Solutions to the Linear Algebra Problems

1: (a) Determine whether the set $\left\{ \frac{1}{\sqrt{2}-a} \middle| a \in \mathbf{Q} \right\} \subseteq \mathbf{R}$ is linearly independent over $\mathbf{Q}$.

Solution: The given set is not linearly independent. Indeed $\frac{1}{\sqrt{2}-1} - \frac{1}{\sqrt{2}+1} = 2\sqrt{2} = \frac{4}{\sqrt{2}-0}$.

(b) Determine whether the set $\left\{ \frac{1}{x-a} \middle| a \in \mathbf{Q} \right\} \subseteq \mathbf{Q}(x)$ is linearly independent over $\mathbf{Q}$.

Solution: This set is linearly independent. Indeed, by the Partial Fractions Decomposition Theorem, the set $\left\{ x^k \middle| k \in \mathbf{N} \right\} \cup \left\{ \frac{1}{(x-a)^k} \middle| k \in \mathbf{Z}^+, a \in \mathbf{C} \right\}$ is a basis for $\mathbf{C}(x)$ over $\mathbf{C}$.

2: (a) Find $\dim U$ where $U = \operatorname{Span}\left\{ \cos(x-a) \middle| a \in \mathbf{R} \right\} \subseteq \mathcal{C}^0(\mathbf{R})$.

Solution: Note that $U = \operatorname{Span}\{ \cos x, \sin x \}$ because $\cos x \in U$ and $\sin x = \cos\left(x - \frac{\pi}{2}\right) \in U$, and for every $a \in \mathbf{R}$ we have $\cos(x-a) = \cos x \cos a + \sin x \sin a \in \operatorname{Span}\{\cos x, \sin x\}$. Also note that $\{\cos x, \sin x\}$ is linearly independent because if $a\cos x + b \sin x = 0$ for all x then taking $x = 0$ gives $a = 0$ and taking $x = \frac{\pi}{2}$ gives $b = 0$. Thus $\dim U = 2$.

(b) Find $\dim U$ where $U = \operatorname{Span}\left\{ \sin^2 x, \cos^2 x, \tan^2 x, \sec^2 x \right\} \subseteq \mathcal{C}^0\left(0, \frac{\pi}{2}\right)$.

Solution: Note that $\operatorname{Span}\left\{ \sin^2 x, \cos^2 x, \tan^2 x, \sec^2 x \right\} = \operatorname{Span}\left\{ \sin^2 x, \cos^2 x, \tan^2 x \right\}$ because we have

$$\sec^2 x = 1 + \tan^2 x = \sin^2 x + \cos^2 x + \tan^2 x\,.$$

Also note that $\left\{ \sin^2 x, \cos^2 x, \tan^2 x \right\}$ is linearly independent because if $a\sin^2 x + b\cos^2 x + c\tan^2 x = 0$ for all x then taking $x = \frac{\pi}{6}$, $\frac{\pi}{4}$ and $x = \frac{\pi}{3}$ gives the three equations $\frac{1}{4}a + \frac{1}{2}b + \frac{3}{4}c = 0$, $\frac{3}{4}a + \frac{1}{2}b + \frac{1}{4}c = 0$ and $\frac{1}{3}a + b + 3c = 0$, and the coefficient matrix is invertible since

$$\det \begin{pmatrix} \frac{1}{4} & \frac{1}{2} & \frac{3}{4} \\ \frac{3}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{3} & 1 & 3 \end{pmatrix} = \frac{3}{8} + \frac{1}{24} + \frac{9}{16} - \frac{1}{16} - \frac{9}{8} - \frac{3}{24} = -\frac{6}{8} - \frac{2}{24} + \frac{8}{16} = -\frac{3}{4} - \frac{1}{12} + \frac{1}{2} = -\frac{1}{3}.$$

Thus $\dim U = 3$.

