Solutions to the Problems on Induction and Recursion

1: Let $a_0 = 0$ and $a_1 = 1$ and for $n \geq 2$ let $a_n = a_{n-1} + 6a_{n-2}$. Show that $a_n = \frac{1}{5}\big(3^n - (-2)^n\big)$ for all $n \geq 0$.

Solution: We claim that $a_n = \frac{1}{5}\big(3^n - (-2)^n\big)$ for all $n \geq 0$. When $n = 0$ we have $a_n = a_0 = 0$ and $\frac{1}{5}\big(3^n - (-2)^n\big) = \frac{1}{5}\big(3^0 - (-2)^0\big) = 0$, so the claim is true when $n = 0$. When $n = 1$ we have $a_n = a_1 = 1$ and $\frac{1}{5}\big(3^n - (-2)^n\big) = \frac{1}{5}\big(3 - (-2)\big) = 1$, so the claim is true when $n = 1$. Let $k \geq 2$ and suppose the claim is true for all $n < k$. In particular we suppose the claim is true when $n = k-1$ and when $n = k-2$, that is we suppose $a_{k-1} = \frac{1}{5}\big(3^{k-1} - (-2)^{k-1}\big)$ and $a_{k-2} = \frac{1}{5}\big(3^{k-2} - (-2)^{k-2}\big)$. Then when $n = k$ we have

$$\begin{aligned}
a_n = a_k &= a_{k-1} + 6a_{k-2} \\
&= \tfrac{1}{5}\big(3^{k-1} - (-2)^{k-1}\big) + \tfrac{6}{5}\big(3^{k-2} - (-2)^{k-2}\big) \\
&= \big(\tfrac{1}{5}\cdot 3^{k-1} + \tfrac{6}{5}\cdot 3^{k-2}\big) - \big(\tfrac{1}{5}(-2)^{k-1} + \tfrac{6}{5}(-2)^{k-2}\big) \\
&= \big(\tfrac{3}{5}\cdot 3^{k-2} + \tfrac{6}{5}\cdot 3^{k-2}\big) - \big(-\tfrac{2}{5}(-2)^{k-2} + \tfrac{6}{5}(-2)^{k-2}\big) \\
&= \tfrac{9}{5}\cdot 3^{k-2} - \tfrac{4}{5}(-2)^{k-2} = \tfrac{1}{5}\cdot 3^k - \tfrac{1}{5}(-2)^k \\
&= \tfrac{1}{5}\big(3^k - (-2)^k\big) = \tfrac{1}{5}\big(3^n - (-2)^n\big)\,.
\end{aligned}$$

Thus the claim is true when $n = k$. By Strong Mathematical Induction, the claim is true for all $n \geq 0$.

2: Let $n \in \mathbf{Z}^+$. Evaluate $\displaystyle\sum_{i=1}^{n}(-1)^i(2i-1)^2$.

Solution: Let $S_n = \displaystyle\sum_{i=1}^{n}(-1)^i(2i-1)^2$. Verify that $S_1 = -1$, $S_2 = 8 = 2\cdot 4$, $S_3 = -17 = 1 - 2\cdot 9$, $S_4 = 32 = 2\cdot 16$, $S_5 = -49 = 1 - 2\cdot 25$ and $S_6 = 72 = 2\cdot 36$. It appears that for all $n \geq 1$, we have

$$S_n = \begin{cases} 2n^2 & \text{when } n \text{ is even,} \\ 1 - 2n^2 & \text{when } n \text{ is odd.} \end{cases}$$

