5: Show that there exist infinitely many positive integers n such that there is a triangle in the plane with sides of length n, $n + 1$ and $n + 2$ whose area is an integer.

6: Let $a_1, a_2, \cdots, a_n \in \mathbf{R} \setminus \{0\}$. Let A be the $n \times n$ matrix with entries $A_{k,l} = a_k/a_l$. Find the characteristic polynomial of A.

7: Let $a \in \mathbf{R}$ with $a \neq 0, \pm 1$. Let A be the $n \times n$ matrix with entries $A_{k,l} = a^{|k-l|}$. Find the entries of the inverse matrix A^{-1}.

8: Show that if $p(x)$ is a polynomial of degree $n \geq 1$ with real coefficients and real roots, then we have $(n-1)p'(x)^2 \geq n\, p(x)p''(x)$ for all $x \in \mathbf{R}$. Also, find every such polynomial for which equality holds.

9: Let $a, d \in \mathbf{Z}^+$. Show that if a is odd and $d \big| (a^2 + 2)$ then either $d = 1 \bmod 8$ or $d = 3 \bmod 8$.

10: Let $n, k \in \mathbf{Z}^+$ and let $r = (2n+1)^k$. Show that $(2n+1)^{k+1}$ is a factor of $\sum\limits_{k=0}^{2n} (a+k)^r$.

11: Let $f : \mathbf{R} \to \mathbf{R}$ be $\mathcal{C}^\infty$. Suppose that for every $x \in \mathbf{R}$ there exists $n \in \mathbf{Z}^+$ such that $f^{(n)}(x) = 0$. Show that f is a polynomial.

12: Let $f, g : [0, 1] \to (0, \infty)$ be continuous. Suppose that f and g/f are increasing. Show that

$$\int_0^1 \frac{\int_0^x f(t)\,dt}{\int_0^x g(t)\,dt}\,dx \leq 2\int_0^1 \frac{f(t)}{g(t)}\,dt.$$

Also, show that the constant 2 on the right is optimal.