SPECIAL K
Saturday November 10, 2018
10:00 am - 1:00 pm

1: Let C be the circle of radius 1 centred at $(0, 1)$. Let D be the circle of radius 2 centred at $(a, 2)$ where $a > 0$ and D is externally tangent to C. Let E be the circle of radius r centred at $(x, 0)$ where $0 < x < a$ and E is externally tangent to both C and D. Find the values of x and r.

2: Let a_n be the n^{th} positive integer k such that $\lfloor \sqrt{k} \rfloor$ divides k. Find n such that $a_n = 600$.

3: A Mersenne prime is a prime of the form $p = 2^k - 1$ for some positive integer k. For a positive integer n, let $\sigma(n)$ be the sum of the positive divisors of n. Show that $\sigma(n)$ is a power of 2 if and only if n is a product of distinct Mersenne primes.

4: Let $\{a_n\}$ be a sequence of positive real numbers such that $\sum\limits_{n=1}^{\infty} a_n < \infty$. Show that there exists a sequence $\{c_n\}$ of positive real numbers with $\lim\limits_{n\to\infty} c_n = \infty$ such that $\sum\limits_{n=1}^{\infty} c_n a_n < \frac{1}{2}$.

5: Find the minimum possible value of $f'(2)$ given that $f(x)$ is a polynomial with nonnegative real coefficients such that $f(1) = 1$ and $f(2) = 3$.

6: Let $a_0 = a_1 = 1$ and let $a_{2n} = a_{n-1} + a_n$ and $a_{2n+1} = a_n$ for $n \geq 1$. Define $f : \mathbf{Z}^+ \to \mathbf{Q}^+$ by $f(n) = \frac{a_n}{a_{n-1}}$. Show that f is bijective.

BIG E
Saturday November 10, 2018
10:00 am - 1:00 pm

1: Let C be the sphere of radius 1 centred at $(0,1,1)$. Let D be the sphere of radius 2 centred at $(a,2,2)$ where $a>0$ and D is externally tangent to C. Let E be the sphere of radius r centred at (x,r,r) where $0<x<a$ and E is externally tangent to both C and D. Find the values of x and r.

2: Let a_n be the n^{th} positive integer k such that $\lfloor \sqrt[3]{k} \rfloor$ divides k. Find n such that $a_n = 600$.

3: Define $f:(1,\infty)\to\mathbf{R}$ by $f(x)=\displaystyle\int_x^{x^2}\frac{dt}{\ln t}$. Find the range of f.

4: Let p be a prime number, let $\mathbf{Z}_p$ be the field of integers modulo p, and let $M_3(\mathbf{Z}_p)$ be the ring of 3×3 matrices with entries in $\mathbf{Z}_p$. Find the number of functions $F:\mathbf{Z}\to M_3(\mathbf{Z}_p)$ such that $F(k+l)=F(k)+F(l)$ and $F(kl)=F(k)F(l)$ for all $k,l\in\mathbf{Z}$.

5: Let $a_0=a_1=1$ and let $a_{2n}=a_{n-1}+a_n$ and $a_{2n+1}=a_n$ for $n\geq 1$. Define $f:\mathbf{Z}^+\to\mathbf{Q}^+$ by $f(n)=\frac{a_n}{a_{n-1}}$. Show that f is bijective.

6: Let $n\in\mathbf{Z}^+$ and let $N=\{1,2,3,\cdots,n\}$. Let S be a set of subsets of N with the property that for all $A,B\subseteq N$, if $A\in S$ and $A\subseteq B$ then $B\in S$. Define $f:[0,1]\to\mathbf{R}$ by $f(x)=\sum_{A\in S} x^{|A|}(1-x)^{|N\setminus A|}$. Show that f is nondecreasing.