Lesson 9: Linear Algebra

1: (a) Determine whether the set $\left\{ \frac{1}{\sqrt{2}-a} \middle| a \in \mathbf{Q} \right\} \subseteq \mathbf{R}$ is linearly independent over $\mathbf{Q}$.

(b) Determine whether the set $\left\{ \frac{1}{x-a} \middle| a \in \mathbf{Q} \right\} \subseteq \mathbf{Q}(x)$ is linearly independent over $\mathbf{Q}$.

2: (a) Find $\dim U$ where $U = \operatorname{Span}\left\{ \cos(x-a) \middle| a \in \mathbf{R} \right\} \subseteq \mathcal{C}^0(\mathbf{R})$.

(b) Find $\dim U$ where $U = \operatorname{Span}\left\{ \sin^2 x, \cos^2 x, \tan^2 x, \sec^2 x \right\} \subseteq \mathcal{C}^0\big(0, \frac{\pi}{2}\big)$.

3: (a) Find A^{-1} where $A \in M_n(\mathbf{R})$ with $A_{i,j} = \begin{cases} 0 \text{ if } i = j \\ 1 \text{ if } i \neq j. \end{cases}$

(b) Find $\det A$ where $A \in M_n(\mathbf{R})$ with $A_{i,j} = \begin{cases} 2 \text{ if } i = j \\ 1 \text{ if } i \neq j. \end{cases}$

4: Let F be a field and let $A, B \in M_n(F)$.
(a) Show that if $\operatorname{trace}(A^TA + B^TB) = \operatorname{trace}(AB + A^TB^T)$ then $A = B^T$.
(b) Show that if $AB \in \operatorname{Span}\{A, B\}$ but $AB \notin \operatorname{Span}\{A\} \cup \operatorname{Span}\{B\}$ then $AB = BA$.

5: Let F be a field and let $A \in M_{k\times l}(F)$, $B \in M_{l\times n}(F)$ and $C \in M_{n\times m}(F)$.
(a) Show that $\operatorname{rank}(AB) \leq \operatorname{rank}(B)$.
(b) Show that $\operatorname{rank}(A) + \operatorname{rank}(B) \leq l + \operatorname{rank}(AB)$.
(c) Show that $\operatorname{rank}(AB) + \operatorname{rank}(BC) \leq \operatorname{rank}(B) + \operatorname{rank}(ABC)$.

6: Let V be a vector space over $\mathbf{R}$. Show that V is finite-dimensional if and only if V is not equal to the union of any countable set of proper subspaces.

7: Let S be a non-empty set and let F be a field. Let U be an n-dimensional subspace of the vector space F^S of all functions $f : S \to F$. Show that there exist elements $a_1, a_2, \cdots, a_n \in S$ and functions $f_1, f_2, \cdots, f_n \in U$ such that $f_j(a_i) = \delta_{i,j}$ for all indices i, j.

8: Let $A, B \in M_n(\mathbf{R})$ with $AB = BA$ and $\det(A + B) \geq 0$. Show that $\det\big(A^n + B^n\big) \geq 0$ for all $n \in \mathbf{Z}^+$.

9: Let $A, B \in M_n(\mathbf{C})$. Suppose that the eigenvalues of A are distinct from the eigenvalues of B. Show that the linear map $L : M_n(\mathbf{C}) \to M_n(\mathbf{C})$ given by $L(X) = AX - XB$ is bijective.

10: Show that the identity map $I : \mathbf{R} \to \mathbf{R}$ given by $I(x) = x$ is equal to the sum of two periodic maps.

Putnam Problems on Linear Algebra

1: (1985 B6) Let G be a finite subgroup of $M_n(\mathbf{R})$ under matrix multiplication. Let A be the sum of the matrices in G. Show that if $\operatorname{trace}(A) = 0$ then $A = 0$.

2: (1986 B6) Let F be a field and let $A, B, C, D \in M_n(F)$. Suppose that AB^T and CD^T are symmetric and that $AD^T - BC^T = I$. Show that $A^T D - C^T B = I$.

3: (1987 B5) Let $A \in M_{2n \times n}(\mathbf{C})$. Suppose that $x^T A \neq 0$ for all $0 \neq x \in \mathbf{R}^{2n}$. Show that for all $x \in \mathbf{R}^{2n}$ there exists $z \in \mathbf{C}^n$ such that $\operatorname{Re}\big(Az\big) = x$.

4: (1988 A6) Let U be an n-dimensional vector space over a field F, and let $L : U \to U$ be a linear map. Suppose that L has a set of $n+1$ eigenvectors any n of which are linearly independent. Show that L is a scalar multiple of the identity map.

5: (1990 A5) Determine whether there exist $A, B \in M_n(\mathbf{R})$ with $ABAB = 0$ but $BABA \neq 0$.

6: (1990 B3) Let M be the set of 2×2 matrices with entries in $\{0^2, 1^2, 2^2, \cdots, 14^2\}$. Let $S \subseteq M$. Show that if $|S| > 15^4 - 15^2 - 15 + 2$ then there exist $A, B \in S$ with $AB = BA$.

7: (1991 A2) Determine whether there exist $A \neq B \in M_n(\mathbf{R})$ with $A^3 = B^3$ and $A^2 B = B^2 A$ such that $A^2 + B^2$ is invertible.

8: (1992 B5) Determine whether the sequence $\left\{ \dfrac{\det(A_n)}{n!} \right\}_{n \geq 2}$ is bounded, where $A_n \in M_{n-1}(\mathbf{R})$ is the matrix with entries

$$A_{i,j} = \begin{cases} i+2 & \text{if } i = j, \\ 1 & \text{if } i \neq j. \end{cases}$$

9: (1992 B6) Let $S \subseteq M_n(\mathbf{R})$ be a subset with the following properties.
(1) $I \in S$,
(2) if $A \in S$ and $B \in S$ then either $AB \in S$ or $-AB \in S$ but not both,
(3) if $A \in S$ and $B \in S$ then $AB = \pm BA$,
(4) if $I \neq A \in S$ then there exists $B \in S$ such that $AB = -BA$.
Show that $|S| \leq n^2$.

10: (1994 B4) Let $A = \begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix}$. For $n \geq 1$, let d_n be the greatest common divisor of the entries of the matrix $A^n - I$. Show that $\lim\limits_{n \to \infty} d_n = \infty$.

11: (1997 B4) Determine whether there exists $A \in M_2(\mathbf{R})$ such that $\sin A = \begin{pmatrix} 1 & 1996 \\ 0 & 1 \end{pmatrix}$.

12: (1999 B5) Let $n \geq 3$ and let $\theta = \frac{2\pi}{n}$. Find $\det(I + A)$ where $A \in M_n(\mathbf{R})$ is the matrix with entries $A_{k,l} = \cos\big((k+l)\theta\big)$.