Solutions to the Problems on Derivatives and Integrals

1: Let $0 < k < 1$, and let $f(x)$ be differentiable with $f'(x) \leq k$ for all $x \in \mathbf{R}$. Show that $f(x)$ has a fixed point.

Solution: If $f(0) = 0$ then 0 is a fixed point of f. Suppose that $f(0) = b > 0$. For any $x > 0$, by the Mean Value Theorem, we can find $c \in (0, x)$ so that $f'(c) = \frac{f(x)-f(0)}{x-0} = \frac{f(x)-b}{x}$ so $f(x) = b + f'(c)\,x \leq b + k\,x$. Notice that the line $y = b + k\,x$ intersects the line $y = x$ when $x = \frac{b}{1-k}$. Let $g(x) = f(x) - x$. Then $g(0) = b > 0$ and $g\left(\frac{b}{1-k}\right) = f\left(\frac{b}{1-k}\right) - \frac{b}{1-k} \leq b + k\left(\frac{b}{1-k}\right) - \frac{b}{1-k} = 0$ so, by the Intermediate Value Theorem, there is a point $x \in \left[0, \frac{b}{1-k}\right]$ such that $g(x) = 0$, and this point x is a fixed point of f. Similarly, if $f(0) = b < 0$ then we have $f(x) \geq b + k\,x$ for all $x < 0$, and $f(x)$ has a fixed point $x \in \left[\frac{b}{1-k}, 0\right]$.

2: Suppose that $f(x)$ is differentiable for all $0 \neq x \in \mathbf{R}$, continuous at $x = 0$, and $\lim\limits_{x\to 0} f'(x)$ exists and is finite. Does it follow that $f(x)$ is differentiable at $x = 0$?

Solution: It does. Let $a = \lim\limits_{x\to 0} f'(x)$. Given $\epsilon > 0$, choose $\delta > 0$ so that $0 < |x| < \delta \Longrightarrow |f'(x) - a| < \epsilon$. Let $0 < u < \delta$. Since $f(x)$ is differentiable in $(0, u)$ and continuous on $[0, u]$, by the Mean Value Theorem we can find $x \in (0, u)$ so that $f'(x) = \frac{f(u)-f(0)}{u-0}$, and so we have $\left|\frac{f(u)-f(0)}{u-0} - a\right| = |f'(x) - a| < \epsilon$. Similarly, when $-\delta < u < 0$ we also have $\left|\frac{f(u)-f(0)}{u-0} - a\right| < \epsilon$. Thus $f'(0) = a$.

3: A person walks 6 kilometers in one hour, at varying speed. Show that at some point along the way, the person walks 1 kilometer in exactly 10 minutes.

Solution: For $t \in [0, 60]$, let $x(t)$ be the distance walked, in kilometers, up until time t, in minutes. We suppose that this person does not know how to teleport, so $x(t)$ is continuous. For $x \in [0, 50]$, let $f(t) = x(t+10) - x(t)$. Consider the six values $f(k)$ with $k \in \{0, 10, 20, 30, 40, 50\}$. We have $f(0)+f(10)+f(20)+f(30)+f(40)+f(50) = \big(x(10) - x(0)\big) + \big(f(20) - f(10)\big) + \cdots + \big(x(60) - x(50)\big) = x(60) - x(0) = 60 - 0 = 6$, so it is not possible that every $f(k) > 1$ and it is not possible that every $f(k) < 1$. Thus we have $f(k) \leq 1$ and $f(l) \geq 1$ for some $k, l \in \{0, 10, 20, 30, 40, 50\}$. By the Intermediate Value Theorem, there is a number c between k and l such that $f(c) = 1$, that is $x(c+10) - x(c) = 1$. In the interval $[c, c+10]$ the person walks 1 kilometer.

4: Let $f(x)$ be $\mathcal{C}^\infty$ on $\mathbf{R}$ with $f\left(\frac{1}{n}\right) = 0$ for all positive integers n. Show that $f^{(k)}(0) = 0$ for all positive integers k.

