Solutions to the Number Theory Problems

1: Show that $\lfloor (2+\sqrt{3})^n \rfloor$ is odd for every positive integer n.

Solution: Notice that $(2+\sqrt{3})^n + (2-\sqrt{3})^n = \sum_{i=0}^{n} \binom{n}{i} 2^{n-i}\sqrt{3}^{\,i} + \sum_{i=0}^{n} (-1)^i \binom{n}{i} 2^{n-i}\sqrt{3}^{\,i} = 2\sum_{j=0}^{\lfloor n/2 \rfloor} \binom{n}{2j} 2^{n-2j} 3^j$, which is an even number, and we have $0 < (2-\sqrt{3})^n < 1$, so $\lfloor (2+\sqrt{3})^n \rfloor$ is odd.

2: Show that there is no non-constant polynomial $f(x)$ with integral coefficients such that $f(n)$ is prime for every positive integer n.

Solution: Suppose, for a contradiction, that f is such a polynomial. Let $p = f(1)$, which is prime. Note that when $n = 1 \bmod p$, we have $f(n) = f(1) = 0 \bmod p$, so p divides $f(n)$. Since $f(n)$ is prime and p divides $f(n)$, we must have $f(n) = p$. Thus there are infinitely many values of n such that $f(n) = p$, and so the polynomial $g(x) = f(x) - p$ has infinitely many roots. But then $g(x) = 0$ for all x and so $f(x) = p$ for all x.

3: Find all prime powers p^k with $k > 1$ such that $p^k = 2^n \pm 1$ for some integer n.

Solution: Suppose first that $p^k = 2^n + 1$. Then $2^n = p^k - 1 = (p-1)(p^{k-1} + p^{k-2} + \cdots + p + 1)$, so $(p-1)$ and $(p^{k-1} + \cdots + p + 1)$ are both powers of 2. Since p is odd and $(p^{k-1} + \cdots + p + 1)$ is even, k must be even, say $k = 2m$. Then we have $2^n = p^{2m} - 1 = (p^m - 1)(p^m + 1)$, so $(p^m - 1)$ and $(p^m + 1)$ are powers of 2 that differ by 2, hence we must have $p^m - 1 = 2$ and $p^m + 1 = 4$, and so $p = 3$ and $m = 1$. Thus there is only one prime power p^k of the form $2^k + 1$, namely $3^2 = 9$.

Next we suppose that $p^k = 2^n - 1$. If n is even, say $n = 2m$, then we have $p^k = 2^{2m} - 1 = (2^m + 1)(2^m - 1)$, but then $p \big| (2^m + 1)$ and $p \big| (2^m - 1)$ so $p = 1$, which is not possible. Thus n must be odd, say $n = 2m + 1$. If k is even, say $k = 2l$, then $p^{2l} = 2^{2m+1} - 1$, but then since p is odd, $p^2 = 1 \bmod 4$ so $p^{2l} = 1 \bmod 4$, so we would have $2^{2m+1} = p^{2l} + 1 = 2 \bmod 4$, so that $m = 0$, which is not possible. Thus k is odd, so we have $p^k = 2^n - 1$ with k and n both odd. We have $2^n = p^k + 1 = (p+1)(p^{k-1} - p^{k-2} + \cdots \pm 1)$, so $(p+1)$ is a power of 2, say $p + 1 = 2^l$. Then $2^n = p^k + 1 = (2^l - 1)^k + 1 = \big((2^l)^k - (2^l)^{k-1} + \cdots + (2^l) - 1\big) + 1 = (2^l)^k - (2^l)^{k-1} + \cdots + (2^l)$, and this is an odd multiple of 2^l. Since 2^n is an odd multiple of 2^l we must have $2^n = 2^l$. So we have $2^n = p^k + 1 = (2^n - 1)^k + 1$. This only happens when $n = 1$ or when $k = 1$, neither of which is allowed. Thus there are no prime powers p^k of the form $2^k - 1$.

We remark that we never made use of the fact that p is prime.

4: Find all positive integers n such that $n^4 + 4^n$ is prime.

