Lesson 4: Probability

1: The digits $1, 2, \cdots, 9$ are written on each of 9 cards. Four cards are drawn at random, without replacement, and the cards are arranged in the order in which they were drawn to form a four-digit number. Find the probability that the number is a multiple of 99.

2: Let l and n be positive integers. Let S be the set of l-tuples $(a_1, a_2, \cdots, a_l)$ of non-negative integers a_i such that $\sum_{i=1}^{l} a_i = n$. For $1 \leq k \leq l$, find the probability that when an element $(a_1, \cdots, a_l)$ is chosen at random from the set S, we have $a_i \neq 0$ for exactly k indices i.

3: A fair coin is tossed repeatedly until two consecutive heads are tossed. Find the probability that the coin was tossed n times.

4: A fair coin is tossed repeatedly. One point is scored for each head that turns up and two points are scored for each tail. Find the probability that exactly n points will have been scored after some number of tosses.

5: Let $p(n)$ be the probability that, when the entries of an $n \times n$ matrix A are chosen at random, with replacement, from $\{1, 2, \cdots, 2006\}$, the determinant of A is odd. Find $p(n)/p(n-1)$.

6: (a) Three points, A, B and C, are chosen at random on the unit circle. Find the probability that the centre of the circle lies inside the triangle ABC.

(b) Three points are chosen at random on the unit circle. Find the probability that all three points lie on some semicircle.

7: (a) A point P is chosen at random inside an equilateral triangle. Find the probability that the three perpendiculars from P to the sides of the triangle can be rearranged to form a triangle.

(b) A point P is chosen at random inside an equilateral triangle ABC. Find the probability that one of the triangles ABP, APC and PBC is acute-angled.

8: (a) Two points are chosen at random in the unit interval $[0, 1]$ and the interval is cut at those two points to form three smaller intervals. Find the probability that the three intervals can be rearranged to form a triangle.

(b) A point is chosen at random in the interval $[0, 1]$ and the interval is cut at that point to form two smaller intervals. Find the probability the two intervals can be arranged so that one of them forms the base of an isosceles triangle, and the other forms the angle bisector at one of the base angles.

9: (a) A disc of radius $\frac{1}{2}$ is tossed at random onto the Cartesian plane. Find the probability that it will cover a point with integral coordinates.

(b) A pin of length 1 is tossed at random onto the plane. Find the probability that it will touch one of the lines $x = n$, where n is an integer.

10: Two points are chosen at random inside the unit ball. Find the probability that the distance between the two points is at most 1.

Putnam Problems on Probability

1: (1985 B4) One point p is chosen randomly on a circle S, and another point q is chosen randomly from inside the circle. Let R be the rectangle which is parallel to the coordinate axes and whose diameter is the line segment pq. Find the probability that R lies inside S.

2: (1989 A4) Determine whether it is possible, given any real number $p \in [0,1]$, to design a game to be played by repeatedly flipping a fair coin which ends after finitely many tosses with the probability of success equal to exactly p.

3: (1989 B1) A point is chosen at random inside a square. Find the probability that the point is nearer to the centre of the square than it is to any edge of the square.

4: (1989 B6) Let $n \in \mathbf{Z}^+$ and let $P = \big\{x \in \mathbf{R}^n \big| 0 < x_1 < x_2 < \cdots, x_n < 1\big\}$. For a continuous function $f : [0,1] \to \mathbf{R}$ and for $x \in P$, set $x_0 = 0$ and $x_{n+1} = 1$, and let $S(f,x)$ be the Riemann sum $S(f,x) = \sum\limits_{i=1}^{n+1} f(x_i)(x_i - x_{i-1})$. Show that there exists a polynomial p of degree n with the property that for every continuous map $f : [0,1] \to \mathbf{R}$ with $f(1) = 0$, the expected value of $S(f,x)$, when x is selected at random from P, is equal to $\displaystyle\int_0^1 p(t)f(t)\,dt$.

5: (1992 A6) Four points a, b, c, d are chosen at random on the surface of a sphere. Find the probability that the centre of the sphere lies inside the tetrahedron with vertices a, b, c, d.

6: (1993 B2) Players A and B play a game. A deck of $2n$ cards, numbered from 1 to $2n$, is shuffled then distributed to the players so that each player has n cards. The players then take turns, beginning with player A, and at each turn a player displays one of their cards. The winner is the first player to display a card which causes the sum of all displayed cards to be a multiple of $2n+1$. Determine the probability that played A will win (assuming that both players adopt an optimal strategy).

7: (1993 B3) Two real numbers are chosen at random from the interval $(0,1)$. Find the probability that the nearest integer to y/x is even.

8: (1995 A6) An $n \times 3$ matrix is constructed by choosing each row, at random, to be one of the 6 permutations of $\{1,2,3\}$. The three column sums a, b and c of the matrix are ordered so that $a \le b \le c$. Show that there exists $n \ge 1995$ such that the probability that $a+1 = b = c-1$ is equal to at least 4 times the probability that $a = b = c$.

9: (1996 A2) Let A be the circle $(x+5)^2 + y^2 = 1$, let B be the circle $(x-5)^2 + y^2 = 9$, and let D be the disc $x^2 + y^2 \le 4$. Find the probability that when a point p is chosen at random inside D, there exists $a \in A$ and $b \in B$ such that $p = \frac{a+b}{2}$.

10: (1998 B3) Let H be the hemisphere $z = \sqrt{1 - (x^2 + y^2)}$. Let P be the regular pentagon with vertices at $\big(\cos\theta_k, \sin\theta_k\big)$ with $\theta_k = \frac{2\pi k}{5}$ for $k = 0,1,2,3,4\}$. Find the probability that, when a point is chosen at random on H, it lies directly above some point in P.