Lesson 3: Combinatorics

1: Find the number of words of length n on the alphabet $\{0,1\}$ with exactly m blocks of the form 01.

2: Find the number of words of length n on the alphabet $\{0,1,2,3\}$ with an even number of zeros.

3: Find the number of words of length n on the alphabet $\{0,1,2\}$ such that neighbours differ by at most 1.

4: Find the number of words on the alphabet $\{0,1,2\}$ with no neighbouring zeros.

5: Find the number of subsets of $\{1,2,\cdots,n\}$ which do not contain two successive numbers.

6: Find the number of ways to choose two disjoint nonempty subsets from the set $\{1,2,\cdots,n\}$.

7: Find the number of surjective maps from the set $\{1,2,3,4,5,6\}$ to the set $\{1,2,3,4\}$.

8: Find the number of permutations of order 6 in the group of all permutations of $\{1,2,\cdots,8\}$.

9: (a) Into how many regions do n great circles divide the surface of a sphere, given that no three of the great circles intersect at a point?

(b) Into how many regions do n spheres divide space, given that any two of the spheres intersect along a circle, no three intersect along a circle, and no four intersect at a point?

10: Find the number of paths in the set $\left\{(x,y)\in\mathbf{Z}^2 \middle| 0\le y\le x\right\}$ which move always to the right or upwards from the point $(0,0)$ to the point (n,n).

11: $2n$ distinct points lie on a circle. In how many ways can the points be paired so that when all pairs are joined by line segments, then resulting n line segments are disjoint.

12: In how many ways can you triangulate a convex n-gon?

Putnam Problems on Combinatorics

1: (1985 A1) Determine the number of ordered triples of sets (A, B, C) such that

$$A \cup B \cup C = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\} \text{ and } A \cap B \cap C = \emptyset .$$

2: (1986 A4) Find the number of $n \times n$ matrices $A = (a_{ij})$ with each $a_{ij} \in \{-1, 0, 1\}$ such that for every permutation σ of $\{1, 2, \cdots, n\}$ we have $\sum\limits_{k=1}^{n} a_{k,\sigma(k)} = \text{trace}(A)$.

3: (1990 A6) For a finite set A, let $|A|$ denote the number of elements in A. We define an ordered pair (A, B) of subsets of $\{1, 2, \cdots, n\}$ to be *admissible* when we have $a > |B|$ for every $a \in A$ and $b > |A|$ for every $b \in B$. Determine the number of admissible ordered pairs of subsets of $\{1, 2, \cdots, 10\}$.

4: (1991 B5) Let p be prime. Find the number of elements $a \in \mathbf{Z}_p$ such that $a = x^2 = y^2 + 1$ for some $x, y \in \mathbf{Z}_p$.

5: (1992 B2) For $n, k \in \mathbf{N}$, let $c_{n,k}$ be the coefficient of x^k in the expansion of $(1+x+x^2+x^3)^n$. Show that $c_{n,k} = \sum\limits_{i=0}^{k} \binom{n}{i}\binom{n}{k-2i}$.

6: (1993 A3) For $n, k \in \mathbf{Z}^+$, Let $P(n)$ be the set of all subsets of $\{1, 2, \cdots, n\}$, and let $c_{n,k}$ be the number of functions $f : P(n) \to \{1, 2, \cdots, k\}$ with the property that for all $A, B \in P(n)$ we have $f(A \cap B) = \min\{f(A), f(B)\}$. Show that $c_{n,k} = \sum\limits_{i=1}^{k} i^n$.

7: (1996 A3) Fix a set of six courses offered at a university. Each of 20 students chooses to take a (possibly empty) subset of this set of six courses. Determine whether it is necessary that there exist some 5 students and some two courses such that either all 5 students took both courses or none of the 5 students took either course.

8: (1996 B1) We define a subset $A \subseteq \{1, 2, \cdots, n\}$ to be *selfish* when $|A| \in A$ (where $|A|$ denotes the number of elements in A), and we define A to be *minimal selfish* when it is selfish and has no proper selfish subsets. Determine the number of minimal selfish subsets of $\{1, 2, \cdots, n\}$.

9: (1996 B5) For a finite string α on the alphabet $\{0, 1\}$, let $\Delta(\alpha)$ denote the number of 1's in α minus the number of 0's in α. We define such a string to be *balanced* when for every substring β in α, we have $-2 \le \Delta(\beta) \le 2$. Determine the number of balanced strings of length n.

10: (1997 A5) Let N be the number of 10-tuples $(a_1, a_2, \cdots, a_{10})$ with each $a_i \in \mathbf{Z}^+$ such that $\sum\limits_{i=1}^{10} \frac{1}{a_i} = 1$. Determine whether N is even or odd.