3: (a) Find A^{-1} where $A \in M_n(\mathbf{R})$ with $A_{i,j} = \begin{cases} 0 \text{ if } i = j \\ 1 \text{ if } i \neq j. \end{cases}$

Solution: Let $B \in M_n(\mathbf{R})$ be the matrix whose entries are all equal to 1. Note that $A = B - I$ and $B^2 = nB$. For $x \in \mathbf{R}$ with x sufficiently near zero, we have

$$\begin{aligned}(xB - I)^{-1} &= -(I - xB)^{-1} = -\left(I + xB + x^2B^2 + x^3B^3 + \cdots\right) = -\left(I + xB + x^2nB + x^3n^2B + \cdots\right) \\ &= -\left(I + \tfrac{x}{1-xn}\,B\right) = \tfrac{x}{xn-1}\,B - I.\end{aligned}$$

By replacing x by 1, we guess that $A^{-1} = (B-I)^{-1} = \frac{1}{n-1}B - I$, and indeed we have

$$A\left(\tfrac{1}{n-1}B - I\right) = (B - I)\left(\tfrac{1}{n-1}B - I\right) = \tfrac{1}{n-1}B^2 - \left(1 - \tfrac{1}{n-1}\right)B + I = \tfrac{n}{n-1}B - \tfrac{n}{n-1}B + I = I.$$

Thus $A^{-1} = \frac{1}{n-1}B - I$.

(b) Let $a \in \mathbf{R}$. Find $\det A$ where $A \in M_n(\mathbf{R})$ with $A_{i,j} = \begin{cases} a \text{ if } i = j \\ 1 \text{ if } i \neq j. \end{cases}$

Solution: Let A_n and B_n denote the $n \times n$ matrices

$$A_n = \begin{pmatrix} a & 1 & 1 & \\ 1 & a & 1 & \\ 1 & 1 & a & \\ & & & \ddots \end{pmatrix}, \quad B_n = \begin{pmatrix} 1 & 1 & 1 & \\ 1 & a & 1 & \\ 1 & 1 & a & \\ & & & \ddots \end{pmatrix}.$$

By first performing the row operation $R_1 \mapsto R_1 - R_2$ on the matrix B_n, and then expanding the determinant along the first row, we find that $\det(B_n) = (a-1)\det B_{n-1}$. Since $\det(B_1) = 1$, it follows that $\det(B_n) = (a-1)^{n-1}$ for all $n \geq 1$. By performing the same row operation on the matrix A_n and then expanding the determinant along the first row, we find that $\det(A_n) = (a-1)\big(\det(A_{n-1}) + \det(B_{n-1})\big)$. Since $\det(A_1) = a$, an easy induction argument shows that $\det(A_n) = (a-1)^{n-1}(a+n-1)$ for all $n \geq 1$.

4: Let $A, B \in M_n(\mathbf{R})$.

(a) Show that if $\text{trace}\,(A^TA + B^TB) = \text{trace}\,(AB + A^TB^T)$ then $A = B^T$.

Solution: Suppose that $\text{trace}\,(A^TA + B^TB) = \text{trace}\,(AB + A^TB^T)$. Then using the inner product on $M_n(\mathbf{R})$ given by $\langle A, B\rangle = \text{trace}\,(B^TA)$ we have

$$\begin{aligned}\left|A - B^T\right|^2 &= \text{trace}\,\big((A - B^T)^T(A - B^T)\big) = \text{trace}\,\big((A^T - B)(A - B^T)\big)\\ &= \text{trace}\,\big(A^TA - A^TB^T - BA + BB^T\big)\\ &= \text{trace}\,(A^TA) + \text{trace}\,(BB^T) - \text{trace}\,(BA) - \text{trace}\,(A^TB^T)\\ &= \text{trace}\,(A^TA) + \text{trace}\,(B^TB) - \text{trace}\,(AB) - \text{trace}\,(A^TB^T)\\ &= \text{trace}\,(A^TA + B^TB) - \text{trace}\,(AB + A^TB^T) = 0.\end{aligned}$$

(b) Show that if $AB \in \text{Span}\,\{A, B\}$ but $AB \notin \text{Span}\,\{A\} \cup \text{Span}\,\{B\}$ then $AB = BA$.