Solution: When $n = 1$, $n^4 + 4^n = 5$, which is prime. We claim that when $n > 1$, $n^4 + 4^n$ is not prime. When n is even, $n^4 + 4^n$ is even and more that 2, so it is not prime. When n is odd, say $n = 2k + 1$, we make use of the factorization $(x^4 + 4y^4) = (x^2 + 2xy + 2y^2)(x^2 - 2xy + 2y^2)$ to get $n^4 + 4^n = (2k+1)^4 + 4^{2k+1} = (2k+1)^4 + 4(2^k)^4 = \big((2k+1)^2 + 2(2k+1)(2^k) + 2(2^k)^2\big)\big((2k+1)^2 - 2(2k+1)(2^k) + 2(2^k)^2\big)$. Note that when $k > 0$, both factors are greater than 1.

5: Find all integral solutions to $x^2+y^2+z^2=x^2y^2$.

Solution: Let (x_0,y_0,z_0) be a solution. Working modulo 4 we have the following: if all three of the numbers x_0, y_0 and z_0 are odd, then ${x_0}^2+{y_0}^2+{z_0}^2=3$ mod 4 but ${x_0}^2{y_0}^2=1$ mod 4; if two of the three numbers are odd then ${x_0}^2+{y_0}^2+{z_0}^2=2$ mod 4 but ${x_0}^2{y_0}^2=1$ or 0 mod 4; if one of the three numbers is odd then ${x_0}^2+{y_0}^2+{z_0}^2=1$ mod 4 while ${x_0}^2{y_0}^2=0$ mod 4. Thus all three of the numbers x_0, y_0 and z_0 must be even, say $x_0=2x_1$, $y_0=2y_1$ and $z_0=2z_1$. Note that we have ${x_1}^2+{y_1}^2+{z_1}^2=4\,{x_1}^2{y_1}^2$. Working modulo 4 we find that all three of the numbers x_1, y_1 and z_1 are even, say $x_1=2x_2$, $y_1=2y_2$ and $z_1=2z_2$. Continuing in this way, we obtain a sequence of triples (x_k,y_k,z_k) with ${x_k}^2+{y_k}^2+{z_k}^2=4^k{x_k}^2{y_k}^2$ and with $x_k=2x_{k+1}$, $y_k=2y_{k+1}$ and $z_k=2z_{k+1}$. Since $x_0=2x_1=2^2x_2=\cdots=2^kx_k=\cdots$ so that x_0 is a multiple of 2^k for every k, we must have $x_0=0$. Similarly, $y_0=z_0=0$. Thus the only solution to the given equation is the zero solution.

6: (a) Prove Wilson's Theorem: for an integer $n>1$, n is prime $\iff$ $(n-1)!=-1$ mod n.

Solution: If $n=4$ then we have $(n-1)!=3!=6=2$ mod 4. If $n=kl$ where $1<k<l<n$ then $(n-1)!=1\cdot 2\cdots k\cdots l\cdots(n-1)$ so $n=kl\big|(n-1)!$ and we have $(n-1)!=0$ mod n. If $n=k^2$ with $k>2$ then we have $1<k<2k<n$ so $(n-1)!=1\cdot 2\cdots k\cdots(2k)\cdots n$ and so $n=k^2\big|(n-1)!$ and hence $(n-1)!=0$ mod n. Thus when n is composite, if $n=4$ then $(n-1)!=2$ mod n, and if $n>4$ then $(n-1)!=0$ mod n.

Suppose that $n=p$ is prime. When $p=2$ we have $(p-1)!=1=-1$ mod p, so suppose p is odd. Let $f(x)=x^{p-1}-1$. By Fermat's Little Theorem, $f(x)=0$ in $\mathbf{Z}_p$ for every $x\in\mathbf{Z}_p$, and so $f(x)$ factors in $\mathbf{Z}_p[x]$ as $f(x)=(x-1)(x-2)\cdots(x-(p-1))$. Put in $x=0$ to get $-1=(-1)^{p-1}(p-1)!\in\mathbf{Z}_p$. Since p is odd this gives $(p-1)!=-1\in\mathbf{Z}_p$.

