Solutions to the Probability Problems

1: The digits $1, 2, \cdots, 9$ are written on each of 9 cards. Four cards are drawn at random, without replacement, and the cards are arranged in the order in which they were drawn to form a four-digit number. Find the probability that the number is a multiple of 99.

Solution: The number of four-digit numbers that can be formed is equal to $9 \cdot 8 \cdot 7 \cdot 6 = 3024$. Let us determine the number of four-digit numbers $abcd$ which are multiples of 99. For $abcd$ to be a multiple of 9 we need $a+b+c+d = 9, 18$ or 27, and in order for $abcd$ to be a multiple of 11, we need $(a+c)-(b+d) = 0$ or ± 11. Write $s = a+c$ and $t = b+d$, so we need $s+t = 9, 18, 27$ and $s-t = 0, \pm 11$. If we had $s-t = \pm 11$ then we would have $\{s,t\} = \{1,12\}, \{2,13\}, \{3,14\}, \{4,15\}, \{5,16\}$ or $\{6,17\}$, but then $s+t = 13, 15, 17, 19, 21$ or 23, so $s+t \neq 9, 18, 27$. Thus we cannot have $s-t = \pm 11$ so we must have $s-t = 0$, that is $s = t$. To get $s+t = 9, 18$ or 27 we must have $s = t = 9$, so each of the pairs $\{a,c\}$ and $\{b,d\}$ must be equal to one of the pairs $\{1,8\}$, $\{2,7\}$, $\{3,6\}$ or $\{4,5\}$. Thus the number of possibilities for the number $abcd$ is equal to $4 \cdot 3 \cdot 2 \cdot 2 = 48$. Thus the required probability is $P = \frac{48}{3024} = \frac{1}{63}$ (which is remarkably high).

2: Nine standard dice are thrown all at once. Find the probability that exactly four distinct numbers are rolled.

Solution: When n fair s-sided dice are thrown at once, the number of possible outcomes is equal to $\binom{n+s-1}{s-1}$. To see this, notice that to each outcome we can associate a word on $\{0,1\}$ of length $n+s-1$ which has exactly $s-1$ of its digits equal to 1 in the following way. For each $i = 1, 2, \cdots, s$, let k_i be the number of dice which rolled the number i (note that $k_1 + k_2 + \cdots + k_s = n$), and let w_i be the word $w_i = 00\cdots 01$ with k_i zeros followed by a one. Note that the word $w = w_1 w_2 \cdots w_s$ is of length $n+s$, has exactly s 1's and ends with a 1. Remove the final 1 to obtain the associated word of length $n+s-1$ with $s-1$ ones.

In a similar way, for $n \geq s$ we can see that when n fair s-sided dice are thrown, the number of outcomes in which each of the s numbers $1, 2, \cdots, s$ is rolled at least once, is equal to $\binom{n-1}{s-1}$. Indeed, we can associate a word as above but replace the words w_i by the shorter words $u_i = 00\cdots 01$ with $k_i - 1$ zeros followed by a one.

Thus when n fair s-sided dice are thrown, the probability that exactly k of the s possible numbers are rolled is equal to

$$P = \frac{\binom{s}{k}\binom{n-1}{k-1}}{\binom{n+s-1}{s-1}}.$$

In particular, when $n = 9$, $s = 6$ and $k = 4$, we have $P = \binom{6}{4}\binom{8}{3} \Big/ \binom{14}{5} = \frac{60}{143}$.

3: A fair coin is tossed repeatedly until two consecutive heads are tossed. Find the probability that the coin was tossed n times.

Solution: Let a_n be the number of ways to toss a coin n times, with no 2 consecutive heads, such that the last toss is a head, and let b_n be the number of ways to toss a head n times, with no 2 consecutive heads, such that the last toss is a tail. Then we have $a_n = b_{n-1}$ and $b_n = b_{n-1} + a_{n-1} = b_{n-1} + b_{n-2}$. Since $b_1 = 1$ and $b_2 = 2$ we see that $b_n = F_{n+1}$ hence $a_n = F_n$, where F_n denotes the n^{th} Fibonacci number. The number of ways to toss a coin n times, with no 2 consecutive heads occurring until the final 2 tosses which are both heads, is equal to $a_{n-1} = F_{n-1}$. Thus the required probability is $P = \frac{F_{n-1}}{2^n}$.

4: A fair coin is tossed repeatedly. One point is scored for each head that turns up and two points are scored for each tail. Find the probability that exactly n points will have been scored after some number of tosses.

