Lesson 1: The Pigeonhole Principle

1: Show that at any party there are two people who have the same number of friends at the party (assume that all friendships are mutual).

2: Show that if 9 distinct points are chosen in the integer lattice $\mathbf{Z}^3$, then the line segment between some two of the 9 points contains another point in $\mathbf{Z}^3$.

3: Let S be a set of n integers. Show that there is a subset of S, the sum of whose elements is a multiple of n.

4: Show that if 101 integers are chosen from the set $\{1, 2, 3, \cdots, 200\}$ then one of the chosen integers divides another.

5: Show that for some integer $k > 1$, 3^k ends with 0001 (in its decimal representation).

6: Let n be a positive integer. Show that there is a positive multiple of n whose digits (in the base 10 representation) are all 0's and 1's.

7: Show that some pair of any 5 points in the unit square will be at most $\frac{\sqrt{2}}{2}$ units apart, and that some pair of any 8 points in the unit square will be at most $\frac{\sqrt{5}}{4}$ units apart.

8: A salesman sells at least 1 car each day for 100 consecutive days selling a total of 150 cars. Show that for each value of n with $1 \leq n < 50$, there is a period of consecutive days during which he sold a total of exactly n cars.

9: Show that there is a Fibonacci number that ends with 9999 (in its base 10 representation).

10: Determine whether the sequence $\left\{ \dfrac{1}{n \sin n} \right\}$ converges.

Putnam Problems Involving the Pigeonhole Principle

1: (1989 A5) Let $n \in \mathbf{Z}^+$. Let P be a regular $(2n+1)$-gon inscribed in the unit sphere. Show that there exists $c > 0$ such that for every point p inside P, there exist two distinct vertices u and v of P such that

$$\Big| |p-u| - |p-v| \Big| < \frac{1}{n} - \frac{c}{n^3}.$$

2: (1990 A3) Show that a convex pentagon with vertices in $\mathbf{Z}^2$ has area at least $\frac{5}{2}$.

3: (1993 A4) Let $n, m \in \mathbf{Z}^+$. Let $M = \{1, 2, \cdots, m\}$, and let P be the set of all subsets of $\{1, 2, \cdots, n\}$. Show that the number of functions $f : P \to M$ with the property that $f(A \cap B) = \min\{f(A), f(B)\}$ is equal to $\sum\limits_{k=1}^{m} k^n$.

4: (1994 A4) Let $A, B \in M_{2\times 2}(\mathbf{Z})$. Suppose that $A + kB$ is invertible in $M_{2\times 2}(\mathbf{Z})$ for all $k \in \{0, 1, 2, 3, 4\}$. Show that $A + kB$ is invertible for all $k \in \mathbf{Z}$.

5: (1994 A6) Let $f_1, f_2, \cdots, f_{10} : \mathbf{Z} \to \mathbf{Z}$ be bijective maps. Suppose that for each integer n, there is some composite $f = f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_m}$, where $m \in \mathbf{Z}^+$ and each $i_j \in \{1, \cdots, 10\}$, with $f(0) = n$. Let

$$F = \Big\{ {f_1}^{e_1} \circ {f_2}^{e_2} \circ \cdots \circ {f_{10}}^{e_{10}} \Big| \text{each } e_i \in \{0, 1\} \Big\}$$

(where ${f_i}^1 = f_i$ and ${f_i}^0$ is the identity). Show that if A is any nonempty finite set of integers, then at most 512 of the 1024 functions in F map A to itself.

6: (1995 B1) Let $S = \{1, 2, \cdots, 9\}$. For a partition $\alpha = \{A_1, \cdots, A_l\}$ of S and an element $x \in S$, let $N(\alpha, x)$ be the number of elements in the set A_i which contains x. Show that for any two partitions α and β of S there exist to distinct elements $x, y \in S$ such that $N(\alpha, x) = N(\alpha, y)$ and $N(\beta, x) = N(\beta, y)$.

7: (1997 B6) Find the least possible diameter of a dissection of the 3-4-5 triangle into four parts. (The diameter of a dissection is the largest of the diameters of the parts).

8: (1999 A5) Show that there exists a constant $c \in \mathbf{R}$ such that for every polynomial $f(x)$ of degree 1999, we have

$$|f(0| \leq c \int_{-1}^{1} |f(x)|\, dx.$$

9: (2000 B1) Let $A \in M_{n\times 3}(\mathbf{Z})$. Suppose that at least one entry in each row of A is odd. Show that for some $x \in \mathbf{Z}^3$, at least $\frac{4n}{7}$ of the entries of Ax are odd.

10: (2000 B6) Let $3 \leq n \in \mathbf{Z}$. Let $S \subseteq \{-1, 1\}^n = \big\{(\pm 1, \pm 1, \cdots, \pm 1)\big\} \subseteq \mathbf{R}^n$ with $|S| > \frac{2^{n+1}}{n}$. Show that there exists an equilateral triangle in $\mathbf{R}^n$ with vertices in S.