In other words, it appears that $S_{2m} = 2(2m)^2$ for all $m \geq 1$ and that $S_{2m-1} = 1 - 2(2m-1)^2$ for all $m \geq 1$. We claim first that $S_{2m} = 2(2m)^2$ for all $m \geq 1$. We have seen that this claim is true when $m = 1$ (and when $m = 2, 3$). Let $k \geq 1$ and suppose that the claim is true when $m = k$, that is suppose that $S_{2k} = 2(2k)^2$. Then when $m = k+1$ we have

$$\begin{aligned}
S_{2m} &= \sum_{i=1}^{2k+2}(-1)^i(2i-1)^2 \\
&= \left(\sum_{i=1}^{2k}(-1)^i(2i-1)^2\right) + (-1)^{2k+1}(4k+1)^2 + (-1)^{2k+2}(4k+3)^2 \\
&= 2(2k)^2 - (4k+1)^2 + (4k+3)^2 = 8k^2 - (16k^2 + 8k + 1) + (16k^2 + 24k + 8) \\
&= 8k^2 + 16k + 8 = 8(k+1)^2 = 2(2m)^2\,.
\end{aligned}$$

Thus the claim is true when $m = k+1$. By Mathematical Induction, the claim is true for all $m \geq 1$. Finally, note that for all $m \geq 1$ we have $1 - 2(2m-1)^2 = 1 - 2(4m^2 - 4m + 1) = -8m^2 + 8m - 1$ and

$$\begin{aligned}
S_{2m-1} &= S_{2m} - (-1)^{2m}(4m-1)^2 = 2(2m)^2 - (4m-1)^2 \\
&= 8m^2 - (16m^2 - 8m + 1) = -8m^2 + 8m - 1 = 1 - 2(2m-1)^2\,.
\end{aligned}$$

3: Let $c,p,q\in\mathbf{R}$ with $p\neq 0$. Let $a_0=c$ and for $n\geq 1$ let $a_n=p\,a_{n-1}+q$. Find a_n.

Solution: We have

$$\begin{aligned}
a_0&=c\\
a_1&=pc+q\\
a_2&=p(pc+q)+q=p^2c+pq+q\\
a_3&=p(p^2c+pq+q)+q=p^3c+p^2q+pq+q\\
a_4&=p(p^3c+p^2q+pq+q)+q=p^4c+p^3q+p^2q+pq+q
\end{aligned}$$

and in general

$$a_n=p^nc+p^{n-1}q+p^{n-2}q+\cdots+p^2q+pq+q=p^nc+(p^{n-1}+p^{n-2}+\cdots p^2+p+1)q\,.$$

We can obtain a (non-recursive) formula for the geometric sum $p^{n-1}+p^{n-2}+\cdots+p^2+p+1$ as follows. Let $S=p^{n-1}+p^{n-2}+\cdots+p^2+p+1$ (1). Note that $pS=p^n+p^{n-1}+p^{n-2}+\cdots+p^2+p$ (2). Subtracting (1) from (2) gives $S(p-1)=p^n-1$ and so $S=\dfrac{p^n-1}{p-1}$. Thus we have

$$a_n=p^n\,c+\frac{p^n-1}{p-1}\,q\,.$$

4: Let $n\in\mathbf{N}$. Evaluate $\displaystyle\sum_{i=0}^{n}\binom{n+i}{i}\frac{1}{2^i}$.

Solution: Let $S_n=\displaystyle\sum_{i=0}^{n}\binom{n+i}{i}\frac{1}{2^i}$. Verify that $S_0=1$, $S_1=2$, $S_2=4$ and $S_3=8$. We claim that $S_n=2^n$ for all $n\geq 0$. When $n=0$ (and also when $n=1$, 2 and 3) we have seen that the claim is true. Let $k\geq 0$ and suppose that the claim is true when $n=k$, that is suppose $S_k=2^k$. Let $n=k+1$. Then we have