Solution: Let p_n denote the required probability, and let $q_n = 1 - p_n$. Note that after n tosses have been made, the number of points scored will be at least n, so to find p_n it suffices to consider sequences of n tosses. Note that of all sequences of n tosses for which a score of $n-1$ was attained at some point, for half of these the next flip was a head so a score of n was then attained, and for the other half the next flip was a tail so a score of n was not attained. On the other hand, for every sequence of n tosses for which a score of $n-1$ was not attained, a score of $n-2$ was attained and the next flip was a tail so a score of n was also attained. It follows that $q_n = \frac{1}{2}\,p_{n-1}$ and that $p_n = \frac{1}{2}\,p_{n-1} + q_{n-1}$, so we have the recurrence equation

$$p_n = \tfrac{1}{2}\,p_{n-1} + \tfrac{1}{2}\,p_{n-2}\,.$$

The characteristic equation is $\lambda^2 - \frac{1}{2}\,\lambda - \frac{1}{2} = 0$, and the roots are $\lambda = -\frac{1}{2}, 1$, and so we have $p_n = a\left(-\frac{1}{2}\right)^n + b$ for some constants a, b. Put in $p_0 = 1$ and $p_1 = \frac{1}{2}$ to get $a + b = 1$ and $-\frac{1}{2}\,a + b = \frac{1}{2}$. Solve these two equations to get $a = \frac{1}{3}$ and $b = \frac{2}{3}$, and so we have $p_n = \frac{1}{3}\left(-\frac{1}{2}\right)^n + \frac{2}{3}$.

5: Let $p(n)$ be the probability that, when the entries of an $n \times n$ matrix A are chosen at random, with replacement, from $\{1, 2, \cdots, 2006\}$, the determinant of A is odd. Find $p(n)/p(n-1)$.

Solution: We can work modulo 2, so $p(n)$ is equal to the probability that when the entries of an $n \times n$ matrix A are chosen at random from the field of two elements $\mathbf{Z}_2 = \{0, 1\}$, the matrix A is invertible. We can count the number c_n of invertible matrices A over $\mathbf{Z}_2$ as follows. The first row must be non-zero so there are $2^n - 1$ choices for the first row. The second row cannot be a multiple of the first row, and there are 2 such multiples, so there are $2^n - 2$ choices for the second row. The third row cannot be a linear combination of the first two rows, and there are 2^2 such linear combinations, so there are $2^n - 2^2$ choices for the third row,. Similarly there are $2^2 - 2^3$ choices for the fourth row and so on. Thus the total number of invertible matrices is

$$\begin{aligned} c_n &= (2^n - 1)(2^n - 2)(2^n - 2^2) \cdots (2^n - 2^{n-1}) \\ &= 1 \cdot 2 \cdot 2^2 \cdot 2^3 \cdots 2^{n-1}(2^n - 1)(2^{n-1} - 1) \cdots (2^1 - 1) \\ &= 2^{n(n-1)/2}(2^n - 1)(2^{n-1} - 1) \cdots (2^2 - 1)(2^1 - 1) \end{aligned}$$

and so, since there are a total of 2^{n^2} $n \times n$ matrices over $\mathbf{Z}_2$, we have

$$p(n) = (2^n - 1)(2^{n-1} - 1) \cdots (2^2 - 1)(2^1 - 1)/2^{n(n+1)/2}\,.$$

Thus $p(n)/p(n-1) = \frac{2^n - 1}{2^n}$.

6: (a) Three points, A, B and C, are chosen at random on the unit circle. Find the probability that the centre of the circle lies inside the triangle ABC.

Solution: By symmetry, we may suppose that $A = (1, 0)$ and that $B = (\cos\theta, \sin\theta)$ with $\theta \in [0, \pi]$. Notice that the centre of the circle lies inside the triangle ABC when we have $C = (\cos\phi, \sin\phi)$ with $\phi \in [\pi, \pi + \theta]$, so the required probability P is the proportion of the rectangle $(\theta, \phi) \in [0, \pi] \times [0, 2\pi]$ where we have $\phi \in [\pi, \pi + \theta]$, and so $P = \frac{1}{4}$.

(b) Three points are chosen at random on the unit circle. Find the probability that all three points lie on some semicircle.

Solution: By symmetry, we may suppose that $A = (1, 0)$ and that $B = (\cos\theta, \sin\theta)$ with $\theta \in [0, \pi]$. Notice that A, B and C lie on some semicircle when we have $C = (\cos\phi, \sin\phi)$ with $\phi \notin [\pi, \pi + \theta]$ (or equivalently when the centre of the circle does not lie in the triangle ABC) so the required probability is $P = \frac{3}{4}$.

7: (a) A point P is chosen at random inside an equilateral triangle. Find the probability that the three perpendiculars from P to the sides of the triangle can be rearranged to form a triangle.

