Lesson 2: Induction and Recursion

1: Let $a_0 = 0$ and $a_1 = 1$ and for $n \geq 2$ let $a_n = a_{n-1} + 6a_{n-2}$. Show that $a_n = \frac{1}{5}\big(3^n - (-2)^n\big)$ for all $n \geq 0$.

2: Let $n \in \mathbf{Z}^+$. Evaluate $\displaystyle\sum_{i=1}^{n} (-1)^i (2i-1)^2$.

3: Let $c, p, q \in \mathbf{R}$ with $p \neq 0$. Let $a_0 = c$ and for $n \geq 1$ let $a_n = p\, a_{n-1} + q$. Find a_n.

4: Let $n \in \mathbf{N}$. Evaluate $\displaystyle\sum_{i=0}^{n} \binom{n+i}{i} \frac{1}{2^i}$.

5: Let $a_0 = 9$ and for $n \geq 0$ let $a_{n+1} = 3{a_n}^4 + 4{a_n}^3$. Show that for all $n \geq 0$, the number a_n has (at least) 2^n nines in its decimal expansion.

6: Let $n \in \mathbf{Z}^+$. Evaluate $\displaystyle\sum_{(k,l)\in A} \frac{1}{kl}$ where A is the set of ordered pairs of integers (k, l) with $1 \leq k \leq n$, $1 \leq l \leq n$, $k + l > n$ and $\gcd(k, l) = 1$.

7: Let $f : \mathbf{Z}^+ \to \mathbf{Z}^+$ be strictly increasing with $f(2) = 2$ and $f(kl) = f(k)f(l)$ for all $k, l \in \mathbf{Z}^+$ with $\gcd(k, l) = 1$. Show that $f(n) = n$ for all $n \in \mathbf{Z}^+$.

8: Let a_n be the n^{th} Fibonacci number (so $a_0 = 0$, $a_1 = 1$ and $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$). Show that ${a_n}^2 + {a_{n+1}}^2 = a_{2n+1}$ for all $n \geq 0$.

9: (a) Show that every positive integer is equal to a sum of distinct Fibonacci numbers.

(b) Show that every positive integer can be expressed uniquely as a sum of distinct non-consecutive Fibonacci numbers.

10: Let $(a_1, a_2, \cdots, a_n) \in \mathbf{Z}^n$ with $\sum_{i=1}^{n} a_i = 1$. For $k, l \in \{1, 2, \cdots, n\}$, let

$$S_{kl} = \sum_{i=k}^{l} a_i = \begin{cases} a_k + a_{k+1} + \cdots + a_l & \text{if } k \leq l \leq n, \\ a_k + \cdots + a_n + a_1 + \cdots + a_l & \text{if } 1 \leq l < k. \end{cases}$$

Show that there exists a unique k such that $S_{kl} > 0$ for every l.

11: Let $n \in \mathbf{Z}^+$. Suppose that n distinct points are chosen on the unit circle and a line segment is drawn between each of the $\binom{n}{2}$ pairs of points and suppose that no three of the line segments are coincident. Let a_n be the number of regions into which the unit disc is divided by these line segments.

(a) Find $a_1, a_2, \cdots, a_5$ and conjecture a formula for a_n.

(b) The obvious conjecture from Part (a) is incorrect. Find the correct formula for a_n.

12: Let p be an odd prime and suppose that $U_{p^2} = \langle a \rangle$. Show that $U_{p^k} = \langle a \rangle$ for all $k \geq 2$.

Putnam Problems Involving Induction

1: (1986 B3) Let p be prime. Let $f, g, h, r, s \in \mathbf{Z}[x]$. Suppose that $rf + sg = 1 \in \mathbf{Z}_p[x]$ and $fg = h \in \mathbf{Z}_p[x]$. Show that for all $n \in \mathbf{Z}^+$, there exist $k, l \in \mathbf{Z}[x]$ with $k = f \in \mathbf{Z}_p[x]$ and $l = g \in \mathbf{Z}_p[x]$ and $kl = h \in \mathbf{Z}_{p^n}$.

2: (1987 B2) Let $k, l, n \in \mathbf{N}$ with $k + l \le n$. Show that $\displaystyle\sum_{i=0}^{l} \frac{\binom{l}{i}}{\binom{n}{k+i}} = \frac{n+1}{(n-l+1)\binom{n-l}{k}}$.

3: (1987 B4) Let $a_0 = \frac{4}{5}$ and $b_0 = \frac{3}{5}$, and for $n \ge 0$ let $a_{n+1} = a_n \cos(b_n) - b_n \sin(a_n)$ and $b_{n+1} = a_n \sin(b_n) + b_n \cos(a_n)$. Determine whether $\{a_n\}$ converges and if so find the limit, and determine whether $\{b_n\}$ converges and if so find the limit.

: (1988 B5) Let $n \in \mathbf{Z}^+$. Find the rank of the matrix $A \in M_{2n+1}(\mathbf{R})$ with entries

$$A_{i,j} = \begin{cases} 1 \text{ if } n < j - i \le 2n \text{ or } -n \le j - i < 0 \\ -1 \text{ if } 0 < j - i \le n \text{ or } -2n \le j - i < n \\ 0 \text{ if } 0 = j - i. \end{cases}$$

5: (1990 A1) Let $a_0 = 2$, $a_1 = 3$ and $a_2 = 6$, and let $a_n = (n+4)a_{n-1} - (4n)a_{n-2} + (4n-8)a_{n-3}$ for $n \ge 3$. Find a formula for a_n.

6: (1990 B5) Determine whether there exists a sequence $a_0, a_1, a_2, \cdots$, with $0 \ne a_i \in \mathbf{R}$ for all $i \ge 0$, such that for every $n \in \mathbf{Z}^+$, the polynomial $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ has n distinct real roots.

7: (1992 A4) Let $f : \mathbf{R} \to \mathbf{R}$ be $\mathcal{C}^\infty$. Suppose that $f\left(\frac{1}{k}\right) = \frac{k^2}{k^2+1}$ for all $k \in \mathbf{Z}^+$. Find the n^{th} derivative $f^{(n)}(0)$ for all $n \in \mathbf{Z}^+$.

8: (1997 A2) Let $5 \le n \in \mathbf{Z}$. There are n players seated around a circle. Initially, the players sit in counterclockwise order from Player 1 to Player n, and each player has 1 dollar. They play the following game. Player 1 passes his 1 dollar to Player 2 and leaves the circle. Player 2 passes his 2 dollars to Player 3 and leaves the circle. Player 3 passes 1 of his 3 dollars to Player 4 and remains seated. Player 4 passes his 2 dollars to Player 5 and leaves the circle. The game continues with each player passing alternately 1 then 2 dollars counterclockwise to the next player at the table, and any player who passes all his money leaves the table. Show that there exist infinitely many values of n for which one player ends up with all the money.

9: (1998 A4) For $k, l \in \mathbf{Z}$ let $k * l$ denote the concatenation of k followed by l. Let $a_0 = 0$ and $a_1 = 1$, and for $n \ge 2$, let $a_n = a_{n-1} * a_{n-2}$. Find all values of $n \in \mathbf{Z}^+$ such that $11 \big| a_n$.

10: (1999 A3) Let $T(x) = \displaystyle\sum_{k=0}^{\infty} c_n x^n$ be the Taylor series centred at 0 for $f(x) = \dfrac{1}{1 - 2x - x^2}$. Show that for every $n \in \mathbf{N}$ there exists $m \in \mathbf{N}$ such that ${a_n}^2 + {a_{n+1}}^2 = a_m$.