$$\begin{aligned}
S_n=S_{k+1}&=\tbinom{k+1}{0}+\tbinom{k+2}{1}\tfrac{1}{2}+\tbinom{k+3}{2}\tfrac{1}{2^2}+\tbinom{k+4}{3}\tfrac{1}{2^3}+\cdots+\tbinom{2k+1}{k}\tfrac{1}{2^k}+\tbinom{2k+2}{k+1}\tfrac{1}{2^{k+1}}\\
&=1+\left(\tbinom{k+1}{0}+\tbinom{k+1}{1}\right)\tfrac{1}{2}+\left(\tbinom{k+2}{1}+\tbinom{k+2}{2}\right)\tfrac{1}{2^2}+\left(\tbinom{k+3}{2}+\tbinom{k+3}{3}\right)\tfrac{1}{2^2}\\
&\quad+\cdots+\left(\tbinom{2k}{k-1}+\tbinom{2k}{k}\right)\tfrac{1}{2^k}+\left(\tbinom{2k+1}{k}+\tbinom{2k+1}{k+1}\right)\tfrac{1}{2^{k+1}}\\
&=\left(\tbinom{k+1}{0}\tfrac{1}{2}+\tbinom{k+2}{1}\tfrac{1}{2^2}+\tbinom{k+3}{2}\tfrac{1}{2^3}+\cdots+\tbinom{2k}{k-1}\tfrac{1}{2^k}+\tbinom{2k+1}{k}\tfrac{1}{2^{k+1}}\right)\\
&\quad+\left(1+\tbinom{k+1}{1}\tfrac{1}{2}+\tbinom{k+2}{2}\tfrac{1}{2^2}+\tbinom{k+3}{3}\tfrac{1}{2^3}+\cdots+\tbinom{2k}{k}\tfrac{1}{2^k}+\tbinom{2k+1}{k+1}\tfrac{1}{2^{k+1}}\right)\\
&=\left(\tfrac{1}{2}S_n-\tbinom{2k+2}{k+1}\tfrac{1}{2^{k+2}}\right)+\left(\sum_{i=0}^{k}\tbinom{k+i}{i}\tfrac{1}{2^i}+\tbinom{2k+1}{k+1}\tfrac{1}{2^{k+1}}\right).
\end{aligned}$$

Subtract $\frac{1}{2}S_n$ from each side to get

$$\tfrac{1}{2}S_n=\sum_{i=0}^{k}\tbinom{k+i}{i}\tfrac{1}{2^i}+\tbinom{2k+1}{k+1}\tfrac{1}{2^{k+1}}-\tbinom{2k+2}{k+1}\tfrac{1}{2^{k+2}}\,.$$

Notice that

$$\tbinom{2k+2}{k+1}=\tfrac{(2k+2)!}{(k+1)!(k+1)!}=\tfrac{(2k+2)(2k+1)!}{(k+1)k!(k+1)!}=\tfrac{2(2k+1)!}{k!(k+1)!}=2\tbinom{2k+1}{k+1}$$

and so we have $\frac{1}{2}S_n=\displaystyle\sum_{i=0}^{k}\tbinom{k+i}{i}\tfrac{1}{2^i}=S_k=2^k$, that is $S_n=2^{k+1}=2^n$. Thus the claim holds when $n=k+1$, and so by Mathematical Induction, the claim holds for all $n\geq 0$.

5: Let $a_0 = 9$ and for $n \geq 0$ let $a_{n+1} = 3{a_n}^4 + 4{a_n}^3$. Show that for all $n \geq 0$, the number a_n has (at least) 2^n nines in its decimal expansion.

Solution: Note first that a positive integer m ends with (at least) l nines $\iff$ $m+1$ ends with l zeros $\iff$ $m+1 = 10^l q$ for some positive integer q $\iff$ $m = 10^l q - 1$ for some positive integer q.

We claim that for all $n \geq 0$, a_n ends with (at lest) 2^n nines. When $n = 0$. the claim is true since $a_0 = 9$ which ends with $2^0 = 1$ nine(s). Let $k \geq 0$ and suppose (inductively) that a_k ends with 2^k nines, say $a_k = 10^{2^k} q - 1$. Then when $n = k+1$ we have