Solution: Let A, B and C be the vertices of the equilateral triangle. Let D, E and F be the midpoints the sides opposite A, B and C, respectively. Let a, b and c be the length of the three perpendiculars from P to the sides opposite A, B and C respectively. Note that $a = b + c$ when P lies on the line segment EF with $a \leq b + c$ when P lies in the trapezoid $BCEF$. In order that the three perpendiculars can be rearranged to form the sides of a triangle, we need to have $a \leq b + c$ and $b \leq a + c$ and $c \leq a + b$, and this occurs when P lies in the triangle DEF. The probability of this occurring is equal to $\frac{1}{4}$.

(b) A point P is chosen at random inside an equilateral triangle ABC. Find the probability that one of the triangles ABP, APC and PBC is acute-angled.

Solution: The angle at P in triangle ABP is equal to $90°$ when P lies on the circle with diameter AB. In order for triangle ABP to be acute-angled, the point P must lie outside the circle with diameter AB. In order for one of the three triangles ABP, APC and PBC to be acute-angled, P must lie outside the intersection of the three discs with diameters AB, BC and CA. If we suppose the sides of ABC have length 2, then the area of the region inside the triangle and outside the intersection of these three discs is $3\left(\frac{\sqrt{3}}{2} - \frac{\pi}{6}\right)$ and the area of triangle ABC is $\sqrt{3}$, and so the required probability is equal to $\sqrt{3}\left(\frac{\sqrt{3}}{2} - \frac{\pi}{6}\right) = \frac{9-\sqrt{3}\,\pi}{6}$

8: (a) Two points are chosen at random in the unit interval $[0, 1]$ and the interval is cut at those two points to form three smaller intervals. Find the probability that the three intervals can be rearranged to form a triangle.

Solution: Let the chosen points be a and b, and consider the case that $a \leq b$ (the other case being symmetrical). The three intervals then have lengths a, $b - a$ and $1 - b$. These can be rearranged to form a triangle provided that the length of the longest does not exceed the sum of the two shorter lengths. We need $a \leq (b - a) + (1 - b) = 1 - a$, that is $a \leq \frac{1}{2}$, and we need $b - a \leq a + (1 - b)$, that is $b \leq a + \frac{1}{2}$, and we need $1 - b \leq a + (b - a) = b$, that is $\frac{1}{2} \leq b$. Thus the required probability P is equal to the proportion of the triangle given by $0 \leq a \leq 1$, $a \leq b \leq 1$ with $a \leq \frac{1}{2}$, $b \leq a + \frac{1}{2}$ and $\frac{1}{2} \leq b$. Thus $P = \frac{1}{4}$.

(b) A point is chosen at random in the interval $[0, 1]$ and the interval is cut at that point to form two smaller intervals. Find the probability the two intervals can be arranged so that one of them forms the base of an isosceles triangle, and the other forms the angle bisector at one of the base angles.

Solution: To begin with, let us consider the isoceles triangle with vertices $a = (-1, 0)$, $b = (1, 0)$ and $c = (0, h)$. The length of ac is $\sqrt{h^2 + 1}$, so the angle bisector at a is the line through $a = (-1, 0)$ and the point $c + (\sqrt{h^2 + 1}, 0) = (\sqrt{h^2 + 1}, h)$; it has equation $y = \frac{h}{1+\sqrt{h^2+1}}$ (1). The line through $b = (1, 0)$ and $c = (0, h)$ has equation $y = -h(x - 1)$ (2). Solve equations (1) and (2) for x to get $(x, y) = \left(\frac{\sqrt{h^2+1}}{2+\sqrt{h^2+1}}, \frac{2h}{2+\sqrt{h^2+1}}\right)$. Verify that $x'(h) = \frac{2h}{(2+\sqrt{h^2+1})^2\sqrt{h^2+1}}$ and $y'(h) = \frac{2(1+2\sqrt{h^2+1})}{(2+\sqrt{h^2+1})^2\sqrt{h^2+1}}$, so we have $x'(h) > 0$ and $y'(h) > 0$ for $h > 0$, and so x and y are both increasing with h. Also $\lim\limits_{h\to 0}(x, y) = \left(\frac{1}{3}, 0\right)$ and $\lim\limits_{x\to\infty}(x, y) = (1, 2)$. Thus the length of the angle bisector from a increases from a minimum of $\frac{4}{3}$ when $h = 0$ to a maximum of $2\sqrt{2}$ when as $h \to \infty$. This shows that the ratio of the length of the angle bisector to the length of the base may be any value between $\frac{2}{3}$ and $\sqrt{2}$. Thus, since $\frac{3}{2} > \sqrt{2}$, the ratio of the shorter of these two segments to the longer of the two can take any value between $\frac{2}{3}$ and 1.

