Week 8: Assorted Problems

1: There are 4 distinct points u_1, u_2, u_3, u_4 in the plane. Suppose that for 3 of the 6 pairs of indices (i, j) with $i < j$ we have $|u_i - u_j| = a$ and for the other 3 pairs we have $|u_i - u_j| = b$. Find all possible values for the ratio $r = \frac{a}{b}$.

2: Let $a, b \in \mathbf{Z}$ with $\gcd(a, b) = 1$. Show that there exists $n \in \mathbf{Z}^+$ such that $a^n + b^n = 1 \bmod ab$.

3: Let $f : \mathbf{R} \to \mathbf{R}$ be continuous with $f(x+1) = f(x)$ for all $x \in \mathbf{R}$. Suppose that $\displaystyle\int_0^1 f = 1$. Find the maximum possible value of $\displaystyle\int_0^1 \int_0^x f(x+y)\, dy\, dx$.

4: Let $n \in \mathbf{Z}^+$. Evaluate $\sum\limits_{k=0}^{3n} (-1)^k \binom{6n-k}{k}$.

5: Let $m \in \mathbf{Z}^+$. Evaluate $\displaystyle\sum_{n=0}^{m} \binom{m}{n} \sum_{k=0}^{n} (-1)^k \frac{n!}{k!}$.

6: Find a formula for a function $f(x)$ such that when a square with sides of length 2 rolls in the xy-plane without slipping along the curve $y = f(x)$, the centre of the square moves along the horizontal line $y = \sqrt{2}$.

7: Find the maximum possible number of elements that can be contained in a set of positive integers S with the property that for all $x, y \in S$ with $x < y$ we have $25(y - x) \geq xy$.

8: Find the largest possible cardinality of a set $A \subseteq \{1, 2, 3, \cdots, 30\}$ with the property that no product of two distinct elements in A is a square.

9: Show that for every prime number $p \geq 5$ we have $p^2 \,\Big|\, \sum\limits_{k=1}^{\lfloor 2p/3 \rfloor} \binom{p}{k}$.

10: Let $n \in \mathbf{Z}^+$. Find $\displaystyle\sum_{k=1}^{n} \phi(k) \left\lfloor \frac{n}{k} \right\rfloor$ where ϕ is Euler's totient function.

11: Show that for every integer $n \geq 2$ we have $\prod\limits_{k=2}^{n} \ln k < \frac{\sqrt{n!}}{n}$.

12: Let $k, n \in \mathbf{Z}^+$ with $k < n$. Let $f(x)$ be a polynomial of degree n whose coefficients all lie in the set $\{-1, 0, 1\}$. Suppose that $(x-1)^k \big| f(x)$. Let p be a prime number with $\frac{p}{\ln p} < \frac{k}{\ln(n+1)}$. Show that the complex p^{th} roots of 1 are all roots of $f(x)$.