$$\begin{aligned}
a_n = a_{k+1} &= 3{a_k}^4 + 4{a_k}^3 \\
&= 3\left(10^{2^k} q - 1\right)^4 + 4\left(10^{2^k} q - 1\right)^3 \\
&= 3\left(10^{4\cdot 2^k} q^4 - 4\cdot 10^{3\cdot 2^k} q^3 + 6\cdot 10^{2\cdot 2^k} q^2 - 4\cdot 10^{2^k} q + 1\right) \\
&\quad + 4\left(10^{3\cdot 2^k} q^3 - 3\cdot 10^{2\cdot 2^k} q^2 + 3\cdot 10^{2^k} q - 1\right) \\
&= 3\cdot 10^{4\cdot 2^k} q^4 - 8\cdot 10^{3\cdot 2^k} q^3 + 6\cdot 10^{2\cdot 2^k} q^2 - 1 \\
&= 10^{2\cdot 2^k}\left(3\cdot 10^{2\cdot 2^k} q^4 - 8\cdot 10^{2^k} q^3 + 6q^2\right) - 1 \\
&= 10^{2^{k+1}} r - 1 \text{ , where } r = 3\cdot 10^{2\cdot 2^k} q^4 - 8\cdot 10^{2^k} q^3 + 6q^2 \,,
\end{aligned}$$

which ends with 2^{k+1} nines. Thus for all $n \geq 0$, a_n ends with 2^n nines, by mathematical induction.

6: Let $n \in \mathbf{Z}^+$. Evaluate $\displaystyle\sum_{(k,l)\in A} \frac{1}{kl}$ where A is the set of ordered pairs of integers (k,l) with $1 \leq k \leq n$, $1 \leq l \leq n$, $k + l > n$ and $\gcd(k,l) = 1$.

Solution: Let $A_n = \{(k,l)\big| 1 \leq k \leq n, 1 \leq l \leq n, k+l > n, \gcd(k,l) = 1\}$ and let $S_n = \sum_{(k,l)\in A_n} \frac{1}{kl}$. Note that $A_1 = \{(1,1)\}$ so that $S_1 = 1$. Fix $n \in \mathbf{Z}^+$ and suppose, inductively, that $S_n = 1$. We have

$$\begin{aligned}
A_n &= \Big\{(k,l)\Big| 1 \leq k \leq n, 1 \leq l \leq n, k+l > n, \gcd(k,l) = 1\Big\}, \\
A_{n+1} &= \Big\{(k,l)\Big| 1 \leq k \leq n+1, 1 \leq l \leq n+1, k+l > n+1, \gcd(k,l) = 1\Big\}, \\
A_n \setminus A_{n+1} &= \Big\{(k,l)\Big| 1 \leq k \leq n, 1 \leq l \leq n, k+l = n+1, \gcd(k,l) = 1\Big\}, \\
&= \Big\{(k, n+1-k)\Big| 1 \leq k \leq n, \gcd(k, n+1) = 1\Big\}, \\
A_{n+1} \setminus A_n &= \Big\{(k,l)\Big| 1 \leq k \leq n+1, 1 \leq l \leq n+1, \text{ either } k = n+1 \text{ or } l = n+1 \text{ , } \gcd(k,l) = 1\Big\}, \\
&= \Big\{(n+1,l)\Big| 1 \leq l \leq n, \gcd(n+1,l) = 1\Big\} \cup \Big\{(k,n+1)\Big| 1 \leq k \leq n, \gcd(k,n+1) = 1\Big\}, \text{ and} \\
&= \Big\{(n+1,n+1-j)\Big| 1 \leq j \leq n, \gcd(n+1,j) = 1\Big\} \cup \Big\{(k,n+1)\Big| 1 \leq k \leq n, \gcd(k,n+1) = 1\Big\},
\end{aligned}$$

and so

$$\begin{aligned}
\sum_{(k,l)\in A_{n+1}\setminus A_n} \frac{1}{kl} &= \sum_{\substack{1\leq j\leq n \\ \gcd(k,n+1)=1}} \frac{1}{(n+1)(n+1-j)} + \sum_{\substack{1\leq k\leq n \\ \gcd(k,n+1)=1}} \frac{1}{k(n+1)} \\
&= \sum_{\substack{1\leq k\leq n \\ \gcd(k,n+1)=1}} \left(\frac{1}{(n+1)(n+1-k)} + \frac{1}{k(n+1)}\right) \\
&= \sum_{\substack{1\leq k\leq n \\ \gcd(k,n+1)=1}} \frac{1}{k(n+1-k)} = \sum_{(k,l)\in A_n\setminus A_{n+1}} \frac{1}{kl}.
\end{aligned}$$