Solution: Suppose that $AB \in \text{Span}\,\{A, B\}$ but $AB \notin \text{Span}\,\{A\} \cup \text{Span}\,\{B\}$. Then we have $AB = sA + tB$ for some non-zero real numbers $0 \neq s, t \in \mathbf{R}$. Note that

$$(A - tI)(B - sI) = AB - sA - tB + stI = AB - AB + stI = stI$$

and so we see that $(A - tI)$ is invertible with $(A - tI)^{-1} = \frac{1}{st}(B - sI)$. It follows that

$$I = \tfrac{1}{st}(B - sI)(A - tI) = \tfrac{1}{st}(BA - tB - sA + stI) = \tfrac{1}{st}(BA - AB + stI)$$

so that $stI = BA - AB + stI$, and hence $BA - AB = 0$.

5: Let F be a field and let $A \in M_{k\times l}(F)$, $B \in M_{l\times m}(F)$ and $C \in M_{m\times n}(F)$.

(a) Show that $\text{rank}\,(AB) \leq \text{rank}\,(B)$.

Solution: Note that $\text{Range}\,(B^TA^T) \subseteq \text{Range}\,(B^T)$, indeed if $x \in \text{Range}\,(B^TA^T)$ then $x = B^TA^Ty$ for some $y \in \mathbf{R}^k$ and then we have $x = B^Tz$ for $z = A^Ty$ so that $x \in \text{Range}\,(B^T)$. Thus $\text{rank}\,(B^TA^T) \leq \text{rank}\,(B^T)$, so

$$\text{rank}\,(AB) = \text{rank}\,(B^TA^T) \leq \text{rank}\,(B^T) = \text{rank}\,(B).$$

(b) Show that $\text{rank}\,(A) + \text{rank}\,(B) \leq l + \text{rank}\,(AB)$.

Solution: Note that

$$\begin{aligned}\text{Range}\,(A) = A(\mathbf{R}^l) &= A\Big(\text{Range}\,(B) \oplus \text{Range}\,(B)^\perp\Big)\\ &= A\big(\text{Range}\,B\big) + A\big((\text{Range}\,B)^\perp\big)\\ &= \text{Range}\,(AB) + A\big((\text{Range}\,B)^\perp\big)\end{aligned}$$

so we have

$$\begin{aligned}\text{rank}\,(A) = \dim\big(\text{Range}\,A\big) &\leq \dim\big(\text{Range}\,(AB)\big) + \dim A\big((\text{Range}\,B)^\perp\big)\\ &= \text{rank}\,AB + \dim A\big(\text{Range}\,B)^\perp\big) \leq \text{rank}\,AB + \dim(\text{Range}\,B)^\perp\\ &= \text{rank}\,AB + l - \text{rank}\,B.\end{aligned}$$

(c) Show that $\text{rank}\,(AB) + \text{rank}\,(BC) \leq \text{rank}\,(B) + \text{rank}\,(ABC)$.

Solution: Applying Part (b) to the matrices $A \in M_{k\times l}(F)$ and $BC \in M_{l\times n}(F)$ gives

$$\text{rank}\,(A) + \text{rank}\,(BC) \leq l + \text{rank}\,(ABC).$$

In the case that B is onto, we have $\text{rank}\,(A) = \text{rank}\,(AB)$ and $l = \text{rank}\,(B)$ and so

$$\text{rank}\,(AB) + \text{rank}\,(BC) \leq \text{rank}\,(B) + \text{rank}\,(ABC)$$

as required. When B is not onto, replace the matrices C, B and A by the linear maps $C' : \mathbf{R}^n \to \mathbf{R}^m$ given by $C'(x) = Cx$, and $B' : \mathbf{R}^m \to \text{Range}\,(B)$ given by $B'(y) = By$, and $A' : \text{Range}\,(B) \to \mathbf{R}^k$ given by $A'(z) = Az$. The linear map B' is onto, and applying the above inequality to the linear maps A', B' and C' gives

$$\text{rank}\,(A'B') + \text{rank}\,(B'C') \leq \text{rank}\,(B') + \text{rank}\,(A'B'C').$$

