Week 9: Assorted Problems

1: Show that $\cot^{-1}(-1) + \cot^{-1} 2 + \cot^{-1} 5 + \cot^{-1} 8 = 0$.

2: Let P be a point inside an equilateral triangle ABC with sides of length L. Let D, E and F be the points where the lines PA, PB and PC meet the edges opposite A, B and C respectively. Show that $|PD| + |PE| + |PF| < L$.

3: Find the maximum possible area for a triangle ABC which contains a point P such that $|PA| = 2$, $|PB| = \sqrt{2}$ and $|PC| = \sqrt{3} - 1$.

4: For $n \in \mathbf{Z}^+$, find the number of non-congruent triangles with perimeter $6n$ whose side lengths are integers.

5: Find a formula, in terms of n and k, for the number of binary sequences of length n which contain the block 01 exactly k times.

6: Show that for p prime and $n \in \mathbf{Z}^+$ we have $\binom{np}{p} = n \bmod p^2$.

7: For $n \in \mathbf{Z}^+$, let $\sigma(n)$ denote the sum of the positive divisors of n. Show that

$$\sum_{k=1}^{n} \int_0^\infty \cos\left(\tfrac{2\pi n}{k} \lfloor x + 1 \rfloor\right) dx = \sigma(n).$$

8: Let $2 \leq n \in \mathbf{Z}$. Let $x_1, x_2, \cdots, x_n \in \mathbf{R}$ with $x_1 < x_2 < \cdots < x_n$. For which values of n are the numbers $x_1, x_2, \cdots, x_n$ uniquely determined from the set of sums $S = \left\{x_k + x_l \middle| 1 \leq k < l \leq n\right\}$.

9: Let $\{a_n\}_{n \geq 1}$ be a sequence of positive real numbers such that $3a_n \geq a_{n-1} + 2a_{n-2}$ for all $n \geq 3$. Show that either $\{a_n\}$ converges or $\lim\limits_{n \to \infty} a_n = \infty$.

10: Let R be a ring of characteristic 0. Let $a, b, c \in R$ Suppose that $a^2 = a$, $b^2 = b$, $c^2 = c$ and $a + b + c = 0$. Show that $a = b = c = 0$.

11: Let n be a positive integer with at least two distinct prime factors. Show that there exists an n-gon whose internal angles are all equal and whose sides are of length $1, 2, 3, \cdots, n$, in some order.

12: Let $a, b \in \mathbf{R}$ and let $f, g, h, k : \mathbf{R} \to \mathbf{R}$ be differentiable with

$$\lim_{x \to \infty} f(x) = \infty\,,\ \lim_{x \to \infty} g(x) = \infty\,,\ \lim_{x \to \infty} h(x) = a \text{ and } \lim_{x \to \infty} k(x) = b.$$

Suppose that $\dfrac{f'(x)}{g'(x)} + h(x)\dfrac{f(x)}{g(x)} = k(x)$ for all $x \in \mathbf{R}$. Show that $\lim \dfrac{f(x)}{g(x)} = \dfrac{b}{a+1}$.