Thus $S_{n+1} = \displaystyle\sum_{(k,l)\in A_{n+1}} \frac{1}{kl} = \sum_{(k,l)\in A_n} \frac{1}{kl} + \sum_{(k,l)\in A_{n+1}\setminus A_n} \frac{1}{kl} - \sum_{(k,l)\in A_n\setminus A_{n+1}} \frac{1}{kl} = \sum_{(k,l)\in A_{n+1}} \frac{1}{kl} = S_n = 1$. By induction, $S_n = 1$ for all $n \in \mathbf{Z}^+$.

7: Let $f : \mathbf{Z}^+ \to \mathbf{Z}^+$ be strictly increasing with $f(2) = 2$ and $f(kl) = f(k)f(l)$ for all $k, l \in \mathbf{Z}^+$ with $\gcd(k, l) = 1$. Show that $f(n) = n$ for all $n \in \mathbf{Z}^+$.

Solution: Since $f(1) \in \mathbf{Z}^+$ and $f(1) < f(2) = 2$ we must have $f(1) = 1$. Since $f(3) > f(2) = 2$ and since $f(3)f(5) = f(15) < f(18) = f(2)f(9) < f(2)f(10) = f(2)^2 f(5) = 4f(5)$ so that $f(3) < 4$ we have $f(3) = 3$. Since $f(6) = f(2)f(3) = 2 \cdot 3 = 6$ and since $1 = f(1) < f(2) < \cdots < f(6) = 6$ it follows that $f(k) = k$ for all $k \le 6$. Let $l \ge 2$ and suppose, inductively, that $f(k) = k$ for all $1 \le k \le 2(2l - 1)$. Note that $2 < 2(2l - 1)$ and $2l + 1 < 2(2l - 1)$ and so we have $f\big(2(2l + 1)\big) = f(2)f(2l + 1) = 2(2l + 1)$. Since $1 = f(1) < f(2) < \cdots < f\big(2(2l + 1)\big) = 2(2l + 1)$ it follows that $f(k) = k$ for all $1 \le k \le 2(2l + 1)$. By induction, we have $f(k) = k$ for all $k \in \mathbf{Z}^+$.

8: Let a_n be the n^{th} Fibonacci number (so $a_0 = 0$, $a_1 = 1$ and $a_n = a_{n-1} + a_{n-2}$ for $n \ge 2$). Show that ${a_n}^2 + {a_{n+1}}^2 = a_{2n+1}$ for all $n \ge 0$.

Solution: We begin by trying (and failing) to use induction to prove that ${a_n}^2 + a_{n+1}^2 = a_{2n+1}$ for all $n \ge 1$. When $n = 1$, we have $LS = {a_1}^2 + {a_2}^2 = 1^2 + 1^2 = 2$ and $RS = a_3 = a_2 + a_1 = 1 + 1 = 2 = LS$, so the equality holds. Let $k \ge 1$ and suppose (inductively) that ${a_k}^2 + {a_{k+1}}^2 = a_{2k+1}$. Then when $n = k + 1$ we have

$$\begin{aligned} LS &= {a_{k+1}}^2 + {a_{k+2}}^2 \\ &= {a_{k+1}}^2 + (a_{k+1} + a_k)^2 \\ &= {a_{k+1}}^2 + {a_{k+1}}^2 + 2a_k a_{k+1} + {a_k}^2 \\ &= \big({a_{k+1}}^2 + 2a_k a_{k+1}\big) + \big({a_k}^2 + {a_{k+1}}^2\big) \\ &= \big({a_{k+1}}^2 + 2a_k a_{k+1}\big) + a_{2k+1} \end{aligned}$$

(where the last inequality follows from the induction hypothesis), and we have

$$RS = a_{2k+3} = a_{2k+2} + a_{2k+1}\,.$$

If we could show that $\big({a_{k+1}}^2 + 2a_k a_{k+1}\big) = a_{2k+2}$ then we would have $LS = RS$ and our induction proof would work. We shall modify this abortive proof by proving two equalities at once.

