Lesson 7: Derivatives and Integrals

1: Let $0 < k < 1$, and let $f(x)$ be differentiable with $f'(x) \leq k$ for all $x \in \mathbf{R}$. Show that $f(x)$ has a fixed point.

2: Suppose that $f(x)$ is differentiable for all $0 \neq x \in \mathbf{R}$, continuous at $x = 0$, and $\lim_{x \to 0} f'(x)$ exists and is finite. Does it follow that $f(x)$ is differentiable at $x = 0$?

3: A person walks 6 kilometers in one hour, at varying speed. Show that at some point along the way, the person walks 1 kilometer in exactly 10 minutes.

4: Let $f(x)$ be $\mathcal{C}^\infty$ on $\mathbf{R}$ with $f\left(\frac{1}{n}\right) = 0$ for all positive integers n. Show that $f^{(k)}(0) = 0$ for all positive integers k.

5: A car with tires of radius r drives at constant velocity v. Find the maximum height which can be reached by a particle which is thrown from the tire.

6: Let $y = f(x)$ be the solution to the differential equation $y^2 y'' + 1 = 0$ with $y(0) = 2$ and $y'(0) = 0$. Find the value of $x > 0$ such that $f(x) = 1$.

7: Let $f(x)$ be differentiable with $f(0) = 0$ and $0 \leq f'(x) \leq \left|f(x)\right|$ for all $x \in \mathbf{R}$. Show that $f(x) = 0$ for all $x \in \mathbf{R}$.

8: Let $f(x)$ be integrable on $[0,1]$ with $\displaystyle\int_0^1 f(x)\,dx = 1$ and $\displaystyle\int_0^1 x\,f(x)\,dx = 1$. Show that $\displaystyle\int_0^1 \big(f(x)\big)^2\,dx \geq 4$.

9: Let $f(x)$ be continuous on $[0,1]$. Show that $\displaystyle\int_{x=0}^1 \int_{y=x}^1 \int_{z=x}^y f(x)f(y)f(z)\,dx\,dy\,dz = \frac{1}{3!}\left(\int_{t=0}^1 f(t)\,dt\right)^3$.

10: Evaluate each of the following integrals.

(a) $\displaystyle\int_0^{\pi/2} \frac{dx}{1+\sqrt{\tan x}}$ (b) $\displaystyle\int_0^{\pi/2} \frac{dx}{1+(\tan x)^{\sqrt{2}}}$ (c) $\displaystyle\int_0^{\pi} \ln(\sin x)\,dx$

(d) $\displaystyle\int_0^{\infty} \frac{\ln x}{1+x^2}\,dx$ (e) $\displaystyle\int_0^{\infty} \frac{\tan^{-1}(\pi x) - \tan^{-1}(x)}{x}\,dx$ (f) $\displaystyle\int_0^1 \int_0^1 \frac{dx\,dy}{1-xy}$

Putnam Problems on Derivatives and Integrals

1: (1987 A3) Let $y = f(x)$ be a real-valued function such that $y'' - 2y' + y = 2e^x$ for all $x \in \mathbf{R}$.
(a) If $f(x) > 0$ for all $x \in \mathbf{R}$ then must we also have $f'(x) > 0$ for all $x \in \mathbf{R}$?
(b) If $f'(x) > 0$ for all $x \in \mathbf{R}$ then must we also have $f(x) > 0$ for all $x \in \mathbf{R}$?

2: (1987 B1) Find $\displaystyle\int_2^4 \frac{\sqrt{\ln(9-x)}\,dx}{\sqrt{\ln(9-x)} + \sqrt{\ln(x+3)}}$.

3: (1989 A2) Let $a, b > 0$. Evaluate $\displaystyle\int_{x=0}^{a} \int_{y=0}^{b} e^{\max\{b^2x^2, a^2y^2\}}\,dy\,dx$.

4: (1990 B1) Find every real-valued continuously differentiable function $f(x)$ such that we have $f(x)^2 = 1990 + \displaystyle\int_0^x f(t)^2 + f'(t)^2\ dt$ for all $x \in \mathbf{R}$.

5: (1991 A5) Find the maximum possible value of $\displaystyle\int_0^y \sqrt{x^4 + (y - y^2)^2}\ dx$ where $0 \le y \le 1$.

6: (1992 A2) For $a \in \mathbf{R}$, let $c(a)$ be the coefficient of x^{1992} in the binomial series for $(1+x)^a$. Evaluate $\displaystyle\int_0^1 c(-y-1)\left(\tfrac{1}{y+1} + \tfrac{1}{y+2} + \tfrac{1}{y+3} + \cdots + \tfrac{1}{y+1992}\right) dy$.

7: (1994 B3) Find the set of all real numbers k with the property that for every differentiable function $f(x)$ with $f'(x) > f(x) > 0$ for all $x \in \mathbf{R}$, there exists a number N such that $f(x) > e^{kx}$ for all $x > N$.

8: (1995 B2) An ellipse which is congruent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ rolls without slipping along the curve $y = c\sin\left(\frac{x}{a}\right)$. Find the relationship between a, b and c which ensures that the ellipse completes exactly one revolution as it traverses one period of the curve.

9: (1997 A3) Find $\displaystyle\int_0^\infty \left(x - \tfrac{x^3}{2} + \tfrac{x^5}{2\cdot 4} - \tfrac{x^7}{2\cdot 4\cdot 6} + \cdots\right)\left(1 + \tfrac{x^2}{2^2} + \tfrac{x^4}{2^2\cdot 4^2} + \tfrac{x^6}{2^2\cdot 4^2\cdot 6^2} + \cdots\right) dx$

10: (1998 A3) Let $f(x)$ be a real-valued function such that $f'''(x)$ is continuous for all $x \in \mathbf{R}$. Show that there exists a point $a \in \mathbf{R}$ such that $f(a)\cdot f'(a)\cdot f''(a)\cdot f'''(a) \ge 0$.