Solution: By the Mean Value Theorem, we can find points $c_n^1 \in \left(\frac{1}{n+1}, \frac{1}{n}\right)$ with $f'\left(c_n^1\right) = \frac{f\left(\frac{1}{n}\right)-f\left(\frac{1}{n+1}\right)}{\frac{1}{n}-\frac{1}{n+1}} = 0$. Since the sequence $\{c_n^1\}$ decreases to 0, and $f'(x)$ is continuous, we have $f'(0) = 0$. Again by the Mean Value Theorem, we can find points $c_n^2 \in (c_{n+1}^1, c_n^1)$ with $f''(c_n^2) = \frac{f'(c_{n+1}^1)-f'(c_n^1)}{c_{n+1}^1-c_n^1} = 0$. Note that this second sequence $\{c_n^2\}$ also decreases to 0, so since $f''(x)$ is continuous, we have $f''(0) = 0$. This argument can be repeated.

5: A car with tires of radius r drives at constant velocity v. Find the maximum height which can be reached by a particle which is thrown from the tire.

Solution: Note that when an object is thrown vertically from an initial height of h with an initial velocity of v, its height is given by $y(t) = -\frac{g}{2}\,t^2 + v\,t + h = -\frac{g}{2}\left(t - \frac{v}{g}\right)^2 + \left(h + \frac{v^2}{2g}\right)$, so it rises to a maximum height $h + \frac{v^2}{2g}$. A particle on the tire follows the cycloid $(x, y) = r\,(\theta - \sin\theta, 1 - \cos\theta)$ where $r\theta = vt$. Its velocity is $(x', y') = r\theta'(1 - \cos\theta, \sin\theta) = v(1 - \cos\theta, \sin\theta)$. If the particle is thrown from the tire when the angle is θ, then its height is $h = r(1 - \cos\theta)$ and the vertical component of its velocity is $v = v\sin\theta$, so it rises to a maximum height of $H(\theta) = r(1 - \cos\theta) + \dfrac{(v\sin\theta)^2}{2g}$. We have $H'(\theta) = r\sin\theta + \frac{v^2}{g}\sin\theta\cos\theta = \sin\theta\left(r + \frac{v^2}{g}\cos\theta\right)$. For $\theta \in [0, \pi]$, $H'(\theta) = 0$ when $\theta = 0$ or $\theta = \pi$ and when $\cos\theta = -\frac{rg}{v^2}$. Note that $H(0) = 0$ and $H(\pi) = 2r$. If $\frac{rg}{v^2} > 1$ then $\cos\theta \neq -\frac{rg}{v^2}$ so the maximum value of H is $H(\pi) = 2r$. If $\frac{rg}{v^2} < 1$ then when $\cos\theta = -\frac{rg}{v^2}$ we have $H(\theta) = r(1 - \cos\theta) + \frac{v^2}{2g}\sin^2\theta = r - r\cos\theta + \frac{v^2}{2g} - \frac{v^2}{2g}\cos^2\theta = r + \frac{r^2 g}{v^2} + \frac{v^2}{2g} - \frac{v^2}{2g}\,\frac{r^2 g^2}{v^4} = r + \frac{r^2 g}{2v^2} + \frac{v^2}{2g} > 2r$, so the maximum value of H is $r + \frac{r^2 g}{2v^2} + \frac{v^2}{2g}$.

6: Let $y = f(x)$ be the solution to the differential equation $y^2 y'' + 1 = 0$ with $y(0) = 2$ and $y'(0) = 0$. Find the value of $x > 0$ such that $f(x) = 1$.