We claim that ${a_n}^2 + {a_{n+1}}^2 = a_{2n+1}$ and ${a_{n+1}}^2 + 2a_n a_{n+1} = a_{2n+2}$ for all $n \ge 1$. When $n = 1$ we have ${a_n}^2 + {a_{n+1}}^2 = {a_1}^2 + {a_2}^2 = 1^2 + 1^2 = 2$ and $a_{2n+1} = a_3 = 2$ so the first equality holds, and we also have ${a_{n+1}}^2 + 2a_n a_{n+1} = {a_2}^2 + 2a_1 a_2 = 1^2 + 2 \cdot 1 \cdot 1 = 3$ and $a_{2n+2} = a_4 = 3$ so the second equality holds.

Let $k \ge 1$ and suppose (inductively) that both equalities hold when $n = k$, that is ${a_k}^2 + {a_{k+1}}^2 = a_{2k+1}$ and ${a_{k+1}}^2 + 2f_k a_{k+1} = a_{2k+2}$.

When $n = k + 1$ we have

$$\begin{aligned} {a_n}^2 + {a_{n+1}}^2 &= {a_{k+1}}^2 + {a_{k+2}}^2 \\ &= {a_{k+1}}^2 + (a_{k+1} + a_k)^2 \\ &= {a_{k+1}}^2 + {a_{k+1}}^2 + 2a_k a_{k+1} + {a_k}^2 \\ &= ({a_{k+1}}^2 + 2a_k a_{k+1}) + ({a_k}^2 + {a_{k+1}}^2) \\ &= a_{2k+2} + a_{2k+1} \\ &= a_{2k+3} = a_{2n+1} \end{aligned}$$

and we have

$$\begin{aligned} {a_{n+1}}^2 + 2a_n a_{n+1} &= {a_{k+2}}^2 + 2a_{k+1} a_{k+2} \\ &= {a_{k+2}}^2 + 2a_{k+1}(a_{k+1} + a_k) \\ &= {a_{k+2}}^2 + 2{a_{k+1}}^2 + 2a_k a_{k+1} \\ &= ({a_{k+1}}^2 + 2a_k a_{k+1}) + ({a_{k+1}}^2 + {a_{k+2}}^2) \\ &= a_{2k+2} + a_{2k+3} \\ &= a_{2k+4} = a_{2n+2}\,. \end{aligned}$$

Thus both equalities hold when $n = k + 1$, and hence both equalities hold for all $n \ge 1$ by mathematical induction.

9: (a) Show that every positive integer is equal to a sum of distinct Fibonacci numbers.

Solution: We omit a solution for Part (a) as it follows from Part (b).

(b) Show that every positive integer can be expressed uniquely as a sum of distinct non-consecutive Fibonacci numbers.

Solution: Let a_n denote the n^{th} Fibonacci number (so $a_1 = a_2 = 1$ and $a_n = a_{n-1} + a_{n-2}$ for $n \geq 3$). We interpret the statement of the problem to mean that every $n \in \mathbf{Z}^+$ can be represented uniquely in the form $n = a_{j_1} + a_{j_2} + \cdots + a_{j_m}$ for some $m \in \mathbf{Z}^+$ and some j_i with

$$2 \leq j_1 \ , \ j_1 + 2 \leq j_2 \ , \ j_2 + 2 \leq j_3 \ , \ \cdots \ , \ j_{m-1} + 2 \leq j_m \, .$$