(b) Show that for a positive integer $n>1$, $\left\lfloor\frac{(n-1)!+1}{n}-\left\lfloor\frac{(n-1)!}{n}\right\rfloor\right\rfloor=\begin{cases}1 \text{ if } n \text{ is prime}\\ 0 \text{ if } n \text{ is composite}\end{cases}$

Solution: Suppose that n is prime. Then we have $(n-1)!=-1$ mod n, say $(n-1)!=-1+kn$. Then $\left\lfloor\frac{(n-1)!+1}{n}-\left\lfloor\frac{(n-1)!}{n}\right\rfloor\right\rfloor=\left\lfloor k-\left\lfloor k-\frac{1}{n}\right\rfloor\right\rfloor=\left\lfloor k-(k-1)\right\rfloor=1$. Now suppose that $n=4$. Then $\left\lfloor\frac{(n-1)!+1}{n}-\left\lfloor\frac{(n-1)!}{n}\right\rfloor\right\rfloor=\left\lfloor\frac{7}{4}-\left\lfloor\frac{3}{2}\right\rfloor\right\rfloor=\left\lfloor\frac{7}{4}-1\right\rfloor=0$. Finally, suppose that n is composite with $n>4$. Then $(n-1)!=0$ mod n, say $(n-1)!=kn$. Then $\left\lfloor\frac{(n-1)!+1}{n}-\left\lfloor\frac{(n-1)!}{n}\right\rfloor\right\rfloor=\left\lfloor k+\frac{1}{n}-\lfloor k\rfloor\right\rfloor=\left\lfloor k+\frac{1}{n}-k\right\rfloor=\left\lfloor\frac{1}{n}\right\rfloor=0$.

7: (a) Show that there are infinitely many primes of the form $4n+3$ where n is an integer.

Solution: Let $p_1,p_2,\cdots,p_k$ be any list of primes of the form $4n+3$. Consider the number $m=4\,p_1p_2\cdots p_n-1$. Since m is odd, its prime factors are odd, and every odd number is equal to 1 or 3 mod 4. It is not possible that every prime factor of m is equal to 1 mod 4, since $m=3$ mod 4. Thus m must have some prime factor, say p, which is equal to 3 mod 4. Note that p is not equal to any of the primes $p_1,p_2,\cdots,p_k$ since they are not factors of m. Thus given any k primes of the form $4n+3$, there exists another such prime.

(b) Show that there are infinitely many primes of the form $4n+1$ where n is an integer.

Solution: We claim that for any integer a, the number a^2+1 has no prime factors of the form $4n+3$. To prove this, let p be any odd prime factor of a^2+1. Then $a^2=-1$ mod p. Raise both sides to the power of $(p-1)/2$ to get $a^{p-1}=(-1)^{(p-1)/2}$. Since $a^2=-1$ mod p, p is not a factor of a, so by Fermat's Little Theorem $a^{p-1}=1$ mod p, and so we have $1=(-1)^{(p-1)/2}$ mod p. Thus $(p-1)/2$ must be even and so p must be of the form $4n+1$. This proves the claim.

Now let $p_1,p_2,\cdots,p_k$ be any primes of the form $4n+1$. Consider the number $m=(2p_1p_2\cdots p_k)^2+1$. Since m is odd it has an odd prime factor, say p, and since m is of the form a^2+1, the prime p must be of the form $4n+1$. Note that p is not equal to any of the primes $p_1,p_2,\cdots,p_k$ since they are not factors of m.

8: For a positive integer n, let $\tau(n)$ denote the number of positive divisors of n and let $\sigma(n)$ denote the sum of the positive divisors of n.

(a) Show that if $n = {p_1}^{k_1}{p_2}^{k_2}\cdots{p_m}^{k_m}$ is the prime factorization of n then $\tau(n) = \prod_{i=1}^{m}(k_i+1)$ and $\sigma(n) = \prod_{i=1}^{m}\frac{{p_i}^{k_i+1}-1}{p_i-1}$

Solution: The positive factors of $n = {p_1}^{k_1}{p_2}^{k_2}\cdots{p_m}^{k_m}$ are of the form ${p_1}^{j_1}{p_2}^{j_2}\cdots{p_m}^{j_m}$ with $0 \le j_i \le k_i$ for all i. Since there are k_i+1 choices for the exponent j_i, the total number of factors is $\tau(n) = \prod_{i=1}^{m}(k_i+1)$.

The factors of p^k are $1, p, p^2, \cdots, p^k$, so we have $\sigma(p^k) = 1 + p + p^2 + \cdots + p^k = \frac{p^{k+1}-1}{p-1}$. Thus to show that the given formula for $\sigma(n)$ is correct, it suffices to show that $\sigma(rs) = \sigma(r)\sigma(s)$ whenever $\gcd(r,s) = 1$. When $\gcd(r,s) = 1$, the divisors $d|rs$ are of the form $d = ab$ where $a|r$ and $b|s$, so we have $\sigma(rs) = \sum_{d|rs} d = \sum_{a|r}\sum_{b|s} ab = \left(\sum_{a|r} a\right)\left(\sum_{b|s} b\right) = \sigma(r)\sigma(s)$, as desired.