Finally, notice that $\text{Range}\,(A'B') = \text{Range}\,(AB)$, $\text{Range}\,(B'C') = \text{Range}\,(BC)$, $\text{Range}\,(B') = \text{Range}\,(B)$ and $\text{Range}\,(A'B'C') = \text{Range}\,(ABC)$.

6: Let V be a vector space over $\mathbf{R}$. Show that V is finite-dimensional if and only if V is not equal to the union of any countable set of proper subspaces.

Solution: Suppose first that V is infinite dimensional. Choose a countable linearly independent subset of V, say $\mathcal{U} = \{u_1, u_2, u_3, \cdots\} \subseteq \mathcal{V}$. Extend $\mathcal{U}$ (if necessary) to a basis $\mathcal{U} \cup \mathcal{V}$ for V, where $\mathcal{U} \cap \mathcal{V} = \{0\}$. For each $k \in \mathbf{Z}^+$, let $V_k = \mathcal{V} \cup \{u_1, u_2, \cdots, u_k\}$. Then we have $V_1 \subsetneqq V_2 \subsetneqq V_3 \subsetneqq \cdots$ and $V = \bigcup_{k=1}^{\infty} V_k$.

Conversely, suppose that V is finite dimensional. We shall show that no affine space $P \subseteq V$ is equal to the union of a countable set of proper affine subspaces. We prove this by induction on the dimension of P. When $\dim P = 1$, the only proper affine subspaces of P are the one-point sets in P, and since P is uncountable it cannot be equal to the union of a countable set of proper affine subspaces. Let $n \geq 1$, and suppose, inductively, that no affine space $Q \subseteq V$ with $\dim Q = n - 1$ is equal to the union of any countable set of proper affine subspaces. Let $P \subseteq V$ be an affine space with $\dim P = n$. Let $R_1, R_2, R_3, \cdots$ be proper affine subspaces of P. Since P has uncountably many affine subspaces of dimension $n-1$, we can choose an affine subspace $Q \subseteq P$ with $\dim Q = n-1$ and $Q \neq R_i$ for any i. Since each set $R_i \cap Q$ is either empty or is a proper affine subspace of Q, it follows from the induction hypothesis that $Q \neq \bigcup\limits_{i=1}^{\infty} (R_i \cap Q)$. Thus $Q \not\subseteq \bigcup\limits_{i=1}^{\infty} R_i$ and hence $P \neq \bigcup\limits_{i=1}^{\infty} R_i$.

7: Let S be a non-empty set and let F be a field. Let U be an n-dimensional subspace of the vector space F^S of all functions $f : S \to F$. Show that there exist elements $a_1, a_2, \cdots, a_n \in S$ and functions $f_1, f_2, \cdots, f_n \in U$ such that $f_j(a_i) = \delta_{i,j}$ for all indices i, j.

