SPECIAL K
Saturday November 1, 1986
9:00 am - 12:00 noon

1: (a) Find all positive integers n such that $n+1$ divides $n!+1$.

(b) Find all positive integers n such that $\frac{(n-1)!}{k!(n-k)!}$ is an integer for all $1 \leq k \leq n-1$.

2: Let T be a fixed triangle in the plane $\mathbf{R}^2$. For every function $\ell(x,y) = ax+by+c$, let $A(\ell)$ denote the area of the region $\{(x,y) \in T \,|\, \ell(x,y) \geq 0\}$. For any point $p \in \mathbf{R}^2$, let $g(p) = \max\{A(\ell) \,|\, \ell(p) = 0\}$. Find the point p at which $g(p)$ attains its minimum value.

3: Let $S_n = \displaystyle\sum_{k=1}^{n-1} \frac{1}{\sin^2\left(\frac{k\pi}{n}\right)}$ for all $n \geq 1$. Evaluate S_{2^k} for $k \geq 0$.

4: Let $f(x) = \frac{1}{\ln(1+x)} + \frac{1}{\ln(1-x)}$ for $0 < x < 1$. Show that $f(x) \geq 1$ for all x.

5: Consider all octagons with four sides of length 1 and four sides of length 2. Show that among these octagons, the maximum area is attained by the octagon with vertices at $\pm\left(1+\frac{1}{\sqrt{2}}\,,\,\pm 1\right)$ and $\pm\left(\pm 1\,,\,1+\frac{1}{\sqrt{2}}\right)$.

BIG E
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1: Suppose the system of inequalities

$$a_1x + b_1y + c_1z > 0$$
$$a_2x + b_2y + c_2z > 0$$
$$a_3x + b_3y + c_3z > 0$$

has no solutions in $\mathbf{R}^3$. Show that there are no non-negative real numbers X, Y and Z, not all zero, such that

$$a_1X + a_2Y + a_3Z = 0$$
$$b_1X + b_2Y + b_3Z = 0$$
$$c_1X + c_2Y + c_3Z = 0\,.$$

2: Three random points X, Y and Z are independently and uniformly distributed in a disc of radius one. Find the probability that the centre of the disc belongs to the triangle XYZ.

3: How many subsets of $\{1, 2, \cdots, 25\}$ do not contain a pair of integers $\{k, k+2\}$?

4: Consider all octagons with four sides of length 1 and four sides of length 2. Show that among these octagons, the maximum area is attained by the octagon with vertices at $\pm\big(1 + \frac{1}{\sqrt{2}}\,, \pm 1\big)$ and $\pm\big(\pm 1\,, 1 + \frac{1}{\sqrt{2}}\big)$.

5: Prove that

$$\det \begin{pmatrix} 1 & x_1 & {x_1}^2 & \cdots & {x_1}^{2n-1} \\ 0 & 1 & 2x_1 & \cdots & (2n-1){x_1}^{2n-2} \\ 1 & x_2 & {x_2}^2 & \cdots & {x_2}^{2n-1} \\ 0 & 1 & 2x_2 & \cdots & (2n-1){x_2}^{2n-2} \\ & \vdots & & & \vdots \\ 1 & x_n & {x_n}^2 & \cdots & {x_n}^{2n-1} \\ 0 & 1 & 2x_n & \cdots & (2n-1){x_n}^{2n-2} \end{pmatrix} = \prod_{i<j} (x_i - x_j)^4.$$