First we claim that if $n \in \mathbf{Z}^+$ can be represented in this form then we must have $j_m = l$ where l is the index for which $a_l \leq n < a_{l+1}$. Suppose, for a contradiction that $j_m < l$. Then we have $j_m \leq l-1$, $j_{m-1} \leq l-3$, $j_{m-2} \leq l-5$ and so on, and so

$$n = a_{j_m} + a_{j_{m-1}} + a_{j_{m-2}} + \cdots + a_{j_1} \leq a_{l-1} + a_{l-3} + a_{j-5} + \cdots + a_\epsilon$$

where $\epsilon = 2$ when l is odd and $\epsilon = 3$ when n is even. Using induction, it is easy to show that

$$\begin{aligned} a_2 + a_4 + \cdots + a_{2k} &= a_{2k+1} - 1 \\ a_3 + a_5 + \cdots + a_{2k-1} &= a_{2k} - 1 \end{aligned}$$

and so we have $a_l \leq n \leq a_{l-1} + a_{l-3} + \cdots + a_\epsilon = a_l - 1$, giving the desired contradiction.

Now let $n \in \mathbf{Z}^+$ and let l be the index for which $a_l \leq n < a_{l+1}$. If $n = a_l$ then we take $m = 1$ and $j_1 = l$ to get the unique representation $n = a_{j_1} = a_l$. Suppose that $n > a_l$. Then we have $n = a_l + (n - a_l)$ with $1 \leq (n - a_l) < a_{l+1} - a_l = a_{l-1}$. We may suppose, inductively, that $n - a_l$ has a unique representation as a sum of distinct non-consecutive Fibonacci numbers, say

$$n - a_l = a_{j_1} + a_{j_2} + \cdots + a_{j_r}.$$

Note that by our above claim, since $n - a_l < a_{l-1}$ we must have $j_j < l-1$. Thus the unique representation for n as a sum of distinct non-consecutive Fibonacci numbers is

$$n = a_{j_1} + a_{j_2} + \cdots + a_{j_r} + a_l.$$

10: Let $(a_1, a_2, \cdots, a_n) \in \mathbf{Z}^n$ with $\sum\limits_{i=1}^{n} a_i = 1$. For $k, l \in \{1, 2, \cdots, n\}$, let

$$S_{kl} = \sum_{i=k}^{l} a_i = \begin{cases} a_k + a_{k+1} + \cdots + a_l & \text{if } k \leq l \leq n, \\ a_k + \cdots + a_n + a_1 + \cdots + a_l & \text{if } 1 \leq l < k. \end{cases}$$

Show that there exists a unique k such that $S_{kl} > 0$ for every l.

Solution: We introduce some terminology. A *unit-sum n-tuple* is an n-tuple $a = (a_1, a_2, \cdots, a_n) \in \mathbf{Z}^n$ with $\sum a_i = 1$. For $k \in \{1, 2, \cdots, n\}$ we write $k * a = (a_k, a_{k+1}, \cdots, a_n, a_1, \cdots, a_{k-1})$. The sums S_{kl} are called the *partial sums* for $k * a$. A *positive shift* for a is an element $k \in \{1, 2, \cdots, n\}$ such that $S_{kl} > 0$ for all l. Note that there is only one unit-sum 1-tuple, namely $a = (1)$, and it has a unique positive shift in $\{1\}$, namely $k = 1$. Fix $n \geq 1$ and suppose, inductively, that every unit-sum n-tuple has a unique positive shift. Let $b = (b_1, b_2, \cdots, b_{n+1})$ be a unit-sum $(n+1)$-tuple. Note that since each $b_i \in \mathbf{Z}$ and $\sum b_i = 1$, we can choose an index m so that $b_m > 0$ and $b_{m+1} \leq 0$ (where we treat indices modulo $n+1$ so that if $m = n+1$ then $m+1 = 1$). By cyclicly permuting the terms b_i, we may suppose that $m = n$ so we have $b_n > 0$ and $b_{n+1} \leq 0$. Construct a unit-sum n-tuple $a = (a_1, a_2, \cdots, a_n)$ be defining $a_i = b_i$ for $1 \leq i < n$ and $a_n = b_n + b_{n+1}$. Note that $k = n+1$ is not a good shift for b because we have $S_{n+1,n+1} = b_{n+1} \leq 0$. For $k \in \{1, 2, \cdots, n\}$, note that k is a good shift for a if and only if k is a good shift for b because $k*a$ and $k*b$ have the same partial sums except that $k*b$ has the one additional partial sum $b_k + b_{k+1} + \cdots + b_{n-1} + b_n = a_k + \cdots + a_{n-1} + b_n > a_k + \cdots + a_{n-1}$ (in the case that $k = n$, this additional partial sum is equal to $b_n > 0$). Since, by the induction hypothesis, a has a unique positive shift, so does b. By induction, for all $n \in \mathbf{Z}^+$, every unit-sum n-tuple has a unique positive shift.