Solution: Write $y' = u$ and $y'' = u\,u'$, where $u' = \frac{du}{dy}$. The the differential equation becomes $y^2 u\,u' + 1 = 0$. This is separable, so we write it as $u\,du = -\frac{1}{y^2}\,dy$ and integrate both sides to get $\frac{1}{2}u^2 = \frac{1}{y} + a$. Put in $y = 2$ and $u = y' = 0$ to get $0 = \frac{1}{2} + a$, so $a = -\frac{1}{2}$ and we have $\frac{1}{2}u^2 = \frac{1}{y} - \frac{1}{2}$, that is $u = \pm\sqrt{\frac{2}{y} - 1}$. Notice that for $x > 0$ we should use the negative sign since $u = y'$ is initially 0 and $y'' = -\frac{1}{y^2} < 0$. Rewrite $u = -\sqrt{\frac{2}{y} - 1}$ as $y' = -\sqrt{\frac{2}{y} - 1}$. This is again separable, so we write it as $\dfrac{dy}{\sqrt{\frac{2}{y} - 1}} = -dx$ and integrate both sides. To find $\displaystyle\int \frac{dy}{\sqrt{\frac{2}{y} - 1}}$, first let $u = \sqrt{y}$ and then let $\sqrt{2}\sin\theta = u$ to get $\displaystyle\int \frac{dy}{\sqrt{\frac{2}{y} - 1}} = \int \frac{\sqrt{y}\,dy}{\sqrt{2 - y}} = \int \frac{2u^2\,du}{\sqrt{2 - u^2}} = \int 4\sin^2\theta\;d\theta = \int 2 - 2\cos 2\theta\;d\theta = 2\theta - \sin 2\theta = 2\theta - 2\sin\theta\cos\theta = 2\sin^{-1}\frac{u}{\sqrt{2}} - u\sqrt{2 - u^2} = 2\sin^{-1}\sqrt{\frac{y}{2}} - \sqrt{y}\sqrt{2 - y}$. Thus we obtain $2\sin^{-1}\sqrt{\frac{y}{2}} - \sqrt{y}\sqrt{2 - y} = -x + b$. Put in $x = 0$ and $y = 2$ to get $b = \pi$, so we have $2\sin^{-1}\sqrt{\frac{y}{2}} - \sqrt{y}\sqrt{2 - y} = \pi - x$. Finally, put in $y = 1$ to get $2\,\frac{\pi}{4} - 1 = \pi - x$ so $x = 1 + \frac{\pi}{2}$.

7: Let $f(x)$ be differentiable with $f(0) = 0$ and $0 \le f'(x) \le \big|f(x)\big|$ for all $x \in \mathbf{R}$. Show that $f(x) = 0$ for all $x \in \mathbf{R}$.

Solution: Since $f'(x) \ge 0$ for all x, $f(x)$ is non-decreasing, and since $f(0) = 0$ we have $f(x) \ge 0$ for all $x \ge 0$ and we have $f(x) \le 0$ for all $x \le 0$. For $x \ge 0$ we have $f'(x) \le f(x) \Longrightarrow f'(x) - f(x) \le 0 \Longrightarrow e^{-x}f'(x) - e^{-x}f(x) \le 0 \Longrightarrow \frac{d}{dx}\,e^{-x}f(x) \le 0 \Longrightarrow e^{-x}f(x)$ is non-increasing. Since $e^0 f(0) = 0$, we have $e^{-x}f(x) \le 0$, and hence $f(x) \le 0$, for all $x \ge 0$. But earlier we saw that $f(x) \ge 0$ for all $x \ge 0$, and so $f(x) = 0$ for all $x \ge 0$. A similar argument shows that $f(x) = 0$ for all $x \le 0$.

8: Let $f(x)$ be integrable on $[0, 1]$ with $\displaystyle\int_0^1 f(x)\,dx = 1$ and $\displaystyle\int_0^1 x\,f(x)\,dx = 1$. Show that $\displaystyle\int_0^1 \big(f(x)\big)^2\,dx \ge 4$.