(b) For which positive integers n is $\tau(n)$ odd?

Solution: $\tau(n)$ is odd when all primes have an even exponent in the prime factorization of n, that is when n is a square.

(c) For which positive integers n is $\sigma(n)$ odd?

Solution: Note that $\sigma(2^k) = (1 + 2 + 4 + \cdots + 2^k)$ is odd for all values of $k \ge 1$, and note that for an odd prime p, $\sigma(p^k) = (1 + p + p^2 + \cdots + p^k)$ is odd when k is even, so $\sigma(n)$ is odd when all odd primes have an even exponent in the prime factorization of n, that is when n is either a square or twice a square.

(d) For which positive integers n do we have $\phi(n) + \sigma(n) = 2n$?

Solution: We claim that for $n > 1$, $\phi(n) + \sigma(n) = 2n$ when n is prime and $\phi(n) + \sigma(n) > 2n$ when n is composite. When p is prime we have $\phi(p) + \sigma(p) = (p-1) + (1+p) = 2p$ When p is prime and $k \ge 2$ we have $\phi(p^k) + \sigma(p^k) = (p^k - p^{k-1}) + (1 + p + \cdots + p^{k-1} + p^k) = 2p^k + (1 + p + \cdots + p^{k-2}) > 2p^k$. Finally, suppose that $r > 1$ and $s > 1$ are coprime with $\phi(r) + \sigma(r) \ge 2r$ and $\phi(s) + \sigma(s) \ge 2s$. We need to show that $\phi(rs) + \sigma(rs) > 2rs$. Let $\epsilon(r) = r - \phi(r)$ and $\epsilon(s) = s - \phi(s)$. Note that $\epsilon(r) > 0$ and $\epsilon(s) > 0$. Also, since $\phi(r) + \sigma(r) \ge 2r$ we have $r - \epsilon(r) + \sigma(r) \ge 2r$ so $\sigma(r) \ge r + \epsilon(r)$. Similarly $\sigma(s) \ge s + \epsilon(s)$, and so $\phi(rs) + \sigma(rs) = \phi(r)\phi(s) + \sigma(r)\sigma(s) \ge (r - \epsilon(r))(s - \epsilon(s)) + (r + \epsilon(r))(s + \epsilon(s)) = 2rs + 2\epsilon(r)\epsilon(s) > 2rs$.

9: (a) Show that if $2^k + 1$ is prime then k must be a power of 2.

Solution: We remark that when r is odd, $x = -1$ is a root of $x^r + 1$, so $x + 1$ is a factor of $x^r + 1$. Suppose that k is not a power of 2. Then we can write $k = 2^n r$ for some $n \ge 0$ and some odd number $r > 1$, and then we have $2^k + 1 = 2^{2^n r} + 1$. By the above remark, $2^{2^n} + 1$ is a factor of $2^{2^n r} + 1 = 2^k + 1$, so $2^k + 1$ is not prime.

(b) Let $F_k = 2^{2^k} + 1$. Show that if $k \ne l$ then F_k and F_l are coprime.

Solution: We remark that $x = -1$ is a root of $x^{2^n} - 1$ and so $x + 1$ is a factor of $x^{2^n} - 1$. Let $k < l$. We claim that $F_k \big| (F_l - 2)$. Write $n = l - k$. Then $F_l - 2 = 2^{2^l} - 1 = 2^{2^{k+n}} - 1 = \left(2^{2^k}\right)^{2^n} - 1$. By the above remark, $2^{2^k} + 1$ is a factor of $\left(2^{2^k}\right)^{2^n} - 1$, that is $F_k \big| (F_l - 2)$, as claimed. Since $F_k \big| (F_l - 2)$ and F_k and F_l are odd, it follows that F_k and F_l are coprime.

10: (a) Let $a > 1$ and $k > 1$ be integers. Show that if $a^k - 1$ is prime then $a = 2$ and k is prime.