11: Let $n \in \mathbf{Z}^+$. Suppose that n distinct points are chosen on the unit circle and a line segment is drawn between each of the $\binom{n}{2}$ pairs of points and suppose that no three of the line segments are coincident. Let a_n be the number of regions into which the unit disc is divided by these line segments.

(a) Find $a_1, a_2, \cdots, a_5$ and conjecture a formula for a_n.

Solution: By drawing some pictures, you can check that $a_1 = 1$, $a_2 = 2$, $a_3 = 4$ and $a_4 = 8$ and $a_5 = 16$. You will then no doubt be tempted to guess that $a_n = 2^{n-1}$ for all $n \geq 1$, but this is not the case! Indeed you can draw one more picture to see that $a_6 = 31$.

(b) The obvious conjecture from Part (a) is incorrect. Find the correct formula for a_n.

Solution: We claim first that that when a disc is divided into regions by l line segments (no 3 of which intersect) which have p points of intersection inside the circle (not counting the points of intersection that are on the boundary circle), the number of regions is $l+p+1$. We prove this by induction on l. When $l = 0$, we must have $p = 0$ (when there are no line segments, there are certainly no intersection points) so we have $l+p+1 = 0+0+1 = 1$, and indeed when there are no line segments the circle has not been divided so there is 1 region. Thus the claim is true when $l = 0$. Let $k \geq 0$ and suppose (inductively) that the claim is true whenever $l = k$ (that is, whenever there are k line segments). Suppose that we had k line segments with q intersection points in the circle, and then we add one more line segment (so that now there are $l = k+1$ line segments), and suppose that there are r new intersection points which lie along this line (so there are now $p = q+r$ intersection points). By the induction hypothesis, there used to be $k+q+1$ regions before we added the final line. Notice that the r intersection points on the final line segment divide into $r+1$ smaller segments, and each of these segments divides one of the previous regions into two new regions. Thus the number of regions increases by $r+1$. The old number of regions was $k+q+1$, so the new number of regions is $(k+q+1)+(r+1) = (k+1)+(q+r)+1 = l+p+1$, so the claim is still true now that $l = k+1$. By mathematical induction, the claim is true for all $l \geq 1$.

When $n = 1$, so there is one point on the circle, there are no line segments and no points of intersection, and so we have $a_1 = 0+0+1 = 1$. When $n = 2$ there is one line segment and no intersection points, so we have $a_2 = 1+0+1 = 2$. When $n = 3$, there are 3 line segments and no intersection points (inside the circle) so $a_3 = 3+0+1 = 4$. When $n \geq 4$, the number of line segments is $l = \binom{n}{2}$ (since each line segment is determined by its two endpoints, and there are $\binom{n}{2}$ ways to choose the 2 endpoints), and the number of intersection points in the circle is $\binom{n}{4}$ (since each intersection point is determined by the 4 endpoints of the two line segments that contain the point). Thus we have

$$a_n = \binom{n}{2} + \binom{n}{4} + 1\,.$$

If you expand and simplify, you will find that

$$a_n = \frac{n^4 - 6n^3 + 23n^2 - 18n + 24}{24}\,.$$

As you can check, this formula also works when $n = 1$, 2 and 3 (it even works in the case that $n = 0$, that is when there are no points on the circle, and it is not divided, so there is 1 region).

12: Let p be an odd prime and suppose that $U_{p^2} = \langle a \rangle$. Show that $U_{p^k} = \langle a \rangle$ for all $k \geq 2$.

Solution: I may include a solution later.