Solution: Notice first that $\displaystyle\int_0^1 (6x - 2)\,dx = \Big[3x^2 - 2x\Big]_0^1 = 1$ and $\displaystyle\int_0^1 x\,(6x - 2)\,dx = \Big[2x^3 - x^2\Big]_0^1 = 1$ and $\displaystyle\int_0^1 (6x - 2)^2\,dx = \Big[\tfrac{1}{18}(6x - 2)^3\Big]_0^1 = \frac{64}{18} + \frac{8}{18} = 4$. Thus $\displaystyle 0 \le \int_0^1 \big(f(x) - (6x - 2)\big)^2\,dx = \int_0^1 f(x)^2\,dx - 12\int_0^1 x\,f(x)\,dx + 4\int_0^1 f(x)\,dx + \int_0^1 (6x - 2)^2\,dx = \int_0^1 f(x)^2\,dx - 12 + 4 + 4$, and so $\displaystyle\int_0^1 f(x)^2\,dx \ge 4$.

9: Let $f(x)$ be continuous on $[0, 1]$. Show that $\displaystyle\int_{x=0}^1\int_{y=x}^1\int_{z=x}^y f(x)f(y)f(z)\,dx\,dy\,dz = \frac{1}{3!}\left(\int_{t=0}^1 f(t)\,dt\right)^3$.

Solution: Let $g(x,y,z) = f(x)f(y)f(z)$ and let $\displaystyle I = \int_{x=0}^1\int_{y=x}^1\int_{z=x}^y g(x,y,z)\,dx\,dy\,dz = \int_{0\le x\le z\le y\le 1} g(x,y,z)\,dx\,dy\,dz$. Notice that the value of $g(x,y,z)$ is invariant under permutation of the variables x, y and z, and so we have $\displaystyle I = \int_{0\le x\le z\le y\le 1} g(x,y,z)\,dz = \int_{0\le x\le y\le z\le 1} g(x,y,z)\,dz = \int_{0\le y\le x\le z\le 1} g(x,y,z)\,dz = \int_{0\le y\le z\le x\le 1} g(x,y,z)\,dz = \int_{0\le z\le x\le y\le 1} g(x,y,z)\,dz = \int_{0\le z\le y\le x\le 1} g(x,y,z)\,dz$. Adding these 6 integrals gives $\displaystyle 6I = \int_{x=0}^1\int_{y=0}^1\int_{z=0}^1 g(x,y,z)\,dx\,dy\,dz = \int_{x=0}^1 f(x)\,dx\int_{x=0}^1 f(y)\,dy\int_{x=0}^1 f(z)\,dz = \left(\int_0^1 f(t)\,dt\right)^3$, and so we have $I = \frac{1}{6}\left(\displaystyle\int_0^1 f(t)\,dt\right)^3$, as required.

10: Evaluate each of the following integrals.

(a) $\displaystyle\int_0^{\pi/2} \frac{dx}{1+\sqrt{\tan x}}$ (b) $\displaystyle\int_0^{\pi/2} \frac{dx}{1+(\tan x)^{\sqrt{2}}}$

Solution: We can solve parts (a) and (b) together. More generally, we let $I = \displaystyle\int_{x=0}^{\pi/2} \frac{dx}{1+(\tan x)^a}$. Make the substitution $u = \frac{\pi}{2} - x$. Then $I = -\displaystyle\int_{u=\pi/2}^{0} \frac{du}{1+(\cot u)^a} = \int_{u=0}^{\pi/2} \frac{(\tan u)^a\,du}{1+(\tan u)^a} = \int_{x=0}^{\pi/2} \frac{(\tan x)^a\,dx}{1+(\tan x)^a}$ and so $2I = \displaystyle\int_{x=0}^{\pi/2} \frac{1}{1+(\tan x)^a} + \frac{(\tan x)^a}{1+(\tan x)^a}\,dx = \int_{x=0}^{\pi/2} 1\,dx = \frac{\pi}{2}$. Thus $I = \frac{\pi}{4}$.

