SPECIAL K
Saturday November 2, 1985
9:00 am - 12:00 noon

1: Let $a_n = \left(1 - \frac{1}{2^2}\right) \cdot \left(1 - \frac{1}{3^2}\right) \cdot \ldots \cdot \left(1 - \frac{1}{n^2}\right)$. Evaluate $\lim_{n\to\infty} a_n$.

2: Let A_i, for $1 \le i \le 8$, be the vertices of a wire frame forming the edges of a cube. A spider begins at A_1 and, when he is at any vertex, he chooses at random one of the three edges out of that vertex and walks along it to the next vertex. Find the probability that the spider returns to A_1 for the first time in precisely six moves.

3: Consider a triangle $\triangle ABC$. Let P be a point inside the triangle. Let s_1, s_2, s_3 and s be the areas of the circles inscribed in $\triangle PAB$, $\triangle PBC$, $\triangle PCA$ and $\triangle ABC$ respectively. Prove that

$$s_1 + s_2 + s_3 > s/3.$$

4: Let a_0 and a_1 be real numbers and let $a_n = |a_{n-1}| - a_{n-2}$ for $n \ge 2$.

(a) Prove that the sequence $\{a_n\}$ is periodic for all choices of a_0 and a_1.

(b) Is there a least positive integer p such that $a_{n+p} = a_n$ for all n, regardless of the values of a_0 and a_1?

5: Let f be a continuous increasing function from $[0,2]$ to $\mathbf{R}$ such that $f(x) \le e^x$ for all $x \in [0,2]$ and $\int_0^1 f(t)\,dt = 1$. Show that

$$x\,f(x) \ge \int_0^x f(t)\,dt \quad \text{for every } x \in [0,2]\,, \text{ and}$$

$$f(\overline{x}) \le \int_0^{\overline{x}} f(t)\,dt \quad \text{for some } \overline{x} \in [0,2]\,.$$

BIG E
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1: Evaluate $\displaystyle\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{nm(n+m-1)}$.

2: Construct an irreducible polynomial with integer coefficients, with two roots on the unit circle, one root inside the circle, and one root outside the circle.

3: Let $A = I + x\,y^T + u\,v^T$, where I is the $n \times n$ identity matrix, x, y, u and v are column vectors in $\mathbf{R}^n$, and x^T, y^T, u^T and v^T are the corresponding row vectors. Show that

$$\det A = \det \begin{pmatrix} 1 + x^T y & u^T y \\ x^T v & 1 + u^T v \end{pmatrix}.$$

4: Let A_n and B_n be the sets of integers given by

$$A_1 = \emptyset = \{\},\ \ B_1 = \{0\},$$
$$A_{n+1} = \big\{k+1 \big|\, k \in B_n\big\},$$
$$B_{n+1} = (A_n \cup B_n) \setminus (A_n \cap B_n).$$

For which n is $B_n = \{0\}$?

5: Let f be a non-negative continuous function on $[0, \infty)$ such that

(1) $\displaystyle\int_0^x f(t)\,dt > 0$ for all $x \in (0,\infty)$, and

(2) $\displaystyle\Big(\int_0^x f(t)^4\,dt\Big)\Big(\int_0^x f(t)\,dt\Big) = 3\Big(\int_0^x t\,f(t)\,dt\Big)^2$ for all $x \in (0,\infty)$.

(a) Give an interpretation of Equation (2) in terms of centroid and moments of inertia.

(b) Find a function of the form αx^β that satisfies both (1) and (2).

(c) Prove that each function satisfying both (1) and (2) must satisfy the inequality

$$\int_0^x f(t)^4\,dt \le 3\int_0^x t^2 f(t)\,dt \ \text{ for all } x \in (0,\infty).$$