Lesson 8: Number Theory

1: Show that $\lfloor (2+\sqrt{3})^n \rfloor$ is odd for every positive integer n.

2: Show that there is no non-constant polynomial $f(x)$ with integral coefficients such that $f(n)$ is prime for every positive integer n.

3: Find all prime powers p^k with $k > 1$ such that $p^k = 2^n \pm 1$ for some integer n.

4: Find all positive integers n such that $n^4 + 4^n$ is prime.

5: Find all integral solutions to $x^2 + y^2 + z^2 = x^2 y^2$.

6: (a) Prove Wilson's Theorem: for an integer $n > 1$, n is prime $\iff$ $(n-1)! = -1 \bmod n$.

(b) Show that for a positive integer $n > 1$, $\left\lfloor \frac{(n-1)!+1}{n} - \left\lfloor \frac{(n-1)!}{n} \right\rfloor \right\rfloor = \begin{cases} 1 \text{ if } n \text{ is prime} \\ 0 \text{ if } n \text{ is composite} \end{cases}$

7: (a) Show that there are infinitely many primes of the form $4n+3$ where n is an integer.
(b) Show that there are infinitely many primes of the form $4n+1$ where n is an integer.

8: For a positive integer n, let $\tau(n)$ denote the number of positive divisors of n and let $\sigma(n)$ denote the sum of the positive divisors of n.

(a) Show that if $n = {p_1}^{k_1}{p_2}^{k_2}\cdots{p_m}^{k_m}$ is the prime factorization of n then $\tau(n) = \prod_{i=1}^{m}(k_i+1)$ and $\sigma(n) = \prod_{i=1}^{m} \frac{{p_i}^{k_i+1}-1}{p_i - 1}$

(b) For which positive integers n is $\tau(n)$ odd?

(c) For which positive integers n is $\sigma(n)$ odd?

(d) For which positive integers n do we have $\phi(n) + \sigma(n) = 2n$?

9: (a) Show that if $2^k + 1$ is prime then k must be a power of 2.
(b) Let $F_k = 2^{2^k} + 1$. Show that if $k \neq l$ then F_k and F_l are coprime.

10: (a) Let $a > 1$ and $k > 1$ be integers. Show that if $a^k - 1$ is prime then $a = 2$ and k is prime.
(b) Let $M_k = 2^k - 1$. Show that if k and l are coprime then so are M_k and M_l.
(c) Show that if p is prime and q is a prime divisor of $M_p = 2^p - 1$, then $q = 1 \bmod 2p$
(d) List the 6 smallest prime numbers of the form $M_p = 2^p - 1$ with p prime.

11: Show that every rational number p/q, where p and q are integers with $0 < p < q$, can be represented as a sum of distinct fractions of the form $1/n$, where n is a positive integer.

12: Let α and β be positive irrational numbers such that $\frac{1}{\alpha} + \frac{1}{\beta} = 1$. For $n \geq 1$ let $a_n = \lfloor n\alpha \rfloor$ and let $b_n = \lfloor n\beta \rfloor$. Show that the two sequences $\{a_n\}$ and $\{b_n\}$ are disjoint and that every positive integer occurs as a term in one of the two sequences.

Putnam Problems in Number Theory

1: (1985 B3) Let $\{a_{i,j}\}$ with $1 \leq i, j$ be an infinite array of positive integers. Suppose that every positive integer appears exactly eight times in the array. Show that there exists a pair of positive integers (m, n) such that $a_{m,n} > mn$.

2: (1986 A2) Determine the units digit of $\left\lfloor \frac{10^{20000}}{10^{100}+3} \right\rfloor$.

3: (1987 A2) The positive integers are written out in order to produce the sequence of digits $123456789101112\cdots$. For a positive integer n we define $f(n) = m$ when the 10^nth digit in this sequence occurs in the part of the sequence in which the m-digit numbers are placed. For example, $f(2) = 2$ because the 100th digit in the sequence occurs as the first digit of the 2 digit number 55. Find $f(1987)$.

4: (1988 B3) For every integer $n \geq 1$, let r_n be the minimum value of $|c - d\sqrt{3}|$ over all integers $c, d \geq 0$ with $c + d = n$. Find the smallest real number $g > 0$ with $r_n \leq g$ for all $n \geq 1$.

5: (1991 B4) Let p be an odd prime. Show that $\displaystyle\sum_{j=0}^{p} \binom{p}{j}\binom{p+j}{j} \equiv 2^p + 1 \pmod{p^2}$.

6: (1991 B5) Let p be an odd prime and let $\mathbf{Z}_p$ be the field of integers modulo p. Determine the number of elements in the set $\{x^2 | x \in \mathbf{Z}_p\} \cup \{y^2 + 1 | y \in \mathbf{Z}_p\}$.

7: (1993 A6) Let $\{a_n\}$, $n \geq 1$ be the following sequence of 2s and 3s

$$2, 3, 3, 2, 3, 3, 3, 2, 3, 3, 3, 2, 3, 3, 2, 3, 3, 3, 2, 3, 3, 3, 2, 3, 3, 3, 2, 3, 3, 2, 3, 3, 3, 2, 3, 3, 3, 2, \cdots$$

which has the property that if we let b_n be the number of 3s between the n^{th} and the $(n+1)^{\text{st}}$ copies of 2 in the sequence $\{a_n\}$, then we have $b_n = a_n$ for all $n \geq 1$. Show that there exists a real number r with the property that for all $n \geq 1$, we have $a_n = 2$ if and only if $n = 1 + \lfloor rm \rfloor$ for some nonnegative integer m.

8: (1997 B5) Define $f(1) = 2$, and for $n \geq 1$ define $f(n+1) = 2^{f(n)}$. Show that for all $n \geq 2$, we have $f(n) \equiv f(n-1) \pmod{n}$.

9: (1998 B5) Let $n = 111\cdots 1$, where there are 1998 digits all equal to 1. Find the 1000th digit after the decimal point in $\sqrt{n}$.

10: (2000 A2) Prove that there exist infinitely many integers n such that n, $n+1$ and $n+2$ are each equal to the sum of two squares of positive integers.