Solution: Let $\{g_1, g_2, \cdots, g_l\}$ be a basis for U. For each $a \in S$, write $g(a) = \big(g_1(a), g_2(a), \cdots, g_l(a)\big)^T \in F^l$. Let $\mathcal{V} = \big\{g(a) \big| a \in S\big\}$ and let $V = \operatorname{Span} \mathcal{V} \subseteq F^l$. For all $t \in F^l$ we have

$$\begin{aligned} t \in V^\perp &\iff t \cdot g(a) = 0 \text{ for all } a \in S \\ &\iff t_1 g_1(a) + t_2 g_2(a) + \cdots + t_l g_l(a) = 0 \text{ for all } a \in S \\ &\iff t_1 g_1 + t_2 g_2 + \cdots + t_l g_l = 0 \in W \\ &\iff t = 0 \text{ , since } \mathcal{V} \text{ is linearly independent,} \end{aligned}$$

and so we have $V^\perp = \{0\}$ and hence $V = F^l$. Since $\mathcal{V}$ spans F^l, we can select a basis from amongst the elements of $\mathcal{V}$, and so we can choose $a_1, a_2, \cdots, a_l \in S$ so that $\big\{g(a_1), g(a_2), \cdots, g(a_l)\big\}$ is a basis for F^l. Let

$$A = \big(g(a_1), g(a_2), \cdots, g(a_l)\big) = \begin{pmatrix} g_1(a_1) & g_1(a_2) & \cdots & g_1(a_l) \\ g_2(a_1) & g_2(a_2) & \cdots & g_2(a_l) \\ & \vdots & & \\ g_l(a_1) & g_l(a_2) & \cdots & g_l(a_l) \end{pmatrix}$$

and note that A is invertible since $\big\{g(a_1), g(a_2), \cdots, g(a_l)\big\}$ is a basis for F^l. Let $B = A^{-1}$, say B has entries $b_{i,j} = B_{i,j}$, and define $f_1, f_2, \cdots, f_l \in U$ by $f_j = \sum\limits_{i=1}^{l} b_{j,i} g_i$. Then for $k = 1, 2, \cdots, l$ we have

$$f_j(a_k) = \big(\textstyle\sum b_{j,i} g_i\big)(a_k) = \sum b_{j,i} g_i(a_k) = (BA)_{j,k} = \delta_{j,k}.$$

8: Let $A, B \in M_n(\mathbf{R})$ with $AB = BA$ and $\det(A+B) \geq 0$. Show that $\det\big(A^n + B^n\big) \geq 0$ for all $n \in \mathbf{Z}^+$.

Solution: First we suppose that n is even, say $n = 2k$. Since the characteristic polynomial $f_A(x) = \det(A - xI)$ has finitely many roots, we can choose $\delta > 0$ so that for all $x \in (0, \delta)$ we have $f_A(x) \neq 0$ so that the matrix $A_x = A - xI$ is invertible. Then for all $x \in (0, \delta)$ we have

$$\begin{aligned} \det({A_x}^n + B^n) &= \det(A_x^{2k} + B^{2k}) = \det\big(A_x^{2k}(I + A_x^{-2k} B^{2k})\big) \\ &= \det\big({A_x}^k\big)^2 \det\big(I + i\, {A_x}^{-k} B^k\big) \det\big(I - i\, {A_x}^{-k} B^k\big) \\ &= \det\big({A_x}^k\big)^2 \big|\det\big(I + i\, {A_x}^{-k} B^k\big)\big|^2 \geq 0. \end{aligned}$$

Taking the limit as $x \to 0^+$ we obtain $\det(A^n + B^n) \geq 0$.

Next we suppose that n is odd, say $n = 2k+1$. Let $\alpha = e^{i\,\pi/n}$ so that $\alpha^n + 1 = 0$ and so that $x^n + 1$ factors as $x^n + 1 = (x+1) \prod\limits_{j=1}^{k} (x - \alpha^j)(x - \overline{\alpha}^j)$. Since $AB = BA$ we have $(A^n + B^n) = (A+B) \prod\limits_{j=1}^{k} (A - \alpha^j B)(A - \overline{\alpha}^j B)$ and so

$$\det(A^n + B^n) = \det(A+B) \prod_{j=1}^{k} \big(\det(A - \alpha^j B) \det(A - \overline{\alpha}^j B)\big) = \det(A+B) \prod_{j=1}^{k} \big|\det(A - \alpha^j B)\big|^2 \geq 0.$$

9: Let $A, B \in M_n(\mathbf{C})$. Suppose that the eigenvalues of A are distinct from the eigenvalues of B. Show that the linear map $L : M_n(\mathbf{C}) \to M_n(\mathbf{C})$ given by $L(X) = AX - XB$ is bijective.