Solution: Suppose that $a > 2$. Then $a^k - 1 = (a-1)(a^{k-1} + a^{k-2} + \cdots + a + 1)$, and $(a-1) > 1$ and $(a^{k-2} + a^{k-1} + \cdots + a + 1) > 1$, so $a^k - 1$ is not be prime. Thus if $a^k - 1$ is prime then we must have $a = 2$. Now suppose that $a = 2$ and that $k = lm$ with $1 < l$ and $1 < m$. Then $a^k - 1 = 2^{lm} - 1 = \left(2^l\right)^m - 1 = (2^m - 1)\left((2^m)^{l-1} + \cdots + (2^m) + 1\right)$, so $2^m - 1$ is a factor of $2^{lm} - 1$. Since $1 < 2^m - 1 < 2^{lm} - 1$ it follows that $2^{lm} - 1$ is not prime.

(b) Let $M_k = 2^k - 1$. Show that if k and l are coprime then so are M_k and M_l.

Solution: Suppose that M_k and M_l are not coprime. Let $d = \gcd(M_k, M_l)$. Note that d is odd (since M_k and M_l are odd), so 2 is an invertible element in $\mathbf{Z}_d$. Let n be the order of 2 in $\mathbf{Z}_d$ (so n is the smallest positive integer such that $2^n = 1$ in $\mathbf{Z}_d$). Since $d \big| M_k = 2^k - 1$ we have $2^k = 1 \in \mathbf{Z}_d$ and so $n \big| k$. Similarly $n \big| l$ and so $\gcd(k, l) \geq n > 1$.

(c) Show that if p is prime and q is a prime divisor of $M_p = 2^p - 1$, then $q = 1 \bmod 2p$

Solution: Let q be a prime divisor of M_p. By Fermat's Little Theorem we have $2^{q-1} = 1 \bmod q$ and so $q \big| (2^{q-1} - 1) = M_{q-1}$. Since $q \big| M_{q-1}$ and $q \big| M_p$, we have $\gcd(M_{q-1}, M_p) \neq 1$, so by part (b) $\gcd(q-1, p) \neq 1$. Since p is prime, this implies that $p \big| (q-1)$ so $q = 1 \bmod p$. Since q and p are both odd, $q = 1 \bmod 2p$.

(d) List the 6 smallest prime numbers of the form $M_p = 2^p - 1$ with p prime.

Solution: We have $M_2 = 3$, $M_3 = 7$, $M_5 = 31$, $M_7 = 127$, $M_{11} = 2047$, $M_{13} = 8191$ and $M_{17} = 131071$. The first 4 of these, 3, 7, 31 and 127 are easily seen to be prime. If q is a prime factor of M_{11} then by part (c) we have $q = 1 \bmod 22$, that is $q = 1, 23, 45, \cdots$. We try $q = 23$ and find that $M_{11} = 23 \cdot 89$, so M_{11} is not prime. If q is a prime factor of M_{13} then $q = 1 \bmod 26$. The only such primes with $q \leq \sqrt{M_{13}}$ are $q = 53$ and 79. We test 53 and 79 and find they are not factors of M_{13}, so M_{13} is prime. Finally, if q is a prime factor of M_{17} then by part (c) we have $q = 1 \bmod 34$, and the only primes q with $q = 1 \bmod 34$ and $q \leq \sqrt{M_{17}}$ are $q = 103, 137, 239$ and 307. We try each of these and find they are not factors of M_{17}

11: Show that every rational number p/q, where p and q are integers with $0 < p < q$, can be represented as a sum of distinct fractions of the form $1/n$, where n is a positive integer.

Solution: Given the rational number $\frac{p}{q}$ with $0 < p < q$, we choose that smallest positive integer n_1 with $\frac{1}{n_1} \leq \frac{p}{q}$, then if $\frac{1}{n_1} \neq \frac{p}{q}$ we choose the smallest positive integer n_2 such that $\frac{1}{n_1} + \frac{1}{n_2} \leq \frac{p}{q}$, then if $\frac{1}{n_1} + \frac{1}{n_2} \neq \frac{p}{q}$ we choose the smallest positive integer n_3 such that $\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} \leq \frac{p}{q}$, and so on. Note that we have $1 < n_1 < n_2 < n_3 < \cdots$ since if we had $n_{k+1} \leq n_k$ then we would have $\frac{p}{q} \geq \frac{1}{n_1} + \cdots + \frac{1}{n_{k-1}} + \frac{1}{n_k} + \frac{1}{n_{k+1}} \geq \frac{1}{n_1} + \cdots + \frac{1}{n_{k-1}} + \frac{2}{n_k} \geq \frac{1}{n_1} + \cdots + \frac{1}{n_{k-1}} + \frac{1}{n_k - 1}$ contradicting our choice of n_k. Also note that by our choice of n_1 we have $\frac{1}{n_1} \leq \frac{p}{q} < \frac{1}{n_1 - 1}$ so $n_1 - 1 < \frac{q}{p}$ and so $pn_1 - p < q$ and hence $pn_1 - q < p$, and so the numerator of $\frac{p}{q} - \frac{1}{n_1} = \frac{pn_1 - q}{qn_1}$ is smaller than the numerator of $\frac{p}{q}$. Similarly, we see that the numerators of the fractions $\frac{p}{q} - \frac{1}{n_1} - \cdots - \frac{1}{n_k}$ decrease with each value of k. Eventually, the numerator becomes zero, and we obtain $\frac{p}{q} = \frac{1}{n_1} + \frac{1}{n_2} + \cdots + \frac{1}{n_k}$.