When x is chosen at random in $[0, 1]$, the two resulting segments have lengths x and $1 - x$, and they can be rearranged to form the base and an angle bisector from one of the base angles, in an isosceles triangle, when we have $x \geq \frac{2}{3}(1 - x)$, that is $x \geq \frac{2}{5}$, and also $\frac{2}{3}x \leq (1 - x)$, that is $x \leq \frac{3}{5}$. Thus the probability is $P = \frac{3}{5} - \frac{2}{5} = \frac{1}{5}$.

9: (a) A disc of radius $\frac{1}{2}$ is tossed at random onto the Cartesian plane. Find the probability that it will cover a point with integral coordinates.

Solution: By symmetry, we may assume the centre of the disc lands in the square $[0,1]\times[0,1]$. The disc covers one of the vertices when its centre lies in one of the four quarter-discs centred at each vertex. The combined area of these four quarter-discs is $\frac{\pi}{4}$ and so the required probability is $P=\frac{\pi}{4}$.

(b) A pin of length 1 is tossed at random onto the plane. Find the probability that it will touch one of the lines $x=n$, where n is an integer.

Solution: The head of the pin is equally likely to fall between any of the two lines $x=n$ so we may assume the x-coordinate where the head of the pin lands is chosen at random from $x\in[0,1]$. Let θ be the angle in which the pin points (from head to tip, measured counterclockwise from the positive x-axis). We may suppose that $\theta\in[0,\pi]$, as the case that $\theta\in[\pi,2\pi]$ is symmetrical. Then if $0\le\theta\le\cos^{-1}(1-x)$ then the pin will touch the line $x=1$, while if $0\le\pi-\theta\le\cos^{-1}x$, that is if $\pi-\cos^{-1}x\le\theta\le\pi$, then the pin touches the line $x=0$. Thus the required probability P is equal to the proportion of the rectangle $(x,\theta)\in[0,1]\times[0,2\pi]$ with $0\le\theta\le\cos^{-1}(1-x)$ or $\pi-\cos^{-1}x\le\theta\le\pi$. By symmetry, we may use the bottom half of the rectangle, given by $\theta\in[0,\pi]$, and by turning the rectangle on its side, we see that P is equal to the proportion of the rectangle $(\theta,x)\in[0,\pi]\times[0,1]$ with $0\le x\le\cos\theta$. Thus the probability the pin touches a line is

$$P=\frac{\int_0^{\pi/2}\cos\theta\,d\theta}{\pi/2}=\frac{2}{\pi}\,.$$

10: Two points are chosen at random inside the unit sphere. Find the probability that the distance between the two points is at most 1.

Solution: First we note that the volume of the spherical cap which is obtained by revolving the region $a\le x\le 1$, $0\le y\le\sqrt{1-x^2}$ is equal to

$$V(a)=\int_a^1\pi(1-x^2)\,dx=\pi\Big[x-\tfrac{1}{3}\,x^3\Big]_a^1=\pi\left(\tfrac{2}{3}-a+\tfrac{1}{3}\,a^3\right)$$

By symmetry, we may suppose that the first point is chosen along the positive x-axis. For small Δr, the probability that the first point is of the form $(x,0,0)$ with $x\in[r,r+\Delta r]$ is equal to $\frac{4\pi r^2}{\frac{4}{3}\pi}=3r^2\Delta r$, and so the required probability is

$$P=\int_0^1 3r^2\,\rho(r)\,dr$$

where $\rho(r)$ is the probability that the second point lies at most 1 unit away from the first point given that the first point has been chosen at $(r,0,0)$.

Having chosen the first point at $(r,0,0)$, the second point lies at most 1 unit away provided that it lies in the union of two spherical caps; one is the portion of the unit sphere with $\frac{r}{2}\le x\le 1$, and the other is the reflection of the first in the plane $x=\frac{r}{2}$. Thus

$$\rho(r)=\frac{2V\left(\frac{r}{2}\right)}{\frac{4}{3}\,\pi}=\frac{2\pi\left(\frac{2}{3}-\frac{r}{2}+\frac{r^3}{24}\right)}{\frac{4}{3}\,\pi}=1-\tfrac{3}{4}\,r+\tfrac{1}{16}\,r^3\,.$$

Thus the required probability is

$$P=\int_0^1 3r^2\,\rho(r)\,dr=\int_0^1 3r^2-\tfrac{9}{4}\,r^3+\tfrac{3}{16}\,r^5\;dr=\Big[r^3-\tfrac{9}{16}\,r^4+\tfrac{1}{32}\,r^6\Big]_0^1=1-\tfrac{9}{16}+\tfrac{1}{32}=\tfrac{15}{32}$$