Solution: Let $X \in \text{Ker}\,(L)$. Then we have

$$\begin{aligned} &AX = XB \\ &A^2X = AXB = XB^2 \\ &A^3X = A^2XB = AXB^2 = XB^3 \\ &A^4X = A^3XB = A^2XB^2 = AXB^3 = XB^4 \end{aligned}$$

and so on so that $A^kX = XB^k$ for all $k \geq 0$. It follows that $f(A)X = Xf(B)$ for every polynomial $f(x)$. In particular, we have $f_B(A)X = Xf_B(B) = 0$ where $f_B(x)$ is the characteristic polynomial of B. Let $\lambda_1, \cdots, \lambda_n$ be the eigenvalues of A (repeated according to multiplicity) and let $\mu_1, \cdots, \mu_n$ be the eigenvalues of B. Then $f_B(x) = (-1)^n \prod_{i=1}^{n} (x - \mu_i)$ and the eigenvalues of the matrix $f_B(A)$ are the values $f_B(\lambda_j) = (-1)^n \prod_{i=1}^{n} (\lambda_j - \mu_i)$. Note that $f_B(\lambda_j) \neq 0$ since $\lambda_j \neq \mu_i$ for any i, j. Since the eigenvalues of $f_B(A)$ are non-zero, it follows that the matrix $f_B(A)$ is invertible. Since $f_B(A)X = 0$ and $f_B(A)$ is invertible, we have $X = 0$. Thus $\text{Ker}\,(L) = \{0\}$ and so L is invertible.

10: Show that the identity map $I : \mathbf{R} \to \mathbf{R}$ given by $I(x) = x$ is equal to the sum of two periodic maps.

Solution: Let S be a basis for $\mathbf{R}$ over $\mathbf{Q}$. Each $x \in \mathbf{R}$ can be expressed uniquely as a linear combination $x = \sum_{t \in S} x_t \cdot t$ where each $x_t \in \mathbf{Q}$ with $x_t = 0$ for all but finitely many $t \in S$. For each $a \in S$ define a map $\phi_a : \mathbf{R} \to \mathbf{R}$ by $\phi_a(x) = x_a \cdot a$. Note that for every $b \in S$ with $b \neq a$, the function ϕ_a is periodic with period b because for $x \in \mathbf{R}$, if $x = \sum_{t \in S} x_t \cdot t = x_a \cdot a + x_b \cdot b + \sum_{t \neq a,b} x_t \cdot t$ then $x + b = x_a \cdot a + (x_b + 1) \cdot b + \sum_{t \neq a,b} x_t \cdot t$ and so we have $\phi_a(x + b) = x_a \cdot a = \phi_a(x)$.

To express the identity map $I(x)$ as a sum of two periodic functions, partition the basis S into two nonempty sets A and B, then define $f, g : \mathbf{R} \to \mathbf{R}$ by

$$f(x) = \sum_{a \in A} \phi_a(x) = \sum_{a \in A} x_a \cdot a \quad \text{and} \quad g(x) = \sum_{b \in B} \phi_b(x) = \sum_{b \in B} x_b \cdot b\,.$$

Note that the above sums contain only finitely many non-zero terms, so they are well-defined. Also note that f and g are periodic. Indeed for every $b \in B$, we have $f(x + b) = \sum_{a \in A} \phi_a(x + b) = \sum_{a \in A} \phi_a(x) = f(x)$, and so $f(x)$ is periodic with period b, and similarly, for every $a \in A$ the function $g(x)$ is periodic with period a.