12: Let α and β be positive irrational numbers such that $\frac{1}{\alpha}+\frac{1}{\beta}=1$. For $n\geq 1$ let $a_n=\lfloor n\alpha\rfloor$ and let $b_n=\lfloor n\beta\rfloor$. Show that the two sequences $\{a_n\}$ and $\{b_n\}$ are disjoint and that every positive integer occurs as a term in one of the two sequences.

Solution: Suppose, for a contradiction, that $\{a_n\}$ and $\{b_n\}$ are not disjoint, say $\lfloor n\alpha\rfloor=\lfloor mb\rfloor=k$. Since α and β are irrational, we have $k<n\alpha<k+1$ and $k<m\beta<k+1$. Since $\left(k<n\alpha \text{ and } k<m\beta\right)$ we have $\left(\frac{n}{k}>\frac{1}{\alpha} \text{ and } \frac{m}{k}>\frac{1}{\beta}\right)$ and so $\frac{n+m}{k}>\frac{1}{\alpha}+\frac{1}{\beta}=1$, and hence $n+m>k$. Similarly, since $\left(n\alpha<k+1\right.$ and $\left.n\beta<k+1\right)$, we have $n+m<k+1$. This gives the desired contradiction, since we cannot have $k<n+m<k+1$, so the sequences $\{a_n\}$ and $\{b_n\}$ are disjoint.

Note that since $\frac{1}{\alpha}+\frac{1}{\beta}=1$ we have $\frac{1}{\alpha}<1$ so $\alpha>1$, and so $\lfloor n\alpha\rfloor<\lfloor(n+1)\alpha\rfloor$. Thus the elements of $\{a_n\}$ are distinct. Also note that since $\left\lfloor\frac{k}{\alpha}\right\rfloor<\frac{k}{\alpha}$, we have $\left\lfloor\frac{k}{\alpha}\right\rfloor\alpha<k$, and since $\left\lfloor\frac{k}{\alpha}\right\rfloor>\frac{k}{\alpha}-1$ we have $\left\lfloor\frac{k}{\alpha}\right\rfloor\alpha>k-\alpha$ so $\left(\left\lfloor\frac{k}{\alpha}\right\rfloor+1\right)\alpha>k$, and so the number of elements in the sequence $\{a_n\}$ which are less than k is equal to $\left\lfloor\frac{k}{\alpha}\right\rfloor$. Similarly, the number of elements in the sequence $\{b_n\}$ which are less than k is equal to $\left\lfloor\frac{k}{\beta}\right\rfloor$. Thus in order to show that every positive integer occurs as a term in one of the two sequences, it suffices to show that $\left\lfloor\frac{k}{\alpha}\right\rfloor+\left\lfloor\frac{k}{\beta}\right\rfloor=k-1$ for every positive integer k.

Since $\frac{k}{\alpha}-1<\left\lfloor\frac{k}{\alpha}\right\rfloor<\frac{k}{\alpha}$ and $\frac{k}{\beta}-1<\left\lfloor\frac{k}{\beta}\right\rfloor<\frac{k}{\beta}$, we have $k-2=\frac{k}{\alpha}+\frac{k}{\beta}-2<\left\lfloor\frac{k}{\alpha}\right\rfloor+\left\lfloor\frac{k}{\beta}\right\rfloor<\frac{k}{\alpha}+\frac{k}{\beta}=k$, and so $\left\lfloor\frac{k}{\alpha}\right\rfloor+\left\lfloor\frac{k}{\beta}\right\rfloor=k-1$, as required.