Lesson 6: Sequences, Series and Products

1: Let $a_1 = 1$ and for $n \geq 1$ let $a_{n+1} = \dfrac{6}{a_n + 1}$. Determine whether $\{a_n\}$ converges, and if so then find the limit.

2: Let $a_1 = 1$, $a_2 = 2$, and for $n > 2$ let $a_n = \sqrt{a_{n-1}} + \sqrt{a_{n-2}}$. Determine whether $\{a_n\}$ converges, and if so then find the limit.

3: Let $a_1 = \sqrt{2}$, for $n \geq 1$ let $a_{n+1} = \sqrt{2 + a_n}$, and then let $b_n = 4^n(2 - a_n)$. Determine whether $\{b_n\}$ converges, and if so then find the limit.

4: Let $a_n = \left(\dfrac{n^n}{n!}\right)^{1/n}$. Determine whether $\{a_n\}$ converges, and if so then find the limit.

5: For a real number x, let $\langle x \rangle$ denote the fractional part of x, that is $\langle x \rangle = x - \lfloor x \rfloor$. Show that if x is irrational then the sequence $\{\langle nx \rangle\}$ is dense in $[0, 1]$.

6: (a) Find $\displaystyle\sum_{k=2}^{n} \frac{1}{\log_k e}$. (b) Find $\displaystyle\sum_{k=1}^{n} (2k-1)^3$.

7: (a) Find $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{n+i}$. (b) Find $\displaystyle\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{\sqrt{n^2+i^2}}$.

8: (a) Find $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{3n-2}$. (b) Find $\displaystyle\sum_{n=1}^{\infty} \frac{(n+1)^2}{n!}$.

9: Find $\displaystyle\sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} k^2\right)^{-1}$.

10: Find $\displaystyle\int_0^{\pi} \sin x \, dx$ by evaluating the limit of a sequence of Riemann sums.

11: Let $a_n > 0$. Show that $\sum_{n=1}^{\infty} a_n$ converges if and only if $\prod_{n=1}^{\infty} (1 + a_n)$ converges.

12: Find $\displaystyle\prod_{n=0}^{\infty} \left(1 + \frac{1}{2^{2^n}}\right)$.

13: Find $\displaystyle\prod_{n=2}^{\infty} \frac{n^3 - 1}{n^3 + 1}$.

Putnam Problems on Sequences, Series and Products

1: (1985 A3) Fix a real number d. For each positive integer n, let $a_n(0) = d/2^n$ and for $j \geq 0$ let $a_n(j+1) = a_n(j)^2 + 2a_n(j)$. Find $\lim\limits_{n\to\infty} a_n(n)$.

2: (1987 A6) For a positive integer n, let $a(n)$ be the number of zeros in the base 3 representation of n. Find the set of all real numbers x such that the series $\displaystyle\sum_{n=1}^{\infty} \frac{x^{a(n)}}{n^3}$ converges.

3: (1988 B4) Let $\{a_n\}$ be a sequence of positive real numbers. Show that if the series $\sum\limits_{n=1}^{\infty} a_n$ converges then so does $\sum\limits_{n=1}^{\infty} {a_n}^{n/(n+1)}$.

4: (1990 A2) Determine whether there exist sequences $\{a_n\}$ and $\{b_n\}$ of positive integers such that $\lim\limits_{n\to\infty} \left(\sqrt[3]{a_n} - \sqrt[3]{b_n} \right) = \sqrt{2}$.

5: (1993 A2) Let $\{a_n\}$, $n \geq 0$ be a sequence of real numbers such that ${a_n}^2 = 1 + a_{n-1}a_{n+1}$ for all $n \geq 1$. Show that there is a real number c such that $a_n = c\,a_{n-1} - a_{n-2}$ for all $n \geq 2$.

6: (1993 A6) Let $\{a_n\} = \{2, 3, 3, 2, 3, 3, 3, 2, 3, 3, 3, 2, 3, 3, 2, \cdots\}$ be the infinite sequence of 2s and 3s, with $a_1 = 2$, which is determined by the following property: if we form the sequence $\{b_n\}$, $n \geq 1$ by defining b_n to be equal to the number of 3s between the n^{th} and the $(n+1)^{st}$ 2s in $\{a_n\}$, then we have $b_n = a_n$ for all $n \geq 1$. Show that there exists a real number r such that for all n, we have $a_n = 2 \iff n = 1 + \lfloor rk \rfloor$ for some positive integer k.

7: (1995 B4) Find positive integers a, b, c, d such that $\sqrt[8]{2207 - \cfrac{1}{2207 - \cfrac{1}{2207 - \cdots}}} = \dfrac{a + b\sqrt{c}}{d}$.

8: (1996 B2) Show that $\left(\dfrac{2n-1}{e}\right)^{\frac{2n-1}{2}} < \displaystyle\prod_{k=1}^{n} (2k-1) < \left(\dfrac{2n+1}{e}\right)^{\frac{2n+1}{2}}$ for all $1 \leq n \in \mathbf{Z}$.

9: (1997 B1) For a real number x, let $\langle x \rangle$ denote the distance between x and the nearest integer. For a positive integer n, evaluate $\displaystyle\sum_{k=1}^{6n-1} \min\left(\left\langle \frac{k}{6n} \right\rangle, \left\langle \frac{k}{3n} \right\rangle \right)$.

10: (1999 B3) Let $A = \left\{ (x,y) \in \mathbf{R}^2 \middle| 0 \leq x, y < 1 \right\}$. Define $S : A \to \mathbf{R}$ by $S(x,y) = \displaystyle\sum_{\frac{1}{2} \leq \frac{m}{n} \leq 2} x^m y^n$, where the sum ranges over pairs (m,n) of positive integers with $\frac{1}{2} \leq \frac{m}{n} \leq 2$. Evaluate the limit $\lim\limits_{\substack{(x,y)\to(1,1) \\ (x,y)\in A}} (1 - xy^2)(1 - x^2y)S(x,y)$.