(c) $\displaystyle\int_0^{\pi} \ln(\sin x)\,dx$

Solution: Let $I = \displaystyle\int_0^{\pi} \ln(\sin x)\,dx$. By symmetry we have $I = 2\displaystyle\int_0^{\pi/2} \ln(\sin x)\,dx = 2\int_0^{\pi/2} \ln(\cos x)\,dx$. Let $u = \frac{1}{2}x$. Then $I = \displaystyle\int_{u=0}^{\pi/2} 2\ln(\sin 2u)\,du = 2\int_{u=0}^{\pi/2} \ln(2\sin u\cos u)\,du = 2\int_0^{\pi/2} \ln 2 + \ln(\sin u) + \ln(\cos u)\,du =$ $\pi\ln 2 + 2I$. Thus $I = -\pi\ln 2$.

(d) $\displaystyle\int_0^{\infty} \frac{\ln x}{1+x^2}\,dx$

Solution: Let $I = \displaystyle\int_{x=0}^{\infty} \frac{\ln x}{1+x^2}\,dx$. Let $u = \frac{1}{x}$. Then $I = \displaystyle\int_{u=\infty}^{0} \frac{\ln\left(\frac{1}{u}\right)}{1+\frac{1}{u^2}}\left(-\tfrac{1}{u^2}\right)du = \int_{u=0}^{\infty} \frac{-\ln u}{u^2+1}\,du = -I$. Thus $I = 0$.

(e) $\displaystyle\int_0^{\infty} \frac{\tan^{-1}(\pi x) - \tan^{-1}(x)}{x}\,dx$

Solution: More generally, let $I = \displaystyle\int_{x=0}^{\infty} \frac{\tan^{-1}(ax) - \tan^{-1}(x)}{x}\,dx$. Then $I = \displaystyle\int_0^{\infty} \left[\frac{\tan^{-1}(ux)}{x}\right]_{u=1}^{a} dx =$ $\displaystyle\int_{x=0}^{\infty}\int_{u=1}^{a} \frac{1}{1+(ux)^2}\,du\,dx = \int_{u=1}^{a}\int_{x=0}^{\infty} \frac{1}{1+(ux)^2}\,dx\,du = \int_{u=1}^{a} \left[\frac{\tan^{-1}(ux)}{u}\right]_{x=0}^{\infty} du = \int_{u=1}^{a} \frac{\pi}{2u}\,du = \left[\tfrac{\pi}{2}\ln u\right]_{u=1}^{a}$ $= \frac{\pi}{2}\ln a$, where we used Fubini's Theorem to interchange the order of integration.

(f) $\displaystyle\int_0^1\int_0^1 \frac{dx\,dy}{1-xy}$

Solution: For $0 < b < 1$, $\displaystyle\int_{y=0}^{b}\int_{x=0}^{1} \frac{1}{1-xy}\,dx\,dy = \int_{y=0}^{b}\int_{x=0}^{1} \sum_{n=0}^{\infty}(xy)^n\,dx\,dy = \int_{y=0}^{b}\sum_{n=0}^{\infty}\int_{x=0}^{1}(xy)^n\,dx\,dy =$ $\displaystyle\int_{y=0}^{b}\sum_{n=0}^{\infty}\frac{y^n}{n+1}\,dy = \sum_{n=0}^{\infty}\int_{y=0}^{b}\frac{y^n}{n+1}\,dy = \sum_{n=0}^{\infty}\frac{b^{n+1}}{(n+1)^2} = \sum_{n=1}^{\infty}\frac{b^n}{n^2}$, where we were able to interchange integration and summation twice by uniform convergence. Thus $\displaystyle\int_{y=0}^{1}\int_{x=0}^{1}\frac{1}{1-xy}\,dx\,dy = \lim_{b\to 1^-}\sum_{n=1}^{\infty}\frac{b^n}{n^2} = \sum_{n=0}^{\infty}\frac{1}{n^2}$ by Abel's Theorem. It is well-known that $\displaystyle\sum_{n=0}^{\infty}\frac{1}{n^2} = \zeta(2) = \frac{\pi^2}{6}$, so $\displaystyle\int_{y=0}^{1}\int_{x=0}^{1}\frac{1}{1-xy}\,dx\,dy = \frac